AN EXTENSION OF A THEOREM OF T. ANDÔ

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In [1], T. Andô proved the following result: *Every commutative pair of contractions* $\mathscr{T} = \{T_1, T_2\}$ *in a Hilbert space* H *has a unitary dilation.*

We recall that the pair $\mathscr{U} = \{U_1, U_2\}$ is a unitary dilation (in a Hilbert space $K \supset H$) for the pair $\mathscr{T}$ if the U_j $(j = 1, 2)$ are unitary operators in K, $U_1 U_2 = U_2 U_1$, and

$$P U_1^{n_1} U_2^{n_2} h = T_1^{n_1} T_2^{n_2} h \quad (n_1, n_2 \geq 0,\ h \in H),$$

where P is the orthogonal projection of K onto H.

The aim of this note is to extend this theorem.

We say that a family $\mathscr{T} = \{T_1, \cdots, T_p\}$ of linear bounded operators (in a Hilbert space H) is *cyclic commutative* if

$$\text{(c)} \qquad T_1 T_2 \cdots T_p = T_p T_1 T_2 \cdots T_{p-1} = \cdots = T_2 T_3 \cdots T_p T_1 .$$

THEOREM 1. *Let* $\mathscr{T} = \{T_1, T_2, \cdots, T_p\}$ *be a cyclic commutative family of contractions in the Hilbert space* H. *There exists a cyclic commutative family* $\mathscr{V} = \{V_1, V_2, \cdots, V_p\}$ *of isometries in a Hilbert space* $K \supset H$, *with the property that*

$$\text{(1)} \qquad P V_{i_1}^{n_1} \cdots V_{i_p}^{n_p} h = T_{i_1}^{n_1} \cdots T_{i_p}^{n_p} h \quad (n_j \geq 0,\ h \in H),$$

where $(i_1, i_2, \cdots, i_p)$ *is an arbitrary permutation of* $(1, 2, \cdots, p)$, *and where* P *is the orthogonal projection of* K *onto* H.

Proof. Let $K = \ell^2(H)$; that is, let K be the space of sequences $\{h_i\}_{i=0}^{\infty}$ $(h_i \in H)$ such that $\sum_{i=0}^{\infty} \|h_i\|^2 < \infty$. For $j = 1, 2, \cdots, p$, we define $S_j \in \mathscr{L}(K)$ by the equation

$$S_j\{h_0, h_1, \cdots, h_n, \cdots\} = \{T_j h_0, 0, D_{T_j} h_0, 0, h_1, \cdots, h_n, \cdots\},$$

where $D_{T_j} = (I - T_j^* T_j)^{1/2}$.

It is obvious that the S_j are isometries in K. We consider the products

$$S_1 S_2 \cdots S_p \{h_0, h_1, \cdots\}$$
$$= \{T_1 T_2 \cdots T_p h_0, D_{T_1} T_2 \cdots T_p h_0, 0, D_{T_2} T_3 \cdots T_p h_0, 0, \cdots, D_{T_{p-1}} T_p h_0, 0,$$
$$D_{T_p} h_0, 0, h_1, h_2, \cdots\},$$

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$$S_2 S_3 \cdots S_p S_1 \{h_0, h_1, \cdots\}$$
$$= \{T_2 T_3 \cdots T_p T_1 h_0, D_{T_2} T_3 \cdots T_p T_1 h_0, 0, D_{T_3} T_4 \cdots T_p T_1 h_0, 0, \cdots, D_{T_p} T_1 h_0, 0, D_{T_1} h_0, 0, h_1, h_2, \cdots\},$$
$$\cdots\cdots\cdots\cdots,$$
$$S_p S_1 \cdots S_{p-1} \{h_0, h_1, \cdots\}$$
$$= \{T_p T_1 \cdots T_{p-1} h_0, D_{T_p} T_1 \cdots T_{p-1} h_0, 0, D_{T_1} T_2 \cdots T_{p-1} h_0, 0, \cdots, D_{T_{p-1}} h_0, 0, h_1, \cdots\}.$$

It is easy to see that

$$\|D_{T_1} T_2 T_3 \cdots T_p h_0\|^2 + \|D_{T_2} T_3 \cdots T_p h_0\|^2 + \cdots + \|D_{T_p} h_0\|^2 = \|D_{T_1 \cdots T_p} h_0\|^2,$$
$$\|D_{T_2} T_3 \cdots T_p T_1 h_0\|^2 + \|D_{T_3} T_4 \cdots T_p T_1 h_0\|^2 + \cdots + \|D_{T_1} h_0\|^2 = \|D_{T_2 \cdots T_p T_1} h_0\|^2,$$
$$\cdots\cdots\cdots\cdots,$$
$$\|D_{T_p} T_1 \cdots T_{p-1} h_0\|^2 + \|D_{T_1} T_2 \cdots T_{p-1} h_0\|^2 + \cdots + \|D_{T_{p-1}} h_0\|^2 = \|D_{T_p T_1 \cdots T_{p-1}} h_0\|^2.$$

By cyclic commutativity of the family $\mathscr{T} = \{T_1, \cdots, T_p\}$, all of these expressions are equal.

