SOME n-DIMENSIONAL MANIFOLDS THAT HAVE THE SAME FUNDAMENTAL GROUP

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The formula

$$\begin{aligned} x_1 &\to x_1 \cos\theta + x_2 \sin\theta, \\ x_2 &\to -x_1 \sin\theta + x_2 \cos\theta, \\ x_3 &\to x_3, \\ &\cdots \\ x_n &\to x_n \end{aligned}$$

defines a rotation of n-dimensional euclidean space S about the (n - 2)-dimensional subspace $A = \{(x_1, \cdots, x_n) \mid x_1 = x_2 = 0\}$, which we shall denote by spin_θ. It maps the (n - 1)-dimensional half-space

$$H_\theta = \{(x_1, \cdots, x_n) \mid x_1 = \rho \cos\theta,\ x_2 = \rho \sin\theta,\ \rho \geq 0\}$$

onto the (n - 1)-dimensional half-space $H_0 = \{(x_1, \cdots, x_n) \mid x_1 \geq 0,\ x_2 = 0\}$. The point at infinity is supposed to be included, so that S and A are spheres and each H_θ is a cell whose boundary ∂H_θ is A. An (n - 2)-dimensional sphere L in the finite part of S will be called a *deform-spun sphere* if $L \cap A$ is an (n - 4)-dimensional sphere and if for each θ the intersection of L and H_θ is an (n - 3)-dimensional cell bounded by $L \cap A$. The deformation referred to is the closed isotopical deformation $K_\theta = \text{spin}_\theta\, L \cap H_\theta$ $(0 \leq \theta \leq 2\pi)$ of K_0 in H_0. (During this deformation, the boundary $\partial K_0 = L \cap A$ remains fixed.) The *spun sphere* defined by Artin [1] in 1925 is, of course, the deform-spun sphere whose deformation is the stationary deformation $K_\theta = K_0$. If the deformation K_θ is stationary outside some (n - 1)-dimensional cell C whose boundary ∂C intersects K_0 at diametrically opposite points p, q of ∂C and may be described topologically inside C as the rotation of C about its axis $\overline{pq}$ through the angle $q\theta$, then the deform-spun sphere $L = L_q$ is called a q-*twist-spun sphere.* (The rotation of S is the *spin,* and the rotation of C is the *twist.*) In another paper [3], I have shown that there exist deform-spun spheres that are not twist-spun spheres.

The ν-fold cyclic covering of S branched over L_q is a closed orientable n-dimensional manifold $\Sigma = \Sigma_{\nu,q}(K_0)$. The part of Σ that lies over L is an (n - 2)-dimensional sphere Λ.

THEOREM. *The fundamental group* $\pi(\Sigma_{\nu,q})$ *of* $\Sigma_{\nu,q}$ *depends (for given* K_0*) only on the greatest common divisor* d *of* ν *and* q. *In particular,* $\Sigma_{\nu,q}$ *is simply connected whenever* d = 1.

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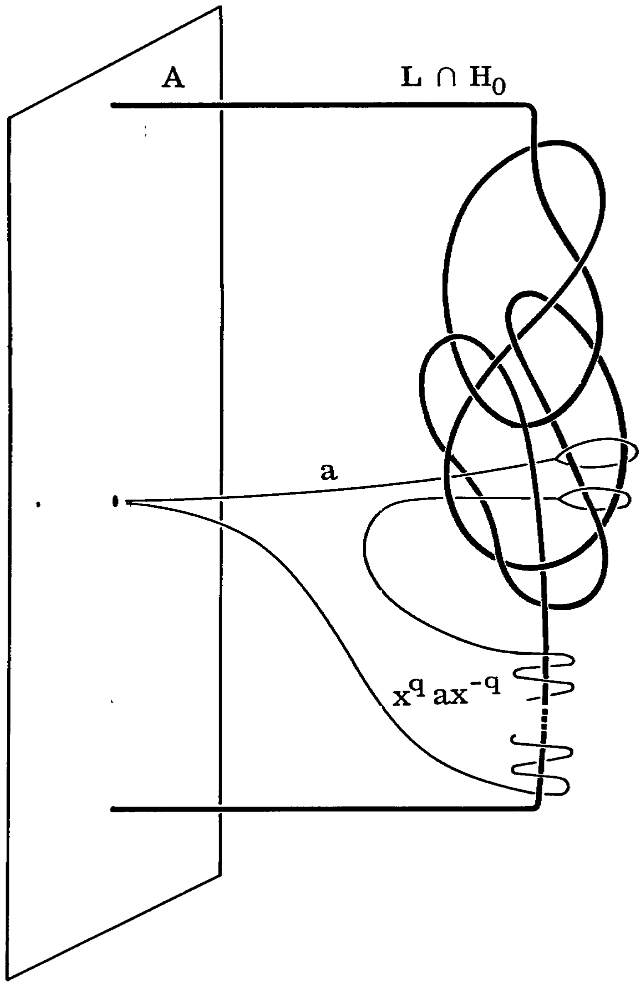

A surprising consequence is that *the ν-fold cyclic covering $\Sigma_{\nu,q}$ of the q-twist spin of K_0 has the same group as the q-fold cyclic covering $\Sigma_{q,\nu}$ of the ν-twist spin of K_0.* This suggests that, in fact, $\Sigma_{\nu,q}$ and $\Sigma_{q,\nu}$ might even be homeomorphic, but I don't know whether this is true.

