MAPPING THE PSEUDO-ARC ONTO CIRCLE-LIKE, SELF-ENTWINED CONTINUA

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The investigation of the continuous images of the pseudo-arc has aroused much activity in this decade. Of particular interest has been the question of deciding which circle-like continua are continuous images of the pseudo-arc.

In this paper, we define the notion of a circle-like, self-entwined continuum. We find that circle-like, self-entwined continua are indecomposable, that all nonplanar, circle-like continua are self-entwined, and that the planar, circle-like, self-entwined continua separate the plane. Two of our results follow.

THEOREM. *No arc-like continuum can be mapped onto a circle-like, self-entwined continuum.*

THEOREM. *The pseudo-circle is self-entwined.*

Several known results follow as corollaries to these theorems. W. T. Ingram [6] has shown that the pseudo-arc cannot be mapped onto a nonplanar, circle-like continuum, and L. Fearnley [5] and the author [11] have shown independently that the pseudo-arc cannot be mapped onto the pseudo-circle.

In this paper, we use the methods of inverse limit spaces. The transition from chains to inverse limits is discussed in [10] and [11]. We use the terminology of [3].

A continuum is a compact, nondegenerate, connected subset of a metric space. A map is a continuous function.

1. THE REVOLVING NUMBER R(f)

Let C denote the unit circle in the plane. Orient C so that a definite sense of rotation exists. Let C_1 and C_2 be triangulations of C, and let $f\colon C_1 \to C_2$ be a surjective, simplicial map.

Let v denote a vertex of C_2, and let $z_0, z_1, \cdots, z_n$ denote the vertices of C_1 that are mapped onto v, ordered by positive rotation. Suppose z_{n+1} is another name for z_0. If $f \mid [z_i, z_{i+1}]$ is surjective, call $[z_i, z_{i+1}]$ an A+ or A-, according to whether the image of $[z_i, z_{i+1}]$ emanates from v in the positive or negative direction.

If $[x_1, x_2]$ is an arc in C_1, oriented in the direction of positive rotation, we define the *degree of* $[x_1, x_2]_f$ to be the number of A+'s of $f \mid [x_1, x_2]$, diminished by the number of A-'s of $f \mid [x_1, x_2]$. Where no confusion is likely, we speak of the degree of $[x_1, x_2]$ without reference to the function f. For greater detail on these concepts, we refer the reader to [10] and [11].

The author [11] has shown that if the map f has positive degree, then there exists an integer i such that if $[z_i, z_j]$ is an arc in C_1, then the degree of $[z_i, z_j]$ is positive. The point z_i is called an *initial point of* C_1 *with respect to* f. If $\deg(f) = 1$, then the initial point is unique [10, Lemma 6].

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If $\deg(f) = 0$, call z_i an initial point of C_1 with respect to f if each arc $[z_i, z_j]$ in C_1 has nonnegative degree. One can prove the existence of an initial point in this case by following Lemma 8 of [11].

The next definition and theorem are basic to the paper. Consider all arcs in C_1 of the form $[z_i, z_j]$ (where z_i is an initial point of C_1 and where $[z_i, z_i]$ denotes C_1). Assume that the degree of f is nonnegative. Define R(f), the *revolving number of* f, to be the maximum of the degrees of the arcs $[z_i, z_j]$. A minimal arc $[z_i, z_j]$ on which such a maximum occurs is called a *defining interval for* R(f).

We note that it follows from [11] that $R(f) \geq \deg(f)$. We also note that R(f) depends on the vertex v of V_2, and that R(f) can vary by 1 if a new vertex is chosen. Hence we adopt the convention that if we consider $R(g \circ f)$, where

$$f: C_1 \to C_2 \quad \text{and} \quad g: C_3 \to C_4$$

are simplicial, surjective maps of nonnegative degree and C_3 is a subdivision of C_2, then for the vertex v of C_3 that determines R(f), we choose a vertex of C_2 that is a first point of a defining interval [v, a] of R(g).

THEOREM 1. *Let* f *and* g *satisfy the conditions of the preceding paragraph. Then*

$$R(g \circ f) \geq R(f) \cdot \deg(g) - \deg(g) + R(g).$$

Proof. Let $[a_1, a_2]$ and $[v, a_3]$ be defining intervals for R(f) and R(g), respectively. Then

$$(1) \qquad \deg[a_1, a_2]_{g \circ f} = \deg[a_1, a_2]_f \cdot \deg(g) = R(f) \cdot \deg(g).$$

Hence, in the case where $R(g) = \deg(g)$, the theorem follows from (1).