Consequently, we can define isometric operators W_j $(j = 1, \cdots, p - 1)$ such that

$$W_j\{D_{T_{j+1}} T_{j+2} \cdots T_{j+p} h_0, 0, D_{T_{j+2}} T_{j+3} \cdots T_{j+p} h_0, 0, \cdots, D_{T_{j+p}} h_0, 0\}$$
$$= \{D_{T_1} T_2 \cdots T_p h_0, 0, D_{T_2} T_3 \cdots T_p h_0, 0, \cdots, D_{T_p} h_0, 0\}$$

for $j = 1, 2, \cdots, p - 1$, where T_k is defined by periodicity:

$$T_k = T_r \quad \text{if } k = qp + r \qquad (1 \le r \le p).$$

Now, by an argument of [1] (see also [3]), the operators W_j can be extended to unitary operators in the Hilbert space

$$\mathscr{S} = \underbrace{H \oplus H \oplus \cdots \oplus H}_{2p \text{ times}},$$

which we denote also by W_j.

We denote by $\mathfrak{W}_j$ the operators in K, defined by the equation

$$\mathfrak{W}_j\{h_0, (h_1, \cdots, h_{2p}), (h_{2p+1}, \cdots, h_{4p}), \cdots\}$$
$$= \{h_0, W_j(h_1, \cdots, h_{2p}), W_j(h_{2p+1}, \cdots, h_{4p}), \cdots\}.$$

Let V_j $(j = 1, \cdots, p)$, the operators in K, be defined by the equations

$$V_1 = S_1 \mathfrak{W}_1^{-1}, \quad V_2 = \mathfrak{W}_1 S_2 \mathfrak{W}_2^{-1}, \quad \cdots, \quad V_{p-1} = \mathfrak{W}_{p-2} S_{p-1} \mathfrak{W}_{p-1}^{-1}, \quad V_p = \mathfrak{W}_{p-1} S_p,$$

and observe that each V_j is an isometry in K.

We shall prove that $\mathscr{V} = \{V_j\}_{j=1}^p$ is cyclic commutative. For this, we note that

$$V_1 V_2 \cdots V_p\{h_0, h_1, \cdots\} = S_1 \mathfrak{W}_1^{-1} \mathfrak{W}_1 S_2 \cdots S_p\{h_0, h_1, \cdots\} = S_1 S_2 \cdots S_p\{h_0, h_1, \cdots\}$$

$$= \{T_1 T_2 \cdots T_p h_0, D_{T_1} T_2 \cdots T_p h_0, 0, \cdots, D_{T_p} h_0, 0, h_1, h_2, \cdots\},$$

$$V_2 V_3 \cdots V_p V_1 \{h_0, h_1, \cdots\} = \mathfrak{W}_1 S_2 \cdots S_p S_1 \mathfrak{W}_1^{-1} \{h_0, h_1, \cdots\}$$

$$= \{T_2 T_3 \cdots T_p T_1 h_0, W_1(D_{T_2} T_3 \cdots T_p T_1 h_0, 0, \cdots, D_{T_1} h_0, 0), h_1, h_2, \cdots\}$$

$$= \{T_2 T_3 \cdots T_p T_1 h_0, D_{T_1} T_2 \cdots T_p h_0, 0, \cdots, D_{T_p} h_0, 0, h_1, h_2, \cdots\},$$

. . . . ,

$$V_p V_1 \cdots V_{p-1} \{h_0, h_1, \cdots\} = \mathfrak{W}_{p-1} S_p S_1 \cdots S_{p-1} \mathfrak{W}_{p-1}^{-1} \{h_0, h_1, \cdots\}$$

$$= \{T_p T_1 \cdots T_{p-1} h_0, W_{p-1}(D_{T_p} T_1 \cdots T_{p-1} h_0, 0, \cdots, D_{T_{p-1}} h_0, 0), h_1, h_2, \cdots\}$$

$$= \{T_p T_1 \cdots T_{p-1} h_0, D_{T_1} T_2 \cdots T_p h_0, 0, \cdots, D_{T_p} h_0, 0, h_1, \cdots\};$$

therefore, by cyclic commutativity of $\mathscr{T}$, it follows that $\{V_j\}_{j=1}^p = \mathscr{V}$ is also cyclic commutative. The relation (1) follows in a natural way.

PROPOSITION. *Every cyclic commutative family* $\mathscr{V} = \{V_j\}_{j=1}^p$ *of isometries in a Hilbert space* K *can be extended to a cyclic commutative family* $\mathscr{U} = \{U_j\}_{j=1}^p$ *of unitary operators in a Hilbert space* $K_1 \supset K$.

This proposition is a variant of a theorem of T. Itô (see [2], [3]). Its proof can be obtained in the same way.

Combining Theorem 1 and the preceding proposition, we obtain the following result.

THEOREM 2. *For every cyclic commutative family* $\mathscr{T} = \{T_j\}_{j=1}^p$ *of contractions in the Hilbert space* H, *there exists a cyclic commutative family* $\mathscr{U} = \{U_j\}_{j=1}^p$ *of unitary operators in a Hilbert space* $K \supset H$, *such that*

$$P_H U_{i_1}^{n_1} U_{i_2}^{n_2} \cdots U_{i_p}^{n_p} h = T_{i_1}^{n_1} T_{i_2}^{n_2} \cdots T_{i_p}^{n_p} h \qquad (n_j \geq 0;\ j = 1, 2, \cdots, p;\ h \in H)$$

for every permutation $(i_1, \cdots, i_p)$ *of* $(1, 2, \cdots, p)$.

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