POWER SERIES WITH MULTIPLY MONOTONIC COEFFICIENTS

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Let

$$s_n^{(k)}(z) = \binom{n+k}{k} + \binom{n+k-1}{k} z + \binom{n+k-2}{k} z^2 + \cdots + z^n$$

$$(k = 1, 2, \cdots; n = 0, 1, 2, \cdots).$$

The polynomials $s_n^{(k)}(z)$ are connected with the Cesàro sums of the geometric series, and

$$s_n^{(k+1)}(z) = s_0^{(k)}(z) + s_1^{(k)}(z) + \cdots + s_n^{(k)}(z).$$

E. Egerváry [1] showed that $s_n^{(k)}(z)$ is univalent in $|z| \leq 1$ for $n > 0$ and $k = 1, 2$, and 3, and that $s_n^{(1)}(z)$ maps $|z| < 1$ onto a domain whose closure is starlike with respect to the boundary point $s_n^{(1)}(1)$. However, $s_n^{(2)}(z)$ is starlike with respect to the interior point $s_n^{(2)}(0)$ for $|z| \leq 1$, and $s_n^{(3)}(z)$ maps the unit disk onto a convex domain. It follows that the functions $s_n^{(k)}(z)$ are close-to-convex in $|z| < 1$ for $k = 1, 2, 3$ and $n > 0$. This implies the existence of regular and univalent functions $\phi_n^{(k)}(z)$ that map the unit disk onto convex domains and for which

$$\Re \left\{ \frac{[s_n^{(k)}(z)]'}{[\phi_n^{(k)}(z)]'} \right\} \geq 0 \qquad (|z| < 1).$$

The question arises whether, for some values k, there exists a $\phi^{(k)}(z)$ that is independent of n.

For $k = 1$, $\phi_n^{(k)}(z)$ cannot be independent of n. This follows from an observation of G. Szegö [9], who constructed a function of the form

$$f(z) = p\, s_m^{(1)}(z) + q\, s_n^{(1)}(z),$$

where p and q are positive constants and m and n are integers, and where p, q, m, n are carefully chosen so that $f'(z)$ vanishes at an interior point of the unit disk, with the consequence that $f(z)$ is not univalent for $|z| < 1$. On the other hand, if $\phi_n^{(1)}(z) = \phi^{(1)}(z)$ were independent of n, it would follow from the equation

$$\Re \left\{ \frac{f'(z)}{[\phi^{(1)}(z)]'} \right\} = p\, \Re \left\{ \frac{[s_m^{(1)}(z)]'}{[\phi^{(1)}(z)]'} \right\} + q\, \Re \left\{ \frac{[s_n^{(1)}(z)]'}{[\phi^{(1)}(z)]'} \right\} \geq 0$$

that $f(z)$ is univalent for $|z| < 1$, and we would be faced with a contradiction. We conclude that $\phi_n^{(1)}(z)$ cannot be independent of n.

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Nevertheless, we shall show that in the case $k = 2$, $\phi_n^{(k)}(z)$ can be chosen independent of n. In fact, we may take $\phi_n^{(2)}(z) = \log 1/(1-z)$ $(n = 1, 2, \cdots)$.

This basic result allows us to sharpen the theorem due to G. Szegö [9], which in turn is an extension of a theorem of L. Fejér [2], [3], for sequences $\{a_n\}$ that are monotonic of order 4.

THEOREM (Szegö). *Let the sequence* $\{a_n\}$ *be monotonic of order* 3. *Then the power series* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is regular for* $|z| < 1$, *and if it is not a constant function, it is univalent for* $|z| < 1$.

Here a sequence $\{a_n\}$ is called *monotonic of order* μ if all the differences

$$\Delta^{(\nu)} a_n = a_n - \binom{\nu}{1} a_{n+1} + \binom{\nu}{2} a_{n+2} - \cdots + (-1)^{\nu} \binom{\nu}{\nu} a_{n+\nu}$$

are nonnegative for $\nu = 0, 1, 2, \cdots, \mu$ and $n = 0, 1, 2, \cdots$.

Our main results are contained in the following two theorems.

