A CHARACTERIZATION OF UNIFORMLY CONTINUOUS UNITARY REPRESENTATIONS OF CONNECTED LOCALLY COMPACT GROUPS

Robert R. Kallman

In this paper, we characterize the uniformly continuous unitary representations of connected, locally compact groups. Roughly stated, the main theorem says that a unitary representation of a connected, locally compact group is uniformly continuous if and only if its support (see J. Dixmier [1, Definition 18.1.7, p. 315]) is a nice bounded set.

In what follows, G denotes a connected, locally compact group whose topology satisfies the second axiom of countability. Let $H = \bigcap_\pi \operatorname{Ker} \pi$, where π ranges over the set of finite-dimensional unitary representations of G. Then H is a closed, normal subgroup of G. Hence, G/H is also a connected, locally compact group whose topology satisfies the second axiom of countability.

The notation used in this paper is that of Dixmier [1]. Specific notation and results from [1] will be recalled as the need arises. To avoid needless circumlocution, we abbreviate "strongly continuous unitary representation" to "unitary representation."

The main result is that $\pi(\cdot)$ is a uniformly continuous unitary representation of G if and only if $\pi(\cdot)$ is quasi-equivalent to a direct integral

$$\sum_{\ell=1}^{n} \oplus \int_{\hat{G}_\ell} \pi(\xi)(\cdot)\, d\mu_\ell(\xi),$$

where n is a positive integer depending on π, and where μ_ℓ is a Borel measure on $\hat{G}_\ell$ $(1 \leq \ell \leq n)$ with compact support. In the process of proving this, we characterize the compact subsets of $\hat{G}_m$ (m a positive integer). Each compact subset of $\hat{G}_m$ is the union of finitely many sets of the form $(\hat{\pi}, C)$. Here $\hat{\pi} \in \hat{G}_n$, C is a compact subset of $\hat{G}_1$ (the set of characters of G), and

$$(\hat{\pi}, C) = [\hat{\pi}\chi \mid \chi \in C].$$

We first prove the theorem for the connected group G/H, making essential use of the fact that G/H is the direct product of a compact group and a vector group. Then we show that $\widehat{(G/H)}_n$ and $\hat{G}_n$ are homeomorphic in a natural manner. The main result then follows from this.

LEMMA 1. G/H *is the direct product of a connected compact group and a vector group.*

Proof. By construction, G/H has a separating collection of finite-dimensional unitary representations. The theorem now follows from a result of R. V. Kadison and I. M. Singer [2, Theorem 1, p. 420]. ■

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LEMMA 2. *Let* π *be a uniformly continuous unitary representation of* G *on some Hilbert space. Then* $H \subseteq \operatorname{Ker} \pi$.

Proof. G/Ker π has a faithful uniformly continuous unitary representation. Hence (see [2, Corollary 4, p. 423]), G/Ker π is the direct product of a compact group and a vector group. But each locally compact group has a separating collection of irreducible unitary representations. Each irreducible representation of a compact group or vector group is finite-dimensional. Hence, Ker π is the intersection of kernels of finite-dimensional unitary representations of G. Hence, $H \subseteq \operatorname{Ker} \pi$. ∎

Remark 3. Lemma 2 shows that each uniformly continuous unitary representation of a connected locally compact group generates a Type I von Neumann algebra. Hence, each uniformly continuous representation is quasi-equivalent to a uniformly continuous, multiplicity-free representation.

Let F be a locally compact group. If n is a cardinal, let $\mathscr{H}_n$ be a fixed Hilbert space of dimension n, and let $\operatorname{Irr}_n(F)$ be the collection of irreducible unitary representations of F into $\mathscr{H}_n$ with the natural topology. Let $\hat{F}$ be the collection of unitary equivalence classes of irreducible unitary representations of F. If π is an irreducible unitary representation of F, denote by $\hat{\pi}$ the corresponding equivalence class in $\hat{F}$. Let

$$\hat{F}_n = [\hat{\pi} \mid \pi \in \operatorname{Irr}_n(F)].$$

Suppose F is of Type I and the topology of F satisfies the second axiom of countability. Then there exists a topology on $\hat{F}$ that generates a nice Borel structure on $\hat{F}$. Recall that there is a one-to-one correspondence between multiplicity-free representations of F and Borel measure classes in $\hat{F}$. For the foregoing, see Dixmier [1] and G. W. Mackey [4].

