MINIMUM CONVEXITY OF A HOLOMORPHIC FUNCTION

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1. Let $w = f(z)$ be a holomorphic function defined in the open unit disc D. An *arc at* 1 is a curve $A \subset D$ such that $A \cup \{1\}$ is a Jordan arc. We say that f has an *asymptotic value at* 1 provided there exists an arc A at 1 on which f has a finite or infinite limit at 1. Let A be an arc at 1, parametrized by $z(t)$ $(0 \leq t < 1)$, and define a family $\mathscr{H}_A$ as follows: $H \in \mathscr{H}_A$ if and only if H is a closed half-plane in the finite w-plane W and there exists t_0 $(0 \leq t_0 < 1)$ such that $f(z(t)) \in H$ if $t_0 \leq t < 1$. If $H_A = \emptyset$, we set $F_A = W$; otherwise, we set $F_A = \bigcap H$, where the intersection is taken over all $H \in \mathscr{H}_A$. (The set F_A was essentially defined by K. Knopp; see [1, p. 113].)

THEOREM 1. *Either* f *has an asymptotic value at* 1, *or there exists an arc* α *at* 1 *such that* $F_\alpha \subset F_A$ *for each arc* A *at* 1.

Remark. If f is bounded on an arc A at 1, then F_A is the convex hull of the cluster set of f on A at 1.

LEMMA 1. *Suppose that* f *does not have an asymptotic value at* 1. *Suppose that* L_0 *is a straight line in* W *such that* L_0 *does not contain the projection of any branch point of the Riemann surface* $\mathscr{S}$ *onto which* f *maps* D, *and such that* $f(A_0) \cap L_0 = \emptyset$ *for some arc* A_0 *at* 1. *Let* H_0 *be the closed half-plane that is bounded by* L_0 *and contains* $f(A_0)$. *Let* A *be an arbitrary arc at* 1, *and let* S *be the smallest connected subset of* L_0 *that contains* $f(A) \cap L_0$. *Then there exists an arc* A' *at* 1 *such that*

$$f(A') \subset (f(A) \cap H_0) \cup S.$$

Proof. Let J be a Jordan curve such that $1 \in J$, $J \subset D \cup \{1\}$, and the interior domain Δ of J contains A_0 and A. Since f has no asymptotic value at 1, each component of the set

$$\Lambda = \{z: z \in \Delta,\ f(z) \in L_0\}$$

is a crosscut of Δ neither endpoint of which is 1. If $\Lambda = \emptyset$, then $f(\Delta) \subset H_0 - L_0$, and in this case we let $A' = A$.

Suppose that $\Lambda \neq \emptyset$. If there exists a sequence $\{c_k\}$ of components of Λ such that, for each k, c_k lies in the component of $\Delta - c_{k+1}$ bounded away from 1 (that is, c_{k+1} separates c_k from 1), then $A_0 \cap c_k \neq \emptyset$ for large k, contrary to the assumption that $f(A_0) \cap L_0 = \emptyset$. Thus some component λ of Λ is not separated from 1 by any other component of Λ. Let U be the component of $\Delta - \Lambda$ that has λ on its boundary and is not bounded away from 1. Any arc at 1 that is contained in Δ intersects U. In particular, A_0 intersects U, and consequently $f(U) \subset H_0 - L_0$. Let A" be an arc at 1 such that $A'' \subset A$ and the initial point of A" is in U. If $A'' \subset U$, let $A' = A''$. Otherwise, let γ_i $(i = 1, 2, \cdots)$ be the finitely or infinitely many components of Λ that are on the boundary of U and intersect A". Note that if there are infinitely many γ_i, the diameter of γ_i tends to 0 as $i \to \infty$. For each i, let γ_i' be

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the (possibly degenerate) closed subarc of γ_i such that the endpoints of γ_i' are on A'' and $A'' \cap \gamma_i \subset \gamma_i'$. It is easy to see that there exists an arc A' at 1 such that

$$A' \subset (A'' \cap U) \cup \left(\bigcup \gamma_i'\right).$$

Note that f is one-to-one on γ_i' and that under f the endpoints of γ_i' correspond to points in S. Thus each rectilinear segment $f(\gamma_i')$ is contained in S, and the proof of Lemma 1 is complete.