THEOREM 1. *Let*

$$s_n^{(2)}(z) = \binom{n+2}{2} + \binom{n+1}{2} z + \binom{n}{2} z^2 + \cdots + z^n.$$

Then $s_n^{(2)}(z)$ *is univalent and close-to-convex in* $|z| < 1$ *relative to the convex function* $\log 1/(1-z)$, *and*

$$\Re\left[(1-z) \frac{d}{dz} s_n^{(2)}(z) \right] \geq 0 \qquad (|z| \leq 1).$$

THEOREM 2. *Let the sequence* $\{a_n\}$ *be monotonic of order* 3, *and let* $a = \lim_{n\to\infty} a_n$. *Then either the function* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is regular, univalent, and close-to-convex for* $|z| < 1$, *relative to the convex function* $\log 1/(1-z)$, *and*

$$\Re[(1-z)f'(z)] > a/2 \qquad (|z| < 1),$$

or else $f(z)$ *is a constant.*

The example $f(z) = a/(1-z)$ $(a > 0)$ shows that the inequality in Theorem 2 is sharp.

Proof of Theorem 1. We obtain successively the equations

$$w(z) = s_n^{(2)}(z) = \frac{1}{2}[(n+2)(n+1) + (n+1)nz + n(n-1)z^2 + \cdots + 2z^n],$$

$$w(z) - \frac{(n+2)(n+1)}{2} = \frac{n(n+1)z - 2n(n+2)z^2 + (n+1)(n+2)z^3 - 2z^{n+3}}{2(1-z)^3},$$

$$(1-z)w'(z) = \frac{1}{2}[(n+1)n + n(n-3)z + (n-1)(n-6)z^2 + \cdots - 2nz^n]$$

$$= \frac{n(n+1) - (2n^2+6n)z + (n^2+5n+6)z^2 - (2n+6)z^{n+2} + 2nz^{n+3}}{2(1-z)^3},$$

$$z\,w'(z^2) = \frac{n(n+1)z - (2n^2+6n)z^3 + (n^2+5n+6)z^5 - (2n+6)z^{2n+5} + 2nz^{2n+7}}{2(1-z^2)^4}.$$

Let $z = e^{i\phi}$. Then

$$z\,w'(z^2) = \frac{n(n+1)z^{-3} - (2n^2+6n)z^{-1} + (n^2+5n+6)z - (2n+6)z^{2n+1} + 2nz^{2n+3}}{32\sin^4\phi},$$

and therefore

$$\Re\,[(1-z^2)w'(z^2)] = 2\sin\phi\,\Im\,[z\,w'(z^2)]$$

$$= \frac{1}{16\sin^3\phi}\,\Im\,[n(n+1)z^{-3} - (2n^2+6n)z^{-1} + (n^2+5n+6)z - (2n+6)z^{2n+1} + 2nz^{2n+3}]$$

$$= \frac{1}{16\sin^3\phi}\,[-n(n+1)\sin 3\phi + (2n^2+6n)\sin\phi + (n^2+5n+6)\sin\phi$$

$$- (2n+6)\sin(2n+1)\phi + 2n\sin(2n+3)\phi]$$

$$= \frac{1}{8\sin^2\phi}\left[4n+3+(2n^2+2n)\sin^2\phi - (n+3)\frac{\sin(2n+1)\phi}{\sin\phi} + \frac{n\sin(2n+3)\phi}{\sin\phi}\right].$$

Thus $\Re\,[(1-z)w'(z)] \geq 0$ for $|z| \leq 1$ if and only if $h(\phi) \geq 0$ for all ϕ, where $h(\phi)$ is defined by the equation

$$h(\phi) = 4n+3+(2n^2+2n)\sin^2\phi - (n+3)\frac{\sin(2n+1)\phi}{\sin\phi} + \frac{n\sin(2n+3)\phi}{\sin\phi}.$$

In proving that $h(\phi) \geq 0$ for all ϕ, we can restrict ϕ to the interval $[0, \pi/2]$, since $h(\phi)$ and $h(\phi - \pi/2)$ are even functions.

