ON THE CONTINUITY OF A CLASS OF UNITARY REPRESENTATIONS

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Let $\{U_n\}$ be a sequence of unitary operators satisfying the conditions $U_{n+1}^2 = U_n$ for $n = 1, 2, \cdots$. Let E^n be the spectral measure on $[-\pi, \pi)$ associated with U_n. In general, E^{n+1} is obtained from an orthogonal splitting of E^n (see the remark at the end of this note). Let D be the group of dyadic rationals, topologized as a subset of the reals. The U_n give rise to the representation V_r of D defined by $V_{m/2^n} = U_n^m$.

In this note, we study the relation between the measures E^n and the continuity of V_r. If for example $E^n(X) = E^{n+1}(X/2)$ for all n and all Borel sets X, then V_r is continuous in the uniform operator topology. If $U_n = \lambda_n I$ and the numbers λ_n satisfy the conditions $\lambda_{n+1}^2 = \lambda_n$ and $\lambda_1 = 1$, and if $\{\lambda_n\}$ has no limit, then the resulting V_r is continuous only on the zero vector. In general, the closed subspace C of vectors x for which $V_r x$ is a strongly continuous function of r is a proper subspace. The theorem we prove below tells how to recapture C from the E^n. The theorem bears out the feeling that the way to get continuity is to use principal square roots, at least asymptotically.

Throughout this note, B denotes the class of Borel sets on the line, and all limits involving projections are in the strong operator topology.

THEOREM. *If* P *is the orthogonal projection on* C *and* I *is any interval having* 0 *in its interior, then*

$$P = \lim_{m \to \infty} \lim_{n \to \infty} E^n(m\, 2^{-n} I).$$

LEMMA. *If* $X \in B$ *and* $X \subset [-\pi, \pi)$, *then* $E^{n+1}(X/2) \leq E^n(X)$.

Proof. By the regularity of the spectral measures [1, p. 63], it suffices to prove the lemma for the case where X is an interval. Pick a sequence of "polynomials" p_m (we allow both positive and negative powers) such that $p_m(e^{i\lambda})$ converges boundedly to the characteristic function $\psi(\lambda)$ of X. Then $p_m(U_n) \to E^n(X)$ strongly. But

$$p_m(U_n) = p_m(U_{n+1}^2) = \int_{-\pi}^{\pi} p_m(e^{2i\lambda})\, dE^{n+1}(\lambda).$$

Let $X_0 = \{\lambda \in [-\pi, \pi): \lambda \in X/2 \pmod{\pi}\}$. Then $p_m(e^{2i\lambda})$ converges to the characteristic function of X_0 for $\lambda \in [-\pi, \pi)$, and therefore $p_m(U_n) = p_m(U_{n+1}^2)$ converges to $E^{n+1}(X_0)$ strongly. Hence $E^{n+1}(X/2) \leq E^{n+1}(X_0) = E^n(X)$.

Proof of the theorem. Define $F^n(X) = E^n(X/2^n)$ for each $X \in B$. Then, if $m \geq n$ and $X \subset [-2^n\pi, 2^n\pi)$, it follows from the lemma that

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$$F^{m+1}(X) = E^{m+1}(2^{-m}X/2) \leq E^m(2^{-m}X) = F^m(X). \tag{1}$$

Also (again for $m \geq n$),

$$U_n = U_m^{2^{m-n}} = \int_{-\infty}^{\infty} e^{i\lambda 2^{m-n}} dE^m(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda 2^{-n}} dF^m(\lambda). \tag{2}$$

For each positive integer m, the sequence $\{F^n(X \cap [-m, m])\}$ is an ultimately decreasing sequence of projections, by (1), and hence

$$\lim_n F^n(X \cap [-m, m]) = G^m(X)$$

defines a projection-valued measure on B. Since $G^{m+1}(X) \geq G^m(X)$, $H(X) = \lim G^m(X)$ likewise defines a projection-valued measure. Thus, since

$$H(X) = \lim_m \lim_n E^n(2^{-n}\{X \cap [-m, m]\}),$$

what we must show is that $H(R) = P$. We may assume that $I = [-1, 1]$.

