BACKWARD LOWER BOUNDS FOR SOLUTIONS OF MIXED PARABOLIC PROBLEMS

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1. A number of recent papers dealt with lower bounds for solutions of abstract first-order differential inequalities in Hilbert space, with applications to parabolic problems. P. J. Cohen and M. Lees [4] investigated an inequality of the form

$$\left|\frac{du}{dt} - Au\right| \leq \phi(t)\,|u|\,, \tag{1}$$

and assuming that A is symmetric and $\phi(t) \in L^p(0, \infty)$, they proved that $|u(t)| \geq K e^{-\mu t}$ for some constants K and μ. In [2], S. Agmon and L. Nirenberg improved these results. Agmon [1] then developed a unified approach to the convexity methods that made their debut in [2], and he made it possible to treat equations of the form (1), where A(t) satisfies rather weak conditions and, in particular, need not be self-adjoint.

One problem, which was pointed out by J.-L. Lions and B. Malgrange [6], is that Agmon's results do not apply to parabolic problems with lower-order terms whose order exceeds half the order of the elliptic operator corresponding to A(t). However, A. Friedman [5] established a forward uniqueness theorem for equations of the form

$$\left|\frac{1}{i}\frac{du}{dt} - A(t)u\right| \leq \eta\,|A(t)u| + K\,|u|\,, \tag{2}$$

where η is sufficiently small, which in applications placed no restriction on the lower-order terms. Friedman also obtained a "uniqueness at $-\infty$" result for such equations.

This paper is devoted to generalizations of Friedman's latter results. We obtain backward lower bounds in a higher norm for solutions of abstract equations in Hilbert space. In the applications to parabolic problems, the norm can be taken to be that in the Sobolev space $H^m(\Omega)$, where 2m is the order of the equation. Our assumptions about the equation and its solution are in most respects less restrictive than those imposed by Friedman. In particular, we need not suppose that the resolvent of A(t) exists.

2. Let H denote a complex Hilbert space with norm $|\cdot|$ and inner product $(\cdot, \cdot)$. We study H-valued functions u(t) that satisfy the vector differential equation

$$\frac{du}{dt} + A(t)u = f(t, u) \tag{3}$$

almost everywhere in (a, b). Here the H-valued function du/dt, defined almost everywhere on (a, b), denotes the derivative of u(t) in the distribution sense of [7], and, for almost all $t \in (a, b)$, we require that u(t) lie in the domain $\mathscr{D}(t)$ of the (in

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general unbounded) operator $A(t)$. The H-valued function $f(t, u)$ is subject to an inequality to be presented shortly.

We suppose that it is possible to write

$$A(t) = A_0(t) + A_1(t)$$

for almost all $t \in (a, b)$, where $A_0(t)$ and $A_1(t)$ both have domain $\mathscr{D}(t)$, and we require that the following five conditions are satisfied.

I. $$\Re(A_1(t)u(t), u(t)) = 0.$$

II. There exists a nonnegative, differentiable function $\lambda(t)$ on (a, b) such that

$$\Re(A_0(t)u(t), u(t)) + \lambda(t)|u(t)|^2 > 0,$$

whenever $u(t) \neq 0$.

III. $$|A_1(t)u(t)|^2 \leq K(\varepsilon, t, u, A_0),$$

where

$$K(\varepsilon, t, u, A_0) = \varepsilon |A_0(t)u(t)|^2 + K(t)[|\Re(A_0 u(t), u(t))| + |u(t)|^2],$$

$K(t)$ is locally integrable on (a, b), and ε is a nonnegative constant not exceeding $1/5$.

So far, we have made no assumptions on the regularity of $A(t)$ as a function of t and very weak assumptions on $u(t)$. Some such assumptions have proved unavoidable in previous work, and we state our version in the following condition.

IV. The functions $|u(t)|^2$ and $\Re(A_0(t)u(t), u(t))$ are absolutely continuous, and for almost all $t \in (a, b)$, we have the relation

$$\frac{d}{dt}|u(t)|^2 = 2\Re\left(u(t), \frac{du}{dt}\right).$$

Moreover, putting

$$\frac{d}{dt}\Re(A_0(t)u(t), u(t)) - \Re\left(A_0(t)u(t), \frac{du}{dt}\right) = a_0'(t; u(t))$$

whenever the derivatives are defined, we obtain the inequality

$$|a_0'(t; u(t))| \leq K(\varepsilon, t, u, A_0).$$

V. $$|f(t, u(t))|^2 \leq K(\varepsilon, t, u, A_0).$$

These conditions should be compared with the corresponding ones in [1] and [6]. The weak condition II replaces the ellipticity condition in [6]. Concerning the regularity assumption IV, one can show that (in the notation of [6]) if $u(t) \in L^2(a, b; V)$ and if $u(t)$ has a distribution derivative in $L^2(a, b; V)$, then $\|u(t)\|^2$ belongs to the Sobolev space $W^{1,1}(a, b)$ and, hence, is absolutely continuous (as is $|u(t)|^2$). If $\Re\, a_0(t; u, v)$ is absolutely continuous in t for fixed $u, v \in V$ and if it has a uniformly bounded derivative almost everywhere, then $a_0(t; v(t))$ is absolutely continuous.