We show first that $h(\phi) \geq 0$ for all $0 \leq \phi \leq \pi/(2n-1)$. Since

$$z\,w'(z^2) = \frac{1}{2}\,[(n+1)nz + 2n(n-1)z^3 + 3(n-1)(n-2)z^5 + \cdots + n\cdot 2\cdot 1\cdot z^{2n-1}],$$

we have for $z = e^{i\phi}$ the relations

$$\Re\,[(1-z^2)w'(z^2)] = 2\sin\phi\,\Im\,[z\,w'(z^2)]$$

$$= \sin\phi\,\Im\,[(n+1)nz + 2n(n-1)z^3 + 3(n-1)(n-2)z^5 + \cdots + 2nz^{2n-1}]$$

$$= \sin\phi\,[(n+1)n\sin\phi + 2n(n-1)\sin 3\phi + 3(n-1)(n-2)\sin 5\phi + \cdots + 2n\sin(2n-1)\phi].$$

In the interval $[0, \pi/(2n-1)]$, each term in the last expression is nonnegative. Hence $h(\phi) \geq 0$ for $0 \leq \phi \leq \pi/(2n-1)$.

Next, let $\pi/(2n-1) < \phi \leq \pi/2$. From the identities

$$\sin(2n+3)\phi - \sin(2n+1)\phi = 2\cos(2n+2)\phi\sin\phi,$$

$$\sin(2n+1)\phi = \sin(2n+2)\phi\cos\phi - \cos(2n+2)\phi\sin\phi,$$

we obtain the equations

$$h(\phi)\sin\phi = (4n+3)\sin\phi + (2n^2+2n)\sin^3\phi + n\sin(2n+3)\phi - (n+3)\sin(2n+1)\phi$$

$$= (4n+3)\sin\phi + (2n^2+2n)\sin^3\phi + (2n+3)\cos(2n+2)\phi\sin\phi - 3\sin(2n+2)\phi\cos\phi$$

$$> (4n+3)\sin\phi + 0 - [(2n+3)^2\sin^2\phi + 9\cos^2\phi]^{1/2}.$$

Thus, $h(\phi) > 0$ for $\pi/(2n-1) < \phi \leq \pi/2$ provided

$$(4n+3)^2\sin^2\phi \geq (2n+3)^2\sin^2\phi + 9\cos^2\phi,$$

that is,

$$\tan^2\phi \geq \frac{3}{4n(n+1)}.$$

In $(\pi/(2n-1), \pi/2]$,

$$\tan^2\phi > \phi^2 \geq \left(\frac{\pi}{2n-1}\right)^2 > \frac{3}{4n(n+1)} \qquad (n = 1, 2, \cdots);$$

consequently, $h(\phi) \geq 0$ for all ϕ, and thus

$$\Re[(1-z)w'(z)] \geq 0 \qquad \text{on } |z| = 1.$$

Because $\Re[(1-z)w'(z)]$ is a harmonic function for $|z| \leq 1$, it assumes its minimum value for $|z| \leq 1$ on the boundary $|z| = 1$. This completes the proof of Theorem 1.

Proof of Theorem 2. Since $\Delta^{(0)}a_n \geq 0$ and $\Delta^{(1)}a_n \geq 0$, the sequence $\{a_n\}$ converges to some nonnegative limit a. For $|z| = r < 1$, the representation

$$f(z) = \sum_{n=0}^{\infty} \Delta^{(3)}a_n \cdot s_n^{(2)}(z) + \frac{a}{1-z} = \sum_0^{\infty} a_n z^n$$

shows that

$$\Re[(1-z)f'(z)] = \sum_{n=0}^{\infty} \Delta^{(3)}a_n \Re\left[(1-z)\frac{d}{dz}s_n^{(2)}(z)\right] + a\Re\left(\frac{1}{1-z}\right)$$

$$\geq 0 + \frac{a}{1+r} > \frac{a}{2} \geq 0.$$

It follows that $f(z)$ is univalent and close-to-convex for $|z| < 1$, relative to the convex function $\phi(z) = \log 1/(1-z)$. Since $F(z) = z\phi'(z) = z(1-z)^{-1}$ is also convex and starlike of order $1/2$ (that is, since $\Re\, zF'(z)/F(z) > 1/2$ for $|z| < 1$), we see that

$$\Re[(1-z)f'(z)] = \Re\left[\frac{zf'(z)}{F(z)}\right] > a/2 \qquad (|z| < 1).$$

Thus, using R. J. Libera's definition of order and type of a close-to-convex function (see [4]), we can say that the function $\frac{f(z) - a_0}{a_1}$ is close-to-convex of order $\frac{a}{2a_1}$ and type $\frac{1}{2}$.

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