MULTIPLICATIONS ON PROJECTIVE SPACES

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In this paper, we consider the collection of multiplications on projective spaces. The results of [1] imply that the only projective spaces that admit a multiplication are the real projective spaces P^n, for $n = 1$, 3, and 7. We describe the collection of multiplications on P^3 as a group and determine the number of multiplications on P^7. C. M. Naylor [4] previously found the number of multiplications on P^3.

1. Let ϕ: $X \vee X \to X$ denote the "folding" map.

A multiplication on a space X is a map μ: $X \times X \to X$ such that $\mu \mid X \vee X = \phi$. Two multiplications on a space X are said to be homotopic if they are homotopic as maps relative to $X \vee X$.

M. Arkowitz and C. R. Curjel showed in [2] that if X is a finite CW-complex admitting a multiplication, then there exists a one-to-one correspondence between the set of homotopy classes of multiplications on X and the homotopy set $[X \wedge X, X]$. When X has a homotopy associative multiplication, $[X \wedge X, X]$ has a group structure.

LEMMA 1. *Let* M *be a smooth, connected manifold of dimension* n, *and let* M_0 *be* M *with an open disc removed. If there exists a smooth embedding* f: $M \to S^{m+n}$ *with trivial normal bundle, then* $S^m M \simeq S^m M_0 \vee S^{n+m}$.

Proof. Let N denote a closed tubular neighbourhood of the embedding, and let T denote the Thom complex of the normal bundle. T is $N/\partial N$ by definition, and it is homotopically equivalent to $S^m \vee S^m M$. If D is a small disc of dimension $m + n$ lying in the interior of N, the space $T - D$ is homotopically equivalent to $S^m \vee S^m M_0$. The attaching map of D is homotopically trivial in S^{n+m} and therefore is also trivial in T. This proves that

$$S^m \vee S^m M \simeq S^m \vee S^m M_0 \vee S^{n+m},$$

and therefore we have that $S^m M \simeq S^m M_0 \vee S^{n+m}$.

I am extremely grateful to Dr. B. J. Sanderson for showing me this lemma. It enables us to avoid a rather long direct proof of the following statement (when $n = 6$).

COROLLARY 2. *The covering map* π: $S^n \to P^n$ *is stably trivial when* $n = 2$ *or* $n = 6$.

Proof. We prove the result for $n = 6$. The case $n = 2$ is similar (and easy to prove directly, anyway).

Embed P^7 in R^{15}. Since P^7 is parallelizable, the normal bundle is trivial. The space P^7 with an open disc removed is homotopically equivalent to P^6, and the attaching map for this disc is $S^8 \pi$. It follows from the proof of the lemma that $S^8 \pi \simeq 0$.

LEMMA 3. *If* K *is an* n*-dimensional complex, then* S: $[K, S^m] \to [SK, S^{m+1}]$ *is an isomorphism provided*

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(1) $n < 2m - 1$, (2) $n = 5$, $m = 3$, *or* (3) $n = 13$, $m = 7$.

Proof. Condition (1) is Freudenthal's suspension theorem.

For (2), look at the following exact sequence, obtained from the Hopf fibration $S^7 \to S^4$:

$$[SK, S^7] \longrightarrow [SK, S^4] \xrightarrow{\partial} [K, S^3] \longrightarrow [K, S^7].$$

The map ∂ is an isomorphism for dimensional reasons. However, the suspension map S splits the sequence, so that S is also an isomorphism.

To prove (3), we use the fibration $S^{15} \to S^8$ and proceed as in the proof of (2).

2. Let $\{X, Y\} = \lim [S^n X, S^n Y]$, the limit maps being suspensions. The following result is due to M. G. Barratt [3].

LEMMA 4. *$\{P^2, P^2\}$ is a cyclic group of order 4 and is generated by the identity 1. The map $2\cdot 1$ is the composite*

$$SP^2 \xrightarrow{p} S^3 \xrightarrow{\eta} S^2 \xrightarrow{i} SP^2,$$

where p collapses $SP^1 \subset SP^2$ to a point, η is the Hopf map, and i is the inclusion map. ($2\cdot 1$ is twice the identity.)

Proof. One can easily show that $\{S^1, P^2\}$ and $\{S^2, P^2\}$ both have exactly two elements. Hence, from the stable Puppe sequence of a map $f\colon S^1 \to S^1$ of degree two, we see that $\{P^2, P^2\}$ has four elements.

To complete the proof of the first statement, we show that $2\cdot 1$ is essential. The cofibre of $2\cdot 1$: $S^3 P^2 \to S^3 P^2$ is the space $S^2 P^2 \wedge P^2$. If $2\cdot 1 \simeq 0$, then $S^2 P^2 \wedge P^2$ would be homotopically equivalent to $S^3 P^2 \vee S^4 P^2$. However, the Steenrod operation Sq^2 is nonzero in $S^2 P^2 \wedge P^2$ and zero in $S^3 P^2 \vee S^4 P^2$, so that $2\cdot 1$ is essential.