By Lemma 1, $G/H = K \times V$, where K is a connected, compact group and V is a vector group. Since V is abelian, each element of $\hat{V}$ is one-dimensional, and therefore we may identify it with a character of V. Hence, there is a bijective map ϕ of $\operatorname{Irr}_n(K) \times \hat{V}$ onto $\operatorname{Irr}_n(K \times V)$, where

$$\phi\colon (\pi, \chi) \to \pi \cdot \chi \qquad (\pi \in \operatorname{Irr}_n(K),\ \chi \in \hat{V}).$$

Here $(\pi \cdot \chi)(k, v) = \pi(k)\chi(v)$ $(k \in K,\ v \in V)$.

LEMMA 4. *The function* $\phi\colon \operatorname{Irr}_n(K) \times \hat{V} \to \operatorname{Irr}_n(K \times V)$ *is a homeomorphism.*

Proof. To show that ϕ is continuous, suppose $\pi_\alpha, \pi \in \operatorname{Irr}_n(K)$, $\pi_\alpha \to \pi$, and $\chi_\beta, \chi \in \hat{V}$, $\chi_\beta \to \chi$. We want to conclude that $\pi_\alpha \cdot \chi_\beta \to \pi \cdot \chi$. Let $x \in \mathscr{H}_n$, and let C be a compact subset of $K \times V$. Choose compact subsets $A \subseteq K$ and $B \subseteq V$ such that $C \subseteq A \times B$. Then

$$\sup_{(k,v)\in C} \|(\pi_\alpha \cdot \chi_\beta)(k, v)x - (\pi \cdot \chi)(k, v)x\|$$

$$\leq \sup_{(k,v)\in A\times B} \|(\pi_\alpha \cdot \chi_\beta)(k, v)x - (\pi \cdot \chi)(k, v)x\|$$

$$\leq \sup_{(k,v)\in A\times B} \|(\pi_\alpha \cdot \chi_\beta)(k, v)x - (\pi \cdot \chi_\beta)(k, v)x\|$$

$$+ \sup_{(k,v)\in A\times B} \|(\pi\cdot\chi_\beta)(k, v)x - (\pi\cdot\chi)(k, v)x\|$$

$$= \sup_{k\in A} \|\pi_\alpha(k)x - \pi(k)x\| + \sup_{v\in B} |\chi_\beta(v) - \chi(v)|\cdot\|x\| \to 0$$

as $\alpha \uparrow \infty$, $\beta \uparrow \infty$, since $\pi_\alpha \to \pi$ and $\chi_\beta \to \chi$. Hence, $\pi_\alpha\cdot\chi_\beta \to \pi\cdot\chi$ as $\alpha \uparrow \infty$, $\beta \uparrow \infty$. Hence, ϕ is continuous.

To see that ϕ^{-1} is also continuous, let $\pi_\alpha\cdot\chi_\alpha \to \pi\cdot\chi$, and let C be compact in K. Then $C \times (0)$ is compact in $K \times V$. Hence, if $x \in \mathscr{H}_n$, then

$$\sup_{k\in C} \|\pi_\alpha(k)x - \pi(k)x\| = \sup_{(k,v)\in C\times(0)} \|(\pi_\alpha\cdot\chi_\alpha)(k, v)x - (\pi\cdot\chi)(k, v)x\| \to 0$$

as $\alpha \uparrow \infty$, since $\pi_\alpha\cdot\chi_\alpha \to \pi\cdot\chi$. Hence $\pi_\alpha \to \pi$. Similarly, let C be a compact subset of V. Then $(e) \times C$ is compact in $K \times V$. Hence, if $x \in \mathscr{H}_n$ is a unit vector, then

$$\sup_{v\in C} |\chi_\alpha(v) - \chi(v)| = \sup_{(k,v)\in(e)\times C} \|(\pi_\alpha\cdot\chi_\alpha)(k, v)x - (\pi\cdot\chi)(k, v)x\| \to 0$$

as $\alpha \uparrow \infty$, since $\pi_\alpha\cdot\chi_\alpha \to \pi\cdot\chi$. Hence, $\chi_\alpha \to \chi$. Hence, ϕ^{-1} is continuous. ∎