Proof. Let $(x, a_1, \cdots, a_n: r_1, \cdots, r_n)$ be a canonical presentation of the group $G_0 = \pi(H_0 - K_0)$ of the knotted (n - 3)-dimensional cell K_0 in the (n - 1)-dimensional half-space H_0. Then x represents a meridian of K_0, which we take to lie outside C, and the elements a_j represent generators of the commutator subgroup G_0' of G_0. We shall need the explicit form

$$\prod_k x^{\lambda(i,k)} a_{j(i,k)}^{\varepsilon(i,k)} x^{-\lambda(i,k)}$$

of r_i.

We can obtain the group $G = \pi(S - L)$ of the q-twist-spun sphere L from G_0 by adjoining the relations $s_j = 1$ $(j = 1, \cdots, n)$, where $s_j = x^q a_j x^{-q} \cdot a_j^{-1}$ [see the figure].

The group $\Gamma = \pi(\Sigma - \Lambda)$ of the ν-fold unbranched cyclic covering $\Sigma - \Lambda$ of S - L is a normal subgroup of G, and G/Γ is cyclic of order ν. The elements $1, x, \cdots, x^{\nu-1}$ form a Schreier system of representatives, and the corresponding nontrivial generators are $\xi = x^\nu$ and

$$a_{j\beta} = x^\beta a_j x^{-\beta} \qquad (j = 1, \cdots, n;\ \beta = 0, \cdots, \nu - 1).$$

The corresponding presentation of Γ is $(\xi, \{a_{j\beta}\}: \{r_{i\alpha}\}, \{s_{j\beta}\})$, where

$$r_{i\alpha} = \prod_k a_{j(i,k)\lambda(i,k)}^{\varepsilon(i,k)} \qquad (i = 1, \cdots, m;\ \alpha = 0, \cdots, \nu - 1),$$

$$s_{j\beta} = a_{j\beta+q} a_{jq}^{-1} \qquad (j = 1, \cdots, n;\ \beta = 0, \cdots, \nu - 1).$$

From the relations $s_{j\beta} = 1$ we deduce that $a_{j\beta} = a_{j\beta+d} = a_{j\beta+2d} = \cdots$. Thus

$$\Gamma = |\xi, \{a_{j0}, a_{j1}, \cdots, a_{jd-1}\}: \bar{r}_{i\alpha}|,$$

where

$$\bar{r}_{i\alpha} = \prod_k a_{j(i,k)\bar{\lambda}(i,k)}^{\varepsilon(i,k)},$$

$\bar{\lambda}$ denoting the residue class of λ modulo d.

We can obtain the group $\pi(\Sigma)$ from Γ by adjoining the relation $\xi = 1$, and hence

$$\pi(\Sigma) = |\{a_{j0}, \cdots, a_{jd-1}\}: \{\bar{r}_{i\alpha}\}|.$$

Clearly, this presentation depends only on the presentation $(x, \{a_j\}: \{r_i\})$ of G_0 and the integer $d = (\nu, q)$.

When $d = 1$, the presentation of $\pi(\Sigma)$ just obtained simplifies to

$$\pi(\Sigma) = |a_{10}, \cdots, a_{n0}: \tilde{r}_1, \cdots, \tilde{r}_m|,$$

where

$$\tilde{r}_i = \prod_k a_{j(i,k)0}^{\varepsilon(i,k)}.$$

But $\pi(H_0)$ is obtained from the group G_0 by adjoining the relation $x = 1$, and hence

$$\pi(H_0) = |a_1, \cdots, a_n: \bar{r}_1, \cdots, \bar{r}_m|,$$

where

$$\bar{r}_i = \prod_k a_{j(i,k)}^{\varepsilon(i,k)}.$$

An isomorphism of $\pi(\Sigma)$ onto $\pi(H_0)$ is therefore determined by the replacement of a_{j0} with a_j $(j = 1, \cdots, n)$. Since H_0 is simply connected, it follows that Σ must also be simply connected. ■

The manifolds Σ have been studied by Giffen [2]. Perhaps the most relevant of his results is that $\Sigma_{\nu,q}$ is always a homotopy sphere if $\nu \equiv \pm 1 \pmod{q}$.

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