A RECURSIVE FUNCTION, DEFINED ON A COMPACT INTERVAL AND HAVING A CONTINUOUS DERIVATIVE THAT IS NOT RECURSIVE

J. Myhill

We shall construct such a function f by placing  -shaped bumps at each number of the form 2^{-n}, where n belongs to a recursively enumerable, nonrecursive set $\mathscr{A}$, and by leaving the neighborhood of all other numbers 2^{-n} flat. For $n \in \mathscr{A}$, the slope of the graph at 2^{-n} can be effectively bounded from below, given n. Thus, if we could compute $f'(2^{-n})$ recursively, we could decide whether $n \in \mathscr{A}$, contradicting the nonrecursiveness of $\mathscr{A}$.

We first define the function f nonconstructively, and then prove that it is actually recursive.

Let

$$\theta(x) \equiv \begin{cases} x(x^2 - 1)^2 & \text{for } -1 \leq x \leq 1, \\ 0 & \text{for } |x| > 1. \end{cases}$$

Then $\theta(x)$ has the required form on $[-1, 1]$, and

$$\theta(-1) = \theta(0) = \theta(1) = 0, \qquad \theta'(-1) = \theta'(1) = 0, \qquad \theta'(0) = 1.$$

The function θ takes its minimum value $-\lambda$ at $x = -1/\sqrt{5}$ and its maximum $+\lambda$ at $x = +1/\sqrt{5}$. We call θ on $[-1, +1]$ a *bump* of length 2 and height λ. Now we define bumps $\theta_{\alpha\beta}$ of length 2α and height β.

The function $\theta_{\alpha\beta}(x) \equiv (\beta/\lambda)\, \theta(x/\alpha)$ satisfies the conditions

$$\theta_{\alpha\beta}(-\alpha) = \theta_{\alpha\beta}(0) = \theta_{\alpha\beta}(\alpha) = 0, \qquad \theta'_{\alpha\beta}(-\alpha) = \theta'_{\alpha\beta}(\alpha) = 0, \qquad \theta'_{\alpha\beta}(0) = \theta/\lambda\alpha,$$

$$-\beta \leq \theta_{\alpha\beta}(x) \leq \beta \qquad (-\alpha \leq x \leq \alpha).$$

For each $n \in \mathscr{A}$, we shall put a bump $\theta_{\alpha_n \beta_n}$ around 2^{-n}; that is, we define f(x) as follows:

If $n \in \mathscr{A}$ and $\delta \in [-\alpha_n, +\alpha_n]$, then $f(2^{-n} + \delta) \equiv \theta_{\alpha_n \beta_n}(\delta)$. Otherwise, $f(x) \equiv 0$. The parameters α_n, β_n for $n \in \mathscr{A}$ will be defined by

$$\alpha_n \equiv 2^{-k-2n-2}, \qquad \beta_n \equiv 2^{-k-n-2},$$

where $n = h(k)$ and h is a function enumerating $\mathscr{A}$ without repetitions.

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To see that the function f is well-defined, we need only prove that no two bumps overlap. For $n > 0$, the half-lengths of the bumps around the two points 2^{-n} and 2^{1-n} are at most 2^{-2n-2} and 2^{-2n}, and the sum of the two half-lengths is less than the distance 2^{-n} between the two points. Since the graph of f consists of alternate bumps and horizontal line-segments, and since the bumps have horizontal half-tangents at their end-points, f has a continuous derivative except possibly at $x = 0$.

Suppose $x < 2^{-n}$. Then, if x is not on a bump, $f(x) = 0$, while if x is on the bump surrounding 2^{-m} $(m \geq n)$, $f(x) \leq \alpha_m = 2^{-k-2n-2}$ and

$$f(x)/x \leq f(x)/2^{-m-1} < 2^{-m} \leq 2^{-n}.$$

Thus $f(x)/x \to 0$ as $x \to 0$; that is, $f'(0) = 0$. Since $|\theta'| \leq 1$ on $[0, 1]$, $|f'(x)| \leq \beta_n/\alpha_n$ on the bump around 2^{-n}; since $\beta_n/\alpha_n \to 0$ as $n \to \infty$, it follows that $f'(x) \to 0$ as $x \to 0$, and therefore f' is continuous at $x = 0$. However, f' cannot be recursive; for if $n \in \mathscr{A}$, then $f'(2^{-n}) = \theta'_{\alpha_n\beta_n}(0) = \beta_n/\lambda\alpha_n = 2^{-n}/\lambda$, while if $n \notin \mathscr{A}$, $f'(2^{-n}) = 0$. Since these alternatives can be decided, this would yield a decision-procedure for $\mathscr{A}$.

It remains only to prove that f is recursive on $[0, 1]$. Let then a number $x \in [0, 1]$ be given, and let it be required to compute $f(x)$ to within 2^{-n}. Let

$$g_M(x) \equiv \sum_{k=0}^{M} \theta_{\alpha_{h(k)}, \beta_{h(k)}}(x - 2^{-h(k)});$$

then it will suffice to prove that (1) $|g_M(x) - f(x)| < 2^{-M}$ and (2) $g_M(x)$ is recursive.

Re (1). By the definition of f,

$$f(x) = \sum_{k=0}^{\infty} \theta_{\alpha_{h(k)}, \beta_{h(k)}}(x - 2^{-h(k)}),$$

where at most one term of the sum is not zero. Therefore $g_n(x) - f(x)$ is zero or consists of a single term

$$\theta_{\alpha_{h(k)}, \beta_{h(k)}}(x - 2^{-h(k)}) = \frac{\beta_{h(k)}}{\lambda}\,\theta\left(\frac{x - 2^{-h(k)}}{\alpha_{h(k)}}\right).$$

But $|\theta_{\alpha_{h(k)}, \beta_{h(k)}}| \leq \beta_{h(k)} = 2^{-k-h(k)-2} < 2^{-k} < 2^{-M}$, q.e.d.

Re (2). It is enough to show that $\theta(x)$ is a recursive function of x. Let x be given. To compute $\theta(x)$, we first determine whether $x < 0$ or $x > -1$. If $x < 0$, we can compute $\theta(x)$, because $\theta(x) = \min(0, x(x^2 - 1)^2)$. If $x > -1$, determine whether $x > 0$ or $x < 1$. If $x > 0$, then $\theta(x) = \max(0, x(x^2 - 1)^2)$; if finally $-1 < x < 1$, then $\theta(x) = x(x^2 - 1)^2$. This completes the proof.

Added in proof. My friend and colleague Milton Parnes has observed that the same method yields an indefinitely differentiable, recursive function (of course, not analytic!) on $[0, 1]$, none of whose derivatives is recursive.

State University of New York at Buffalo
Buffalo, New York 14226