Let $\pi_1, \pi_2 \in \mathrm{Irr}_n(K)$, $\chi_1, \chi_2 \in \hat{V}$. Then $\pi_1\cdot\chi_1$ is unitarily equivalent to $\pi_2\cdot\chi_2$ if and only if $\chi_1 = \chi_2$ and π_1 is unitarily equivalent to π_2. Hence, ϕ induces a bijection $\hat{\phi}$ of $\hat{K}_n \times \hat{V}$ onto $(\widehat{K \times V})_n$.

COROLLARY 5. *$\hat{\phi}$ is a homeomorphism of $\hat{K}_n \times \hat{V}$ onto $(\widehat{K\times V})_n$.*

Proof. Use Lemma 4, the preceding remarks, plus the following: if F is a locally compact group and n is a cardinal, then the quotient map of $\mathrm{Irr}_n(F)$ onto $\hat{F}_n$ is open and continuous. ∎

Remark 6. Let F be a locally compact group. On $\bigcup_{n\geq 1} \hat{F}_n$, consider the unique Borel field of Hilbert spaces $\xi \to \mathscr{H}(\xi)$ such that $\mathscr{H}(\xi) = \mathscr{H}_n$ for $\xi \in \hat{F}_n$. There exists a field $\xi \to \pi(\xi)$ of unitary representations of F on $\mathscr{H}(\xi)$ such that $\pi(\xi) \in \xi$ for all $\xi \in \bigcup_{n\geq 1} \hat{F}_n$ and $\xi \to \pi(\xi)$ is measurable for every positive Borel measure on $\bigcup_{n\geq 1} \hat{F}_n$. (For these results, see Dixmier [1, p. 154].)

LEMMA 7. *Let $\pi(\cdot)$ be a multiplicity-free representation of* G/H. *Then $\pi(\cdot)$ is uniformly continuous if and only if $\pi(\cdot)$ is unitarily equivalent to*

$$\sum_{\ell=1}^{n}\oplus \int_{C_\ell} \pi(\xi)(\cdot)\,d\mu_\ell(\xi) \quad \text{on} \quad \sum_{\ell=1}^{n}\oplus \int_{(\widehat{G/H})_\ell} \mathscr{H}(\xi)\,d\mu_\ell(\xi).$$

Here (1) n *is some positive integer (depending upon* π);

(2) *the mappings $\xi \to \mathscr{H}(\xi)$ and $\xi \to \pi(\xi)$ are described in Remark* 6;

(3) C_ℓ *is a compact subset of* $(\widehat{G/H})_\ell$ $(1 \leq \ell \leq n)$;

(4) μ_ℓ *is a Borel measure on* $(\widehat{G/H})_\ell$, *supported by* C_ℓ.

Proof. $G/H = K \times V$ is separable, of Type I, and has only finite-dimensional irreducible unitary representations. Hence, by changing $\pi(\cdot)$ to a unitarily equivalent representation, we may assume that

$$\pi(\cdot) = \int_{\widehat{K\times V}} \pi(\xi)(\cdot)d\mu(\xi) = \sum_{n\geq 1} \oplus \int_{\widehat{(K\times V)}_n} \pi(\xi)(\cdot)d\mu(\xi)$$

$$= \sum_{n\geq 1} \oplus \int_{\hat{K}_n\times\hat{V}} \pi(\xi)(\cdot)d\mu(\xi)$$

(by Corollary 5). $\pi(\cdot)$ acts on

$$\int_{\widehat{K\times V}} \mathscr{H}(\xi)d\mu(\xi) = \sum_{n\geq 1} \oplus \int_{\widehat{(G/H)}_n} \mathscr{H}(\xi)d\mu(\xi).$$

Here $\xi \to \mathscr{H}(\xi)$ and $\xi \to \pi(\xi)$ are as in Remark 6, and μ is a Borel measure on $\widehat{G/H}$.