Clearly, the Puppe sequence of f implies that $2\cdot 1$ is the composition

$$S^3 P^2 \xrightarrow{p} S^5 \xrightarrow{g} S^3 P^2,$$

where g generates $\pi_5 S^3 P^2$; the generator g is $i\eta$.

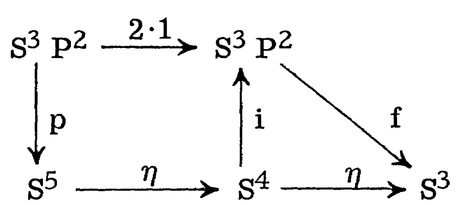
COROLLARY 5. (1) $[S^3 P^2, S^3] \cong \mathbb{Z}_4$.

(2) $[S^3 P^3, S^3] \cong \mathbb{Z}_4 + \mathbb{Z}_{12}$.

(3) $[P^2 \wedge P^2, S^3] \cong \mathbb{Z}_4$.

(4) $[P^3 \wedge P^2, S^3] \cong \mathbb{Z}_4 + \mathbb{Z}_4$.

Proof. It is easy to see that $[S^3 P^2, S^3]$ has four elements. The essential composition $S^3 P^2 \xrightarrow{p} S^5 \xrightarrow{\eta^2} S^3$ can be halved, as the diagram



shows. The map f, an extension of η, exists because $2\eta = 0$. This establishes (1).

Isomorphism (2) follows from (1), because $S^3 P^3 \simeq S^3 P^2 \vee S^6$ by Corollary 2 and because $\pi_6 S^3 \cong \mathbb{Z}_{12}$.

The group $[P^2 \wedge P^2, S^3]$ is stable and has four elements. The group $[S^2 \wedge P^2, S^3]$ has two elements, the nontrivial one being $\eta \circ (1 \wedge p)$. To prove isomorphism (3), it suffices to show that the composition

$$S^2 P^2 \wedge P^2 \xrightarrow{p\wedge 1} S^4 \wedge P^2 \xrightarrow{1\wedge p} S^4 \wedge S^2 \cong S^6 \xrightarrow{\eta} S^5$$

can be halved. This is an immediate consequence of the diagram

$$\begin{array}{ccccc}
S^2 P^2 \wedge P^2 & \xrightarrow{p\wedge 1} & S^4 \wedge P^2 & \xrightarrow{1\wedge p} & S^4 \wedge S^2 \\
\Big\downarrow 2\cdot 1 & & \Big\downarrow \eta\wedge 1 & & \Big\downarrow \eta\wedge 1 \\
 & & S^3 \wedge P^2 & & \\
 & \swarrow i\wedge 1 & & 1\wedge p \searrow & \\
S^2 P^2 \wedge P^2 & & \xrightarrow{\quad h \quad} & & S^3 \wedge S^2
\end{array}$$

The extension h exists because $(1\wedge p)\circ 2\colon S^3 \wedge P^2 \to S^3 \wedge S^2$ is null-homotopic.

The group $[P^3 \wedge P^2, S^3]$ is stable by Lemma 3. Isomorphism (4) now follows from the existence of the homotopy equivalence $S^2 P^3 \wedge P^2 \simeq (S^2 P^2 \wedge P^2) \vee S^5 P^2$, which is a consequence of Corollary 2.

THEOREM 6. $[P^3 \wedge P^3, S^3]$ *is isomorphic with* $\mathbb{Z}_4 + \mathbb{Z}_4 + \mathbb{Z}_4 + \mathbb{Z}_{12}$.

Proof. We look at the Puppe sequence of the map $\pi \wedge 1\colon S^2 \wedge P^3 \to P^2 \wedge P^3$, namely

$$[SP^2 \wedge P^3, S^3] \xrightarrow{p_1} [S^3 \wedge P^3, S^3] \to [P^3 \wedge P^3, S^3]$$
$$\to [P^2 \wedge P^3, S^3] \xrightarrow{p_2} [S^2 \wedge P^3, S^3].$$

We show that p_1 and p_2 are both zero and that the resulting short exact sequence splits. Corollary 2 and Lemma 3 imply that p_2 is zero. In the commutative diagram

$$\begin{array}{ccc}
[S^2 P^2 \wedge P^3, S^4] & \xrightarrow{p_3} & [S^4 \wedge P^3, S^4] \\
\Big\downarrow & & \Big\downarrow \\
[SP^2 \wedge P^3, S^3] & \xrightarrow{p_1} & [S^3 \wedge P^3, S^3],
\end{array}$$

the vertical maps are boundary maps in exact sequences derived from the fibration $S^7 \to S^4$ and are epimorphisms. The map p_3 is zero; therefore p_1 is also zero.