Assuming the five conditions above, we can prove the following theorem.

THEOREM 1. *Suppose* $u(t)$ *is a solution of* (3) *on* (a, b), *and let*

$$p(t) = \Re\,(A_0(t)u(t), u(t)) + \omega(t)\,|u(t)|^2,$$

where $\omega(t) = \lambda(t) + \eta(t)$ *for some positive, differentiable function* $\eta(t)$ *whose inverse and derivative are locally integrable. If* $u(t) \neq 0$ *on* $(\tau, d) \subset (a, b)$, *then*

$$p(\tau) \geq p(d) \exp\left(- \int_\tau^d M(t)\,dt \right),$$

where

$$M(t) = 5\,(1 + \eta(t)^{-1}) \max\{K(t) + 2\varepsilon\,\omega(t), (\omega(t) + 1)K(t) + \varepsilon\,[\omega(t)]^2 + \omega'(t)\} + 2\omega(t).$$

Proof. From II, we obtain the inequality

$$\Re\,(A_0(t)u(t), u(t)) + \omega(t)\,|u(t)|^2 > \eta(t)\,|u(t)|^2, \tag{4}$$

where $\omega(t) = \lambda(t) + \eta(t)$. If we substitute $u(t) = e^{\Omega(t)}v(t)$, where $\Omega(t) = \int_d^t \omega(s)\,ds$, then $v(t)$ satisfies the condition

$$\frac{dv}{dt} + B(t)v = g(t, v),$$

where $B(t) = A(t) + \omega(t)$ and $g(t, v) = e^{-\Omega(t)} f(t, e^{\Omega(t)}v(t))$. Put $B_0(t) = A_0(t) + \omega(t)$ and $B_1(t) = A_1(t)$. Then, from I, we obtain the relation

$$\Re\,(B_1(t)v(t), v(t)) = 0, \tag{5}$$

and II yields the inequality

$$\Re\,(B_0(t)v(t), v(t)) > \eta(t)\,|v(t)|^2. \tag{6}$$

From our assumptions, we obtain after some calculation the following inequalities (for convenience, we suppress the variable t):

$$|B_1 v|^2 \leq K_1(\varepsilon, t, v, B_0), \tag{7a}$$

$$|b_0'(t, v)| \leq K_1(\varepsilon, t, v, B_0), \tag{7b}$$

and

$$|g(t, v)|^2 \leq K_1(\varepsilon, t, v, B_0), \tag{7c}$$

where

$$b_0'(t; v) = \frac{d}{dt}\Re(B_0 v, v) - 2\,\Re\left(B_0 v, \frac{dv}{dt}\right),$$

$$K_1(\varepsilon, t, v, B_0) = \varepsilon\,|B_0 v|^2 + K_1(t)[\,\Re\,(B_0 v, v) + |v|^2],$$

and

$$K_1(t) = \max\{K(t) + 2\varepsilon\omega(t), (\omega(t) + 1)K(t) + \varepsilon\,[\omega(t)]^2 + \omega'(t)\}.$$

Now let $\tilde{p}(t) = \Re(B_0(t)v(t), v(t))$. Then

$$\log \tilde{p}(d) = \log \tilde{p}(\tau) + \int_\tau^d \frac{q(t)}{\tilde{p}(t)}\,dt,$$

where

$$q(t) = \frac{d}{dt}\Re(B_0 v, v) = b_0'(t; v) + 2\Re\left(B_0 v, \frac{dv}{dt}\right)$$

$$= b_0'(t; v) + 2\Re(B_0 v, g) - 2|B_0 v|^2 - 2\Re(B_0 v, B_1 v).$$

The relation

$$|B_0 v + B_1 v - g|^2 = |B_0 v|^2 + |B_1 v|^2 + |g|^2 + 2\Re(B_0 v, B_1 v)$$

$$- 2\Re(B_0 v, g) - 2\Re(B_1 v, g)$$

implies that

$$q(t) = -|B_0 v + B_1 v - g|^2 - |B_0 v|^2 + r(t),$$

where

$$r(t) = |B_1 v|^2 + |g|^2 - 2\Re(B_1 v, g) + b_0'(t; v).$$

Using inequalities (7), we see that

$$|r(t)| \leq 5\varepsilon\,|B_0 v|^2 + \tilde{p}(t)L(t),$$

where

$$L(t) = 5K_1(t)(1 + \eta(t)^{-1}).$$

Now

$$\log \tilde{p}(d) \leq \log \tilde{p}(\tau) - (1 - 5\varepsilon)\int_\tau^d \frac{|B_0 v|^2}{\tilde{p}(t)}\,dt - \int_\tau^d \frac{|B_0 v + B_1 v - g|^2}{\tilde{p}(t)}\,dt + \int_\tau^d L(t)\,dt$$

$$\leq \log \tilde{p}(\tau) + \int_\tau^d L(t)\,dt;$$

hence

$$\tilde{p}(\tau) \geq \tilde{p}(d)\exp\left(-\int_\tau^d L(t)\,dt\right).$$

But $\tilde{p}(t) = e^{-2\Omega(t)}p(t)$, so that

$$p(\tau) \geq p(d)\exp\left(-\int_\tau^d M(t)\,dt\right).$$

This completes the proof.