Now $\hat{K}$, and hence $\hat{K}_n$, is discrete (Dixmier [1, Corollaire 18.4.3, p. 322]).

If $k \in K$ and $v \in V$, then

$$\int_{\hat{K}_n\times\hat{V}} \pi(\xi)(k, v)d\mu(\xi) = \sum_{\xi'\in\hat{S}_n} \oplus \int_{\hat{V}} \chi(v)d\mu_{\xi'}(\chi)\cdot\rho(\xi')(k)$$

where $\hat{S}_n \subseteq \hat{K}_n$ $(n \geq 1)$, $\mu_{\xi'}$ is a nonzero Borel measure on $\hat{V}$, and $\rho(\xi') \in \xi'$. Hence

$$\pi(k, v) = \sum_{\substack{n\geq 1\\ \xi'\in\hat{S}_n}} \oplus \int_{\hat{V}} \chi(v)d\mu_{\xi'}(\chi)\cdot\rho(\xi')(k).$$

Let $\pi' = \pi \mid K$. Then

$$\pi'(k) = \sum_{\substack{n\geq 1\\ \xi'\in\hat{S}_n}} \oplus \rho(\xi')(k) \qquad (k \in K).$$

Suppose $\pi(\cdot)$ is uniformly continuous. We claim that $\bigcup_{n\geq 1} \hat{S}_n$ is finite. There exists a neighborhood N of e in K such that

$$\frac{1}{2} \geq \sup_{a\in N} \|\pi'(a) - \pi'(e)\| = \sup_{\substack{a\in N\\ \xi'\in\bigcup_{n\geq 1}\hat{S}_n}} \|\rho(\xi')(a) - \rho(\xi')(e)\|.$$

Choose a continuous function f on K such that f is zero outside of N, $f \geq 0$, and $\int_K f(a)da = 1$. Then

$$\|\rho(\xi')(f) - \rho(\xi')(e)\| = \left\| \int_N f(a)\rho(\xi')(a)\,da - \int_N f(a)\rho(\xi')(e)\,da \right\|$$

$$\leq \int_N f(a)\,da \cdot \sup_{a \in N} \|\rho(\xi')(a) - \rho(\xi')(e)\| \leq \frac{1}{2}.$$

Therefore, if $\xi' \in \bigcup_{n \geq 1} \hat{S}_n$, then

$$\|\rho(\xi')(f)\| = \|\rho(\xi')(e) + [\rho(\xi')(f) - \rho(\xi')(e)]\|$$

$$\geq \|\rho(\xi')(e)\| - \|\rho(\xi')(f) - \rho(\xi')(e)\| \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

Hence, by [1, Proposition 3.3.7, p. 64], $\bigcup_{n \geq 1} \hat{S}_n$ is contained in a quasi-compact subset of $\hat{K}$. But by [1, Corollaire 18.4.3], $\hat{K}$ is discrete. Hence, $\bigcup_{n \geq 1} \hat{S}_n$ is finite.

Similarly, by restricting π to V, one sees that each of the unitary representations

$$v \to \int_{\hat{V}} \chi(v)\,d\mu_{\xi'}(v) \qquad \left(\xi' \in \bigcup_{n \geq 1} \hat{S}_n\right)$$

is uniformly continuous. Hence, each $\mu_{\xi'}$ has compact support in $\hat{V}$ (see Kallman [3, Lemma 1, p. 251]). Hence, if $\pi(\cdot)$ is uniformly continuous, it has the form described in the lemma.