Proof of Theorem 1. Suppose that f has no asymptotic value at 1. Define a family $\mathscr{H}$ as follows: $H \in \mathscr{H}$ if and only if H is a closed half-plane in W and there exists an arc A at 1 such that $f(A) \subset H$. We need only consider the case where $\mathscr{H} \neq \emptyset$. Choose a sequence $\{H_n\}$ of closed half-planes in W such that for each n the interior H_n^0 of H_n contains some $H \in \mathscr{H}$, such that the boundary of H_n does not contain the projection of any branch point of $\mathscr{S}$, and such that $\bigcap H_n = \bigcap H$, where the last intersection is taken over all $H \in \mathscr{H}$. Let

$$V_n = \bigcap_{j=1}^{n} H_j^0.$$

We define inductively a sequence $\{A_n\}$ of arcs at 1 such that $f(A_n) \subset V_n$. Let A_1 be an arc at 1 such that $f(A_1) \subset V_1$. Suppose that A_{n-1} is an arc at 1 such that $f(A_{n-1}) \subset V_{n-1}$ $(n > 1)$. Choose an $H \in \mathscr{H}$ such that $H \subset H_n^0$. Let L_0 be a straight line in $H_n^0 - H$ that does not contain the projection of any branch point of $\mathscr{S}$, and let A_0 be an arc at 1 such that $f(A_0) \subset H$. Applying Lemma 1 (with $A = A_{n-1}$), we see that there exists an arc A_n at 1 such that

$$f(A_n) \subset (f(A_{n-1}) \cap H_n^0) \cup S,$$

where S is the smallest connected subset of L_0 that contains $f(A_{n-1}) \cap L_0$. Clearly, $S \subset V_{n-1} \cap H_n^0 = V_n$, and we see that $f(A_n) \subset V_n$. Let U_n be the component of $f^{-1}(V_n)$ (that is, of the set $\{z: f(z) \in V_n\}$) that contains A_n.

We prove that $U_{n+1} \subset U_n$ $(n > 1)$. Since $V_{n+1} \subset V_n$, it suffices to prove that for each n, $U_n \cap U_{n+1} \neq \emptyset$. Suppose that for some n, $U_n \cap U_{n+1} = \emptyset$. Join the initial points of A_n and A_{n+1} with a Jordan arc γ that lies (except for its endpoints) in $D - (A_n \cup A_{n+1})$, and let J denote the Jordan curve composed of γ, A_n, and A_{n+1}. Let Δ denote the interior domain of J, and let λ be an arc at 1 that is contained in Δ and in the boundary of U_n. By considering the simple nature of the boundary of V_n, we see that if V_n is unbounded, then f is one-to-one on λ and consequently tends to a limit on λ at 1. Hence V_n is bounded. Again, since f does not have a limit on λ at 1, there exists $w_0 \in W$ and a sequence $\{z_k\} \subset \lambda$ such that $z_k \to 1$ and $f(z_k) = w_0$. Let L be a half-line that begins at w_0, does not intersect V_n, and does not contain the projection of any branch point of $\mathscr{S}$. For each k, let β_k denote the component of $f^{-1}(L)$ that contains z_k. Then $\beta_k \cap \beta_{k'} = \emptyset$ if $z_k \neq z_{k'}$. By routine arguments, only finitely many β_k can intersect γ. Hence some β_k tends to 1, and on this β_k, f tends to a limit at 1. Since this contradicts the hypothesis of the theorem, we see that $U_{n+1} \subset U_n$.

Now let α be an arc at 1, parametrized by $z(t)$ $(0 \leq t < 1)$, and with the property that for each n there exists t_0 such that $z(t) \in U_n$ if $t_0 \leq t < 1$. Then $F_\alpha \subset H_n$ $(n \geq 1)$, and consequently $F_\alpha \subset \bigcap H$, where the intersection is taken over all

$H \in \mathscr{H}$. Clearly, $F_\alpha \subset F_A$ for each arc A at 1, and the proof of Theorem 1 is complete.

2. A simple curve in D parametrized by $z(t)$ $(0 \leq t < 1)$ is an *asymptotic path* of f for the (finite or infinite) value a provided $|z(t)| \to 1$ and $f(z(t)) \to a$ as $t \to 1$. The *end* of an asymptotic path α is $\overline{\alpha} \cap C$, where the bar denotes closure and C is the unit circle. Define the class $\mathscr{A}_p$ as follows: $f \in \mathscr{A}_p$ if and only if f is a nonconstant holomorphic function defined in D, f has an asymptotic value at each point of a set that is dense on C, and the end of each asymptotic path of f consists of a single point of C.

Suppose that $f \in \mathscr{A}_p$, let $\mathscr{S}$ be the Riemann surface onto which f maps D, and define the families $\mathscr{L}$ and $\mathscr{L}_0$ as follows: $L \in \mathscr{L}$ if and only if L is a straight line in W that does not contain the projection of any branch point of $\mathscr{S}$. $L \in \mathscr{L}_0$ if and only if $L \in \mathscr{L}$ and there exists a sequence $\{c_n\}$ of components of $f^{-1}(L)$ (that is, of the set $\{z: f(z) \in L\}$) such that $c_n \to 1$ in the following sense: each c_n is a crosscut of D that joins a point of $\{e^{i\theta}: 0 < \theta < \pi/2\}$ to a point of $\{e^{i\theta}: -\pi/2 < \theta < 0\}$, and the diameter of c_n tends to 0 as $n \to \infty$. The following theorem contains an earlier theorem of the author [2, Theorem 1].