If $R(g) > \deg(g)$, then by Lemma 5 of [10],

$$\deg[a_3, v]_g < 0.$$

Choose a_4 such that $a_1 < a_4 < a_2$ and a_4 is the largest number in the interval $[a_1, a_2]$ whose image under f is a_3. Then $f([a_4, a_2]) = [a_3, v]$. Accordingly,

$$\deg[a_4, a_2]_{g \circ f} = \deg[a_3, v]_g = \deg(g) - R(g).$$

Hence

$$\deg[a_1, a_4]_{g \circ f} = \deg[a_1, a_2]_{g \circ f} - \deg[a_4, a_2]_{g \circ f} = R(f) \cdot \deg(g) - \deg(g) + R(g).$$

Since $R(g \circ f) \geq \deg[a_1, a_4]_{g \circ f}$, the proof of the theorem is complete. ■

2. CIRCLE-LIKE, SELF-ENTWINED CONTINUA

For the rest of this paper, we assume that each factor space of an inverse sequence is a triangulation of the unit circle C, and each bonding map is a piecewise-linear, surjective map of nonnegative degree. We also assume that under these maps, the image of each vertex is either a vertex or the midpoint of a one-simplex, and adjacent vertices are mapped into a simplex. Such inverse sequences are called *barycentric inverse sequences.* Each circle-like continuum is the inverse limit of such an inverse sequence [11, Lemma 8].

We say that the circle-like continuum X is *self-entwined* if X is the inverse limit of an inverse sequence $\{X_i, f_i^{i+1}\}$ and if each bonding map f_i^{i+1} has positive degree and revolving number at least 2. Notice that requiring each f_i^{i+1} to have positive degree implies that no arc-like continuum is self-entwined.

Self-entwined continua are strongly indecomposable.

THEOREM 2. *If the circle-like continuum* X *is self-entwined, then* X *is indecomposable.*

Proof. Let X be the limit of $\{X_i, f_i^{i+1}\}$, where $R(f_i^{i+1}) > 1$ for each i. If X were decomposable, then $X = H + K$, where H and K are proper subcontinua of X. Hence there exist points $p = (p_1, p_2, \cdots)$ in H - K and $q = (q_1, q_2, \cdots)$ in K - H.

Recall that the collection

$$\{f_i^{-1}(O)\colon O \text{ is an open subset of } X_i\}_{i=1}^{\infty}$$

is a basis for the topology of X. Therefore, there exist an integer n and disjoint open sets U and V in X_n such that

$$p \in f_n^{-1}(U) \subset H - K \quad \text{and} \quad q \in f_n^{-1}(V) \subset K - H.$$

Since the revolving number of f_n^{n+1} exceeds 1, there exist an arc $[x_1, x_2]$ in X_{n+1} and points $y_1 < y_2 < y_3 < y_4$ in $[x_1, x_2]$ such that (without loss of generality)

$$f_n^{n+1}(y_1) = f_n^{n+1}(y_3) = p_n, \qquad f_n^{n+1}(y_2) = f_n^{n+1}(y_4) = q_n,$$

and f_n^{n+1} maps both $[y_1, y_3]$ and $[y_3, y_1]$ onto X_n. Hence $f_{n+1}^{-1}(y_1)$ and $f_{n+1}^{-1}(y_3)$ belong to $f_n^{-1}(U)$. Since H is connected, $f_{n+1}(H)$ contains either $[y_1, y_3]$ or $[y_3, y_1]$. Accordingly, $f_n(H) = X_n$. This contradicts the fact that $f_n^{-1}(V) \subset K - H$. Hence X is indecomposable. ■

We remark that there exist non-arc-like, indecomposable, circle-like continua that are not self-entwined; the pseudo-arc with two opposite endpoints identified is an example. The proof of this will follow from Theorem 6. However, such circle-like continua exist only in the plane.

THEOREM 3. *If* X *is a nonplanar, circle-like continuum, then* X *is self-entwined.*

Proof. Since X is nonplanar, X can be represented as the inverse limit of $\{X_i, f_i^{i+1}\}$, where each f_i^{i+1} has degree at least 2.

Since the revolving number of a map is never less than its degree, each bonding map f_i^{i+1} has revolving number at least 2; hence X is self-entwined. ■

Theorem 3 bids us to concentrate our attention on plane continua, and it is natural to examine the most famous (at least in the nonchainable class) circle-like plane continuum, the pseudo-circle [2]. Certainly, the pseudo-circle should be among the circle-like continua that are complicated enough to be self-entwined.

THEOREM 4. *The pseudo-circle is self-entwined.*

The proof of this theorem is an exercise in changing R. H. Bing's original description [2] in terms of circular chains into an inverse limit description. See also [10] and [11].

We pause now to prove some results that will give us easy access to several continua that are not self-entwined.

3. MAPS OF DEGREE ZERO

Suppose that $X = \lim \{X_i, f_i^{i+1}\}$ and $Y = \lim \{Y_i, g_i^{i+1}\}$ are circle-like continua and that h is a continuous map of X onto Y. Let $\{\varepsilon_n\}$ be a sequence of positive numbers converging to zero and bounded above by 1/2. The existence of h implies the existence of an infinite diagram (see [1] and [9])

$$\begin{array}{ccccccccc} X_{n(1)} & \longleftarrow & X_{n(2)} & \longleftarrow & \cdots & \longleftarrow & X_{n(k)} & \longleftarrow & \cdots \\ h_1 \downarrow & & h_2 \downarrow & & & & h_k \downarrow & & \\ Y_{m(1)} & \longleftarrow & Y_{m(2)} & \longleftarrow & \cdots & \longleftarrow & Y_{m(k)} & \longleftarrow & \cdots \end{array} \tag{2}$$

where $\{m(k)\}$ and $\{n(k)\}$ are increasing sequences of positive integers, $\{h_k\}$ is a sequence of surjective maps, and every subdiagram

$$\begin{array}{ccc} X_{n(k)} & \longleftarrow & X_{n(r)} \\ \downarrow & & \downarrow \\ Y_{m(k)} & \longleftarrow & Y_{m(r)} \end{array} \tag{3}$$

is ε_k-commutative, for each $r \geq k$.