Conversely, suppose $\pi(\cdot)$ has the described direct integral decomposition. That $\pi(\cdot)$ is uniformly continuous, then follows if we combine Corollary 5, the discreteness of $\hat{K}$, and [3, Lemma 1, p. 251]. ∎

Now let ψ: G → G/H be the natural quotient mapping. Then ψ induces a bijective mapping ψ^* of Rep(G/H) onto the subset of Rep(G) whose kernels contain H, via $\psi^*(\pi) = \pi \circ \psi$. If n is a positive integer, then ψ^*: $\mathrm{Irr}_n(G/H) \to \mathrm{Irr}_n(G)$ is a bijection. Recall that if F is a locally compact group, then there is a nice topology on $\mathrm{Irr}_n(F)$ (see Dixmier [1, pp. 58-79 and pp. 314-316]).

LEMMA 8. *ψ^* is a homeomorphism of* $\mathrm{Irr}_n(G/H)$ *onto* $\mathrm{Irr}_n(G)$.

Proof. To show that ψ^* is continuous, let C be a compact subset of G; then $\psi(C)$ is a compact subset of G/H (ψ is continuous and G/H is a Hausdorff space, since H is closed). Suppose that $\pi_\alpha, \pi \in \mathrm{Irr}_n(G/H)$ and that $\pi_\alpha \to \pi$. Then, if $x \in \mathscr{H}_n$, $\sup_{a \in \psi(C)} \|\pi_\alpha(a)x - \pi(a)x\| \to 0$ as $\alpha \uparrow \infty$. Hence

$$\sup_{a \in C} \|\psi^*(\pi_\alpha)(a)x - \psi^*(\pi)(a)x\| = \sup_{a \in C} \|\pi_\alpha \circ \psi(a)x - \pi \circ \psi(a)x\|$$

$$= \sup_{a \in \psi(C)} \|\pi_\alpha(a)x - \pi(a)x\| \to 0 .$$

Hence, ψ^* is continuous.

Conversely, suppose $\pi_\alpha, \pi \in \mathrm{Irr}_n(G)$ and $\pi_\alpha \to \pi$. Let $C \subseteq G/H$ be compact. Let $G = \bigcup_\beta \mathscr{O}_\beta$, where each $\mathscr{O}_\beta$ is an open subset of G with compact closure. Then $\psi(\mathscr{O}_\beta)$ is open for all β (ψ is an open mapping). Since $C \subseteq G/H = \bigcup_\beta \psi(\mathscr{O}_\beta)$, there exists a finite set of indices $\beta_1, \cdots, \beta_n$ such that

$$C \subseteq \bigcup_{i=1}^{n} \psi(\mathscr{O}_{\beta_i}) = \psi\left(\bigcup_{i=1}^{n} \mathscr{O}_{\beta_i}\right) \subseteq \psi\left(\bigcup_{i=1}^{n} \overline{\mathscr{O}}_{\beta_i}\right).$$

Note that $\bigcup_{i=1}^{n} \overline{\mathscr{O}}_{\beta_i} = C'$ is compact in G. If $x \in \mathscr{H}_n$, then

$$\sup_{a\in C} \|\psi^{*-1}(\pi_\alpha)(a)x - \psi^{*-1}(\pi)(a)x\| \leq \sup_{a\in C'} \|\pi_\alpha(a)x - \pi(a)x\| \to 0$$

as $\alpha \uparrow \infty$. Hence, ψ^{*-1} is continuous. Therefore,

$$\psi^*: \mathrm{Irr}_n(G/H) \to \mathrm{Irr}_n(G)$$

is a homeomorphism. ■

If $\pi_1, \pi_2 \in \mathrm{Irr}_n(G/H)$, then π_1 and π_2 are unitarily equivalent if and only if $\psi^*(\pi_1)$ and $\psi^*(\pi_2)$ are unitarily equivalent. Hence, ψ^* induces a bijective mapping $\hat{\psi}$ of $\widehat{(G/H)}_n$ onto $\hat{G}_n$.

