ON CERTAIN LINKS IN 3-MANIFOLDS

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1. INTRODUCTION

A. H. Wallace [7] has shown that each closed, orientable 3-manifold can be transformed into the 3-sphere by removing a finite number of solid tori (a regular neighborhood of a compact, connected graph with Euler characteristic $1 - n$) and sewing them back in an appropriate manner. In particular, this can be done with any homotopy 3-sphere. In Theorem 2 we show that if the complement of the solid tori has free fundamental group and if the tori lie "nicely" with respect to some Heegaard splitting of the manifold, then the manifold is the connected sum of lens spaces, and therefore, if it is simply connected, it is the 3-sphere.

2. PRELIMINARIES

All spaces considered will be polyhedra, and all maps will be piecewise linear. We shall denote the n-sphere by S^n. It is known [1] that every closed, orientable 3-manifold can be represented as $H \cup H'$, where H and H' are solid tori of some genus n and $H \cap H' = \partial H = \partial H' = T$ is an orientable surface of genus n. Such a representation, denoted by $(H, H'; T)$, is a *Heegaard splitting of genus* n for the manifold. A manifold that possesses a Heegaard splitting of genus 1 is called a *lens space*. Note that this class of manifolds consists of the classical lens spaces along with the two spaces S^3 and $S^2 \times S^1$. The symbol $\mathrm{cl}(A)$ will denote the closure of A.

Let H be a solid torus of genus n, and let $D_1, \cdots, D_k$ $(k \leq n)$ be pairwise disjoint disks properly embedded in H, that is, let $D_i \cap \partial H = \partial D_i$ for each i. Let $J_1, \cdots, J_k$ be pairwise disjoint, simple closed curves in ∂H such that $J_i \cap D_i$ is a point for each i and $J_i \cap D_j = \emptyset$ for $i \neq j$. Then $\{J_1, \cdots, J_k\}$ is called a *set of* k *canonical longitudes for* H.

Let M and N be oriented, closed 3-manifolds, and let $B \subset M$ and $E \subset N$ be 3-cells. Let $h\colon \partial B \to \partial E$ be an orientation-reversing homeomorphism. Then the orientable, closed 3-manifold $(M - \mathrm{int}(B)) \cup_h (N - \mathrm{int}(E))$ is called the *connected sum of* M *and* N, and it is denoted by $M \# N$. Note that this is independent of the choices of B, E, and h.

3. THE THEOREMS

THEOREM 1. *Let* M *be a closed, orientable* 3-*manifold, and let* $J_1 \cup \cdots \cup J_n = J$ *be a link in* M. *Suppose* $\pi_1(M - J)$ *is a free group. Then* M *is homeomorphic to the connected sum of a homotopy* 3-*sphere with a finite number of lens spaces.*

Proof. First note that if N is a solid torus of genus 1 and E is a 3-cell such that $N \cap E = \partial N \cap \partial E = A$ is an annulus, then $N \cup E$ is a lens space with an open

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3-cell removed. This is true since $\partial(N \cup E)$ is a 2-sphere meeting ∂E in two disjoint disks.

Let N be a regular neighborhood of J_1, and assume that $N \cap \left(\bigcup_{i=2}^n J_i\right) = \emptyset$. Then, by Dehn's lemma [4], the loop theorem [5], and the fact that a subgroup of a free group is free, there exists a nonsingular disk D, properly embedded in $\mathrm{cl}(M - N)$, such that $D \cap \left(\bigcup_{i=2}^n J_i\right) = \emptyset$. If E is a thickening of D, then $M = \mathrm{cl}(M - (N \cup E)) \cup (N \cup E)$ is the connected sum of a lens space with M^*, a manifold containing $J_2, \cdots, J_n$. Therefore $\pi_1\left(M^* - \bigcup_{i=2}^n J_i\right)$ is free, by van Kampen's theorem [2]. Since the theorem is well known for the case $J = \emptyset$, we can now complete the proof by induction on the number of components of J.

It is possible to strengthen this result if the link J is canonical in some Heegaard splitting for M.