Suppose $H(R)x = x$. Then, if J is an interval,

$$\|H(X)x - F^m(X)x\| \leq \|H(X \cap J)x - F^m(X \cap J)x\| + \|H(J')x\| + \|F^m(J')x\|.$$

The last term may be rewritten as $\|x - F^m(J)x\| = \|H(J)x - F^m(J)x + H(J')x\|$, and we get the inequality

$$\|H(X)x - F^m(X)x\| \leq \|H(X \cap J)x - F^m(X \cap J)x\| + 2\|H(J')x\| + \|H(J)x - F^m(J)x\|.$$

Picking first J large, then m large, we see that $F^m(X)x \to H(X)x$ strongly for each $X \in B$. But, by (2),

$$U_n x = \lim_m \int_{-\infty}^{\infty} e^{i\lambda 2^{-n}} dF^m(\lambda)x = \int_{-\infty}^{\infty} e^{i\lambda 2^{-n}} dH(\lambda)x;$$

hence

$$V_r x = \int_{-\infty}^{\infty} e^{i\lambda r} dH(\lambda)x, \tag{3}$$

and therefore $x \in C$, by bounded convergence. Thus $H(R) \leq P$.

To obtain the reverse inequality, we use an elementary Fourier argument. Let $Px = x$, and let $v(\lambda)$ be the continuous extension of $(V_r x, x)$. If $m2^{-n} < \pi$, then

$$\|E^n(m\,2^{-n}I)x\|^2 \geq \int_{-m2^{-n}}^{m2^{-n}} \left(1 - \frac{|\lambda|}{m2^{-n}}\right) d\|E^n(\lambda)x\|^2$$

$$= \sum_{\nu=-\infty}^{\infty} \frac{2\sin^2(m/2)\,2^{-n}\nu}{\pi(m/2)\,2^{-n}\nu^2} \int_{-\pi}^{\pi} e^{i\lambda\nu}\, d\|E^n(\lambda)x\|^2$$

$$= 2^{-n} \sum_{-\infty}^{\infty} \frac{2 \sin^2 (m/2)(\nu/2^n)}{\pi (m/2)(\nu/2^n)^2} (V_{\nu/2^n} x, x).$$

Hence

$$\lim_n \| E^n(m 2^{-n} I)x \|^2 \geq \int_{-\infty}^{\infty} \frac{2 \sin^2 (m\lambda/2)}{\pi (m\lambda^2/2)} v(\lambda) d\lambda,$$

and therefore $\| H(R)x \|^2 \geq v(0) = \| x \|^2$; this completes the proof.

Our proof shows that if V_r is continuous for all x, then, by (3),

$$V_r = \int_{-\infty}^{\infty} e^{i\lambda r} dH(\lambda).$$

This, of course, is Stone's Theorem.

We finish by making some observations on square roots of unitary operators in general. Suppose that $V^2 = U$, where U is unitary, then, for each integer ν,

$$\| V^{\nu} \| \leq \max(\| V \|, \| V^{-1} \|, 1).$$

It follows from a theorem of Sz.-Nagy that there exists a strictly positive operator A such that $V = AWA^{-1}$, where W is unitary (see [2]). From the relation $W^2 = A^{-1} UA$, it follows that $(A^{-1} UA)^* = (A^{-1} UA)^{-1}$ or $A^2 U = UA^2$. Hence A commutes with U, and therefore V is similar to a unitary square root of U.

W can be represented in the following way. There exist projection-valued measures P and N on the unit circle such that $P(X) N(Y) = 0$ and $P(X) + N(X) = E(X)$, where X and Y are any Borel sets and E is the spectral measure of U, and such that

$$W = \int \sqrt{z}\, dP - \int \sqrt{z}\, dN,$$

where $\sqrt{z}$ is the principal square root of z. We omit the proof, since it is easy.

REFERENCES

1. P. R. Halmos, *Introduction to Hilbert space and the theory of spectral multiplicity*. Chelsea, New York, N.Y., 1951.

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