COROLLARY 9. *$\hat{\psi}$ is a homeomorphism of $\widehat{(G/H)}_n$ onto $\hat{G}_n$.*

Proof. Use Lemma 9, the previous discussion, plus the fact that if $\mathscr{A}$ is a C*-algebra, then the quotient mapping of $\mathrm{Irr}_n(\mathscr{A})$ onto $\hat{A}_n$ is continuous and open. ■

Remark 10. Note that Corollary 9 and the proof (not the statement) of Lemma 7 imply the following result. The compact subsets of $\hat{G}_n$ (n a positive integer) are unions of finitely many sets of the form $(\hat{\pi}, C)$, where $\hat{\pi} \in \hat{G}_n$, where $C \subseteq \hat{G}_1$ is a compact subset, and where $(\hat{\pi}, C) = [\hat{\pi}\cdot\chi \mid \chi \in C]$.

THEOREM 11. *Let $\pi(\cdot)$ be a multiplicity-free unitary representation of* G. *Then $\pi(\cdot)$ is uniformly continuous if and only if $\pi(\cdot)$ is unitarily equivalent to*

$$\sum_{\ell=1}^{n} \oplus \int_{C_\ell} \pi(\xi)(\cdot)\,d\mu_\ell(\xi) \quad \text{on} \quad \sum_{\ell=1}^{n} \oplus \int_{\hat{G}_\ell} \mathscr{H}(\xi)\,d\mu_\ell(\xi).$$

Here (1) n *is a positive integer* (*depending on* π);

(2) $\xi \to \mathscr{H}(\xi)$, $\xi \to \pi(\xi)$ *are described in Remark* 6;

(3) C_ℓ *is a compact subset of* $\hat{G}_\ell$ $(1 \leq \ell \leq n)$;

(4) μ_ℓ *is a Borel measure on* $\hat{G}_\ell$, *supported by* C_ℓ.

Proof. Suppose $\pi(\cdot)$ is uniformly continuous. By Lemma 3, $H \subseteq \mathrm{Ker}\,\pi$, and hence $(\psi^{*-1}\pi)(\cdot)$ is a uniformly continuous, unitary representation of G/H. Hence, combining Lemma 7 and Corollary 9, we find that if $a \in G/H$, then

$$(\psi^{*-1}\pi)(a) = \sum_{\ell=1}^{n} \oplus \int_{C_\ell} \pi(\xi)(a)\,d\mu_\ell(\xi) \quad \text{on} \quad \sum_{\ell=1}^{n} \oplus \int_{\hat{G}_\ell} \mathscr{H}(\xi)\,d\mu_\ell(\xi).$$

Here (1), (3), and (4) are satisfied, and $\xi \to \pi(\xi)(\cdot)$ is a field of unitary representations of G/H on $\mathscr{H}(\xi)$ such that $\pi(\xi) \in \hat{\psi}^{-1}(\xi)$ for all $\xi \in \bigcup_{n \geq 1} \hat{G}_n$, and $\xi \to \pi(\xi)$ is measurable for every positive Borel measure on $\bigcup_{n \geq 1} \hat{G}_n$. To complete the argument, let $a = \psi(b)$ $(b \in G)$, and note that

$$\pi(b) = (\psi^{*-1}\pi)(\psi(b)) = \sum_{\ell=1}^{n} \oplus \int_{C_\ell} \pi(\xi) \cdot \psi(b)\, d\mu_\ell(\xi)$$

$$= \sum_{\ell=1}^{n} \oplus \int_{C_\ell} \psi^*(\pi(\xi))(b)\, d\mu_\ell(\xi) \qquad (b \in G).$$

Observe that $\xi \to \psi^*(\pi(\xi))(\cdot)$ is a Borel field of unitary representations of G on $\mathscr{H}(\xi)$ such that $\psi^*(\pi(\xi)) \in \xi$ for all $\xi \in \bigcup_{n \geq 1} \hat{G}_n$. Hence, if $\pi(\cdot)$ is uniformly continuous, it has the required form.

Conversely, suppose $\pi(\cdot)$ has the stated direct integral decomposition. Then Remark 10 and [3, Lemma 1, p. 251] show that $\pi(\cdot)$ is uniformly continuous. ■

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Massachusetts Institute of Technology
Cambridge, Massachusetts 02139