THEOREM 2. *Let* M *and* J *satisfy the conditions in Theorem* 1. *Suppose there exists a Heegaard splitting* $(H, H'; T)$ *of genus* $n \geq 2$ *for* M *such that* $\{J_1, \cdots, J_n\}$ *forms a set of canonical longitudes for* H. *Then* M *is the connected sum of a finite number of lens spaces.*

Proof. First we show that the rank of the free group $\pi_1(M - J)$ is n.

For $i = 1, \cdots, n$, let A_i be an annulus in H such that $A_i \cap A_j = \emptyset$ if $i \neq j$ and $A_i \cap \partial H = J_i \subset \partial A_i$. Let J_i' be $\partial A_i - J_i$ for $i = 1, \cdots, n$. Then $M - \bigcup_{i=1}^n J_i$ is homeomorphic to $M - \bigcup_{i=1}^n J_i'$.

Now choose pairwise disjoint, properly embedded 2-cells $D_1, \cdots, D_{n-1}$ in H so that $\left(\bigcup_{i=1}^n A_i\right) \cap \left(\bigcup_{i=1}^{n-1} D_i\right) = \emptyset$, each D_i separates H, and the closure of each component of $H - \bigcup_{i=1}^{n-1} D_i$ is a solid torus of genus 1. Then $\partial H \cup \left(\bigcup_{i=1}^{n-1} D_i\right)$ is a deformation retract of $H - \bigcup_{i=1}^n J_i'$.

Since ∂D_i represents the trivial element in $H_1(\partial H; Z)$ for $i = 1, \cdots, n - 1$, the inclusion-induced homomorphism

$$H_*(\partial H; Z) \to H_*\left(H - \bigcup_{i=1}^n J_i'; Z\right)$$

is an isomorphism for each $*$. Hence, $H_2\left(M - \bigcup_{i=1}^n J_i'; Z\right)$ is trivial; therefore, from the Mayer-Vietoris sequence in homology for $\left(H - \bigcup_{i=1}^n J_i', H'\right)$, we get the exact sequence

$$0 \to H_1(\partial H; Z) \to H_1\left(H - \bigcup_{i=1}^n J_i'; Z\right) \oplus H_1(H'; Z) \to H_1\left(M - \bigcup_{i=1}^n J_i'; Z\right) \to 0,$$

where both $H_1\left(H - \bigcup_{i=1}^n J_i'; Z\right)$ and $H_1(\partial H; Z)$ are of rank $2n$. Hence, the rank of $H_1\left(M - \bigcup_{i=1}^n J_i'; Z\right)$, and therefore of $\pi_1\left(M - \bigcup_{i=1}^n J_i'; x_0\right)$, is n for each x_0.

Hence, both $\pi_1\left(H' \cup \left(\bigcup_{i=1}^{n-1} D_i\right); x_0\right)$ and $\pi_1(H'; x_0)$ are free of rank n for each x_0 in H'. Van Kampen's theorem and the fact that a finitely generated free group is Hopfian [3, p. 109] therefore imply that ∂D_i represents the trivial element in $\pi_1(H')$. Consequently (again by Dehn's lemma), ∂D_i bounds a nonsingular disk D_i' in H', and we may assume that these disks are pairwise disjoint. Each of these disks separates H'.

We now have 2-spheres $D_1 \cup D_1', \cdots, D_{n-1} \cup D_{n-1}'$ in M, each of which separates M. Each component of $M - \bigcup_{i=1}^{n-1} (D_i \cup D_i')$ is the union of two solid tori of genus 1. These solid tori have common boundary except for the 2-cells D_i and D_i'. Hence, each component of $M - \bigcup_{i=1}^{n-1} (D_i \cup D_i')$ is a lens space with some 3-cells removed. Therefore, M is the connected sum of lens spaces.

COROLLARY. *Let* M *be a closed, simply connected* 3-*manifold. Then* M *is a* 3-*sphere if and only if for each Heegaard splitting* $(H, H'; T)$ *of* M *of genus* n *there exists a set* $\{J_1, \cdots, J_n\}$ *of canonical longitudes for* H *such that* $\pi_1\left(M - \bigcup_{i=1}^{n} J_i\right)$ *is a free group.*

Proof. The sufficiency of the condition follows directly from our Theorem 2, and the necessity from the fact [6] that any two Heegaard splittings of the same genus for the 3-sphere are equivalent.

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