As a corollary, we obtain the following uniqueness theorem.

COROLLARY. *If*

$$\limsup_{\tau \to a^+} p(\tau) \exp\left(\int_\tau^d M(t)\,dt \right) = 0,$$

then $u(t) \equiv 0$ *for* $a < t < d$.

With $a = -\infty$, the corollary should be compared with Theorem 3 in [5].

The translation of mixed problems for parabolic equations into the abstract framework of Theorem 1 is standard. For example, consider the equation

$$\frac{\partial u}{\partial t} + \mathscr{A}(x, t; D)u = \mathscr{B}(x, t; u)$$

on the cylinder $\Omega \times (a, b)$, where Ω is a bounded domain in E^n, and $\mathscr{A}$ and $\mathscr{B}$ are differential operators on Ω whose coefficients depend on $x = (x_1, \cdots, x_n)$ and t. We assume that $\mathscr{A}$ is elliptic of order $2m$ and that the order of $\mathscr{B}$ is less than $2m$. If we require the solutions $u(x, t)$ to satisfy certain null boundary conditions for each t, then we may consider the more general problem

$$\frac{du}{dt} + A(t)u = B(t)u, \tag{8}$$

where the unbounded operators $A(t)$ and $B(t)$ are defined on some subspace $\mathfrak{D}(t)$ of the Sobolev space $H^{2m}(\Omega)$, which is determined by the boundary conditions. For our purposes, it is not necessary that the subspaces $\mathfrak{D}(t)$ are closed or dense in $H^{2m}(\Omega)$, or even that there exists some relationship between the spaces $\mathfrak{D}(t)$ as t varies. In fact, it is relatively easy to check that conditions I to V are satisfied, and this avoids the resolvent questions associated with the assumptions made in [5]. We may suppose that $A_0(t) = A(t)$ and $A_1(t) = 0$, so that I and III hold. Condition II is much weaker than the corresponding assumption

$$\Re(A_0 u, u) + \lambda(t)|u|^2 \geq \alpha(t)\|u\|_m \tag{9}$$

in [6], and the ellipticity of $A(t)$ assures inequality (9). Condition IV will hold if the coefficients of $A(t)$ satisfy rather mild regularity conditions as functions of t (see (ii) below). To establish condition V, we start with the inequality

$$|A_1(t)u| \leq k(t)\|u\|_{2m-1}, \tag{10}$$

where $k(t)$ is a function associated with the maximum norm of the coefficients. Ehrling's Lemma implies that for each $\varepsilon > 0$, there exists a constant K such that

$$\|u(t)\|_{2m-1} \leq \varepsilon\|u(t)\|_{2m} + K\|u\|_0,$$

and finally the standard ellipticity estimates due to Agmon, Douglis, and Nirenberg [3] yield the inequality

$$\|u(t)\|_{2m} \leq C(t)(\|A_0(t)u(t)\|_0 + \|u(t)\|_0). \tag{11}$$

Consequently, condition V holds if $C(t)$ is uniformly bounded in t.

Applying Theorem 1 to (8), we obtain a generalization of Theorem 5 in [5]. As a particular example that can easily be compared with the results in [5], we make the following assumptions:

(i) The ellipticity of the operator $\mathscr{A}(x, t; D)$ is uniformly bounded in t. This means that the coefficients of $\mathscr{A}(x, t; D)$ are smooth enough to ensure that the estimates (9) and (11) are valid, and the functions $\lambda(t)$, $\lambda'(t)$, and $C(t)$ are uniformly bounded in t.

(ii) The coefficients of the leading coefficients in $\mathscr{A}(x, t; D)$ are differentiable in t and have uniformly bounded derivatives. This guarantees that condition IV is satisfied.

(iii) The coefficients of $\mathscr{B}(x, t; D)$ are uniformly bounded in t (so that condition V is satisfied).

Under these assumptions, we obtain the following result.

THEOREM 2. *There exist constants* M *and* K *such that*

$$\limsup_{\tau \to a^+} \|u(\tau)\|_m e^{-M\tau} \geq K.$$

This result strengthens Theorem 5 in [5], where Friedman shows that solutions u that satisfy the condition

$$\lim_{\tau \to -\infty} \|u(\tau)\|_{2m} e^{-\mu\tau} = 0$$

for all $\mu > 0$ must be identically zero. Theorem 2 shows that $2m$ can be replaced by m in Friedman's result. He states that, under additional assumptions which we have not made, $2m$ can be replaced by some j $(j < 2m)$.

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