Let K^5 be the 5-skeleton of $P^3 \wedge P^3$. Using Puppe sequences derived from $S^5 \to K^5 \to P^3 \wedge P^3$, we get the diagram

$$\begin{array}{ccccccccc}
[SK^5, S^3] & \xrightarrow{q_1} & [S^6, S^3] & \to & [P^3 \wedge P^3, S^3] & \to & [K^5, S^3] & \xrightarrow{q_2} & [S^5, S^3] \\
\downarrow \Sigma_1 & & \downarrow \Sigma_2 & & \downarrow \Sigma_3 & & \downarrow \Sigma_4 & & \downarrow \Sigma_5 \\
\{SK^5, S^3\} & \xrightarrow{q_1'} & \{S^6, S^3\} & \to & \{P^3 \wedge P^3, S^3\} & \to & \{K^5, S^3\} & \xrightarrow{q_2'} & \{S^5, S^3\}
\end{array},$$

in which the maps q_2, q_2', q_1' are zero by stability. The map Σ_2 is a monomorphism; hence q_1 is also zero. The map Σ_3 is a monomorphism by the 5-lemma. It is a consequence of Corollary 2 that $\{P^3 \wedge P^3, S^3\}$ is isomorphic with

$$\{S^6, S^3\} + \{S^3 P^2, S^3\} + \{S^3 P^2, S^3\} + \{P^2 \wedge P^2, S^3\} ,$$

which we have already shown to be isomorphic with $\mathbb{Z}_4 + \mathbb{Z}_4 + \mathbb{Z}_4 + \mathbb{Z}_{24}$. Because Σ_2, Σ_3 are monomorphisms and Σ_4 is an isomorphism, we have proved the result.

COROLLARY 7. *There are exactly* 768 *homotopy classes of multiplications on* P^3.

Proof. The set of homotopy classes of multiplications on P^3 is in one-to-one correspondence with the group $[P^3 \wedge P^3, P^3]$. The space $P^3 \wedge P^3$ is simply connected; hence $[P^3 \wedge P^3, P^3]$ is isomorphic with $[P^3 \wedge P^3, S^3]$.

3. We now show how to enumerate the number of multiplications on P^7.

Using the stable cohomotopy spectral sequence, we can prove the following lemma (compare [5]).

LEMMA 8. (1) $[S^5 P^6, S^7] = 0$.

(2) $[S^6 P^6, S^7] = 0$.

(3) $[S^7 P^6, S^7]$ *has order* 8.

LEMMA 9. (1) $[S^7 P^7, S^7]$ *has order* $120 \cdot 2^3$.

(2) $[P^6 \wedge P^6, S^7]$ *has order* 4.

(3) $[P^6 \wedge P^7, S^7]$ *has order* 2^5.

The proof of Lemma 9 is similar to the proofs in Section 2, and we omit it. It uses the Puppe sequences, part (3) of Lemma 3, and Lemma 8.

THEOREM 10. *There are exactly* $120 \cdot 2^8 = 30\,720$ *homotopy classes of multiplications on* P^7.

Proof. We consider the exact sequence

$$[SP^6 \wedge P^7, S^7] \xrightarrow{p_1} [S^7 \wedge P^7, S^7] \to [P^7 \wedge P^7, S^7]$$
$$\to [P^6 \wedge P^7, S^7] \xrightarrow{p_2} [S^6 \wedge P^7, S^7],$$

where p_2 is zero by Corollary 2 and Lemma 3. It only remains to show that p_1 is also zero.

Suppose p_1 is not zero. Choose $x \in [SP^6 \wedge P^7, S^7]$ such that $p_1 x$ is nonzero. The suspension map $S: [S^7 \wedge P^7, S^7] \to [S^8 \wedge P^7, S^8]$ is a monomorphism (it splits the homotopy sequence derived from the Hopf fibration $S^{15} \to S^8$); therefore $Sp_1 x \neq 0$. Lemma 3 and Corollary 2 imply that $S^2 p_1 x = 0$. The kernel of $S: [S^8 P^7, S^8] \to [S^9 P^7, S^9]$ is generated by the element $2\sigma - \sigma'$, where σ, σ'

generate the respective summands in $\pi_{15} S^8 \cong \mathbb{Z} + \mathbb{Z}_{120}$. Hence Sx has infinite order, but it lies in the finite group $[S^2 P^6 \wedge \bar{P}^7, S^8]$, which is a contradiction. Thus $p_1 = 0$.

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