If each map h_k in (2) has degree n, then we say that the map h has *limit degree* n.

THEOREM 5. *Let* X *and* Y *be circle-like continua, and let* Y *be self-entwined. Then there does not exist a map of* X *onto* Y *with limit degree zero.*

Proof. Let $X = \lim \{X_i, f_i^{i+1}\}$ and $Y = \lim \{Y_i, g_i^{i+1}\}$, where the revolving number of each bonding map g_i^{i+1} exceeds 1. Suppose that there exists a map of X onto Y with limit degree zero. Choose the sequence $\{\varepsilon_n\}$ and diagrams (2) and (3) as before, with the additional hypothesis that each map h_i have degree zero.

We show that for large r, the revolving number of $h_k \circ f_{n(k)}^{n(r)}$ is much less than that of $g_{m(k)}^{m(r)} \circ h_r$. Accordingly, we must maximize $R(h_k \circ f_{n(k)}^{n(r)})$. Since the two composite maps may differ by ε_k, it might be possible to stretch a map at both ends of a defining interval and add at most 2 to the revolving number (1 at each end). For this reason, we shall add 2 to $R(h_k \circ f_{n(k)}^{n(r)})$ in the following inequalities.

Since h_k is a map of degree zero, $f_{n(k)}^{n(r)}$ just makes many copies of h_k; hence

$$R(h_k \circ f_{n(k)}^{n(r)}) = R(h_k)$$

for each r.

On the other hand, by Theorem 1,

$$R(g_i^{i+1} \circ g_{i+1}^{i+2}) \geq R(g_{i+1}^{i+2}) \cdot \deg(g_i^{i+1}) - \deg(g_i^{i+1}) + R(g_i^{i+1})$$
$$\geq \deg(g_i^{i+1}) + R(g_i^{i+1}) \geq 3 .$$

Repeated applications of Theorem 1 show that, by choosing a sufficiently large r, we can make $R(g_{m(k)}^{m(r)})$, and hence $R(g_{m(k)}^{m(r)} \circ h_r)$, as large as we please. In particular, if we choose r so large that

$$R(g_{m(k)}^{m(r)} \circ h_r) > R(h_k) + 2 ,$$

then we obtain a contradiction to (3). This contradiction shows that the map h does not exist. ■

The following result is an application of Theorem 5.

THEOREM 6. *The pseudo-arc cannot be mapped onto a circle-like, self-entwined continuum.*

Proof. Let $X = \lim \{X_i, f_i^{i+1}\}$ be the pseudo-arc, and let $Y = \lim \{Y_i, g_i^{i+1}\}$ be a circle-like, self-entwined continuum. Accordingly, we may assume for all i that

$$\deg(g_i^{i+1}) > 0 \quad \text{and} \quad R(g_i^{i+1}) > 1$$

and, since X is arc-like, that $\deg(f_i^{i+1}) = 0$.

Assume that there exists a map h of X onto Y. By Theorem 5, it suffices to show that h has limit degree zero.

Choose a sequence $\{\varepsilon_n\}$ and diagrams (2) and (3) as in the introduction to this section. Diagram (3) and Lemma 4 of [10] assure us that

$$\deg(h_k \circ f_{n(k)}^{n(r)}) = \deg(g_{m(k)}^{m(r)} \circ h_r) \qquad (r > k) .$$

Since $\deg(f_{n(k)}^{n(r)}) = 0$, we have that

$$\deg(h_k \circ f_{n(k)}^{n(r)}) = 0 ,$$

and hence

$$\deg(g_{m(k)}^{m(r)} \circ h_r) = 0 .$$

Finally, since $g_{m(k)}^{m(r)}$ has positive degree, the degree of h_r must be 0. Hence $\deg(h_r) = 0$ if $r > 1$; therefore, h has limit degree zero. ■

Theorem 6 has several important corollaries. First, we can exhibit circle-like continua that are not self-entwined.

COROLLARY 1. *If* Y *is a circle-like continuum that is formed by identifying two opposite endpoints of an arc-like continuum, then* Y *is not self-entwined.*

Next we obtain new proofs of two known mapping relations. The first was proved independently by Fearnley [5] and the author [11]; the second was proved by Ingram [6].

COROLLARY 2. *The pseudo-circle is not a continuous image of the pseudo-arc.*

COROLLARY 3. *The pseudo-arc cannot be mapped onto a nonplanar, circle-like continuum.*

Finally, we state a more general form of Theorem 6, which follows at once from [4], [7], or [8].

COROLLARY 4. *No arc-like continuum can be mapped onto a circle-like, self-entwined continuum.*

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