THEOREM 2. *Suppose that* $f \in \mathscr{A}_p$. *Then either* f *has an asymptotic value at* 1, *or there exists an arc* α *at* 1 *such that the following two statements hold:*

(I) $F_\alpha \subset F_A$ *for each arc* A *at* 1.

(II) *If* $L \in \mathscr{L}$ *and both components of* $W - L$ *intersect* F_α, *then* $L \in \mathscr{L}_0$.

LEMMA 2. *Suppose that* $f \in \mathscr{A}_p$ *and* $L \in \mathscr{L}$. *Then, for each positive number* ε, *the diameters of at most finitely many components of* $f^{-1}(L)$ *are greater than* ε.

Proof. This lemma was proved in [2]; here we sketch a simpler proof. Suppose that the conclusion is false. Then there exist an arc $\gamma \subset C$ and a sequence $\{\gamma_n\}$ of pairwise disjoint Jordan arcs such that $\gamma_n \subset f^{-1}(L)$ and $\gamma_n \to \gamma$ in such a way that each arc at a point of the interior γ^0 of γ intersects all but finitely many γ_n. We shall repeatedly use the fact that f is one-to-one on each γ_n. If f had the asymptotic value ∞ at a point $\zeta \in \gamma^0$, then one component of $\gamma - \{\zeta\}$ would be contained in the end of an asymptotic path of f for the value ∞, contrary to the assumption that $f \in \mathscr{A}_p$. Thus there exist distinct points $\zeta_j \in \gamma^0$ $(j = 1, 2)$ such that at ζ_j, f has a finite asymptotic value a_j. If $a_1 = a_2$, then the subarc of γ joining ζ_1 and ζ_2 is contained in the end of an asymptotic path of f for the value a. Thus $a_1 \neq a_2$. We can construct a Jordan curve J in D that is partitioned into four (closed) subarcs Γ_j $(j = 1, 2, 3, 4)$ such that $\Gamma_1 \cap \Gamma_3 = \emptyset$, $\Gamma_1 \cup \Gamma_3 \subset f^{-1}(L)$, and

$$f(\Gamma_{2j}) \subset \{w: |w - a_j| < |a_1 - a_2|/3\} \qquad (j = 1, 2).$$

There exists $z_0 \in \Gamma_1$ such that $f(z_0) = (a_1 + a_2)/2$. Let L' be a straight line through $f(z_0)$ that is distinct from L, does not intersect $f(\Gamma_2) \cup f(\Gamma_4)$, and does not contain the projection of any branch point of $\mathscr{S}$. The component of $f^{-1}(L')$ that contains z_0 crosses Γ_1 but does not intersect $J - \{z_0\}$. Since its ends must tend to C, this is impossible. The proof of Lemma 2 is complete.

Proof of Theorem 2. Suppose that f does not have an asymptotic value at 1. By Theorem 1, there exists an arc α at 1 such that (I) holds. Suppose that $L \in \mathscr{L} - \mathscr{L}_0$. Applying Lemma 2 and the argument used to find U in the proof of Lemma 1, we see that the point 1 is (curvilinearly) accessible through $D - f^{-1}(L)$, and consequently F_α intersects at most one component of $W - L$. Thus (II) holds, and the proof of Theorem 2 is complete.

Remark. If $f \in \mathcal{A}_p$ *and* f *does not have an asymptotic value at* 1, *then for each arc* α *at* 1, *the statements* (I) *and* (II) *are equivalent.* In the proof of Theorem 2, we saw that (I) implies (II). We sketch the proof that (II) implies (I). Let H_0 be a closed half-plane such that for some arc A at 1, $f(A) \subset H_0$. Choose any $L \in \mathcal{L}$ that does not intersect H_0, and let H be the closed half-plane bounded by L such that $H_0 \subset H$. If F_α does not intersect H, we can use the argument used to construct A' in the proof of Lemma 1 to construct an arc at 1 on which f tends to ∞. Thus F_α intersects H. If $F_\alpha \not\subset H$, we choose $L' \in \mathcal{L}$ such that $L' \cap H = \emptyset$ and both components of W - L' intersect F_α. By (II), $L' \in \mathcal{L}_0$, and this is inconsistent with the relation $f(A) \cap L' = \emptyset$. Hence $F_\alpha \subset H$ for each such H, and $F_\alpha \subset H_0$. Thus (II) implies (I).

REFERENCES

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