LENGTH DISTORTION OF CURVES UNDER CONFORMAL MAPPINGS

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1. INTRODUCTION

Let α denote the open upper half of the unit circle C: $|z| = 1$, let α^* denote the open real diameter of the unit disk K: $|z| < 1$, and let us consider a conformal mapping f of K onto a simply connected domain D in the finite plane. The image $\beta^* = f\alpha^*$ of α^* is a locally rectifiable curve with length $\|\beta^*\| \leq \infty$, and α corresponds to a "curve" β on the boundary of D to which we can assign a "length" $\|\beta\| \leq \infty$.

An unpublished but widely circulated conjecture by Piranian states that there exists a *finite constant* A such that $\|\beta^*\| \leq A \cdot \|\beta\|$, and that the *best possible value* A_0 of the constant is π. Gehring and Hayman [1, Theorem 1] proved the first part of the conjecture, and they showed that $\pi \leq A_0 < 74$.

In Section 7, we disprove the second part of Piranian's conjecture: by means of an example, we show that $A_0 \geq 4.5$. In Section 6, we reduce the upper estimate of A_0 to 17.5. Since the first part of the conjecture can not be extended to quasiconformal mappings, the proof for the upper estimate must involve conformality in an essential way; indeed, we use the distortion $|dw|/|dz| = |f'(z^*)|$ under the mapping $w = f(z)$ at the points z^* of α^* in order to get the length $\|\beta^*\|$.

We shall consider all circular arcs α^* in K on which the harmonic measure of α has the constant value ω; the original problem is the special case $\omega = 1/2$. In Sections 3 to 5, we give certain lower estimates for the length of β; they depend on the harmonic measure ω of β at an interior point $w^* = f(z^*)$ of D, and they are either proportional to the distance of w^* from β or proportional to the distortion $|f'(z^*)|$ at the point z^*.

In Section 2 we discuss "curves" on the boundary of an arbitrary simply connected domain. Without this generality, we should repeatedly be forced to put clumsy restrictions on the domains D and on the conformal mappings f to be admitted.

2. CURVES ON THE BOUNDARY

We consider a simply connected domain D in the extended complex plane, the *abstract boundary* ∂D consisting of the *prime ends* $\mathscr{W}$, and the *projections* $W = p\mathscr{W}$ into the plane. We define a *semidistance* $\rho(\mathscr{W}_1, \mathscr{W}_2)$ for prime ends as the infimum of those constants r such that for each point of $p\mathscr{W}_1$ and of $p\mathscr{W}_2$ there is a point of the other set within euclidean distance r.

If f denotes a conformal mapping of K onto D, we shall use the same symbol f for the mapping that carries the points of C onto the corresponding prime ends of D. The induced *cyclic ordering* of ∂D allows us to speak of *intervals* on the abstract boundary. We regard an open interval β on ∂D as a generalized *curve*, because it is the image $f\alpha$ under f of an interval α on the unit circle C.

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Since we have an ordering on C and a semidistance on ∂D, the *total variation* $V(f \mid \alpha)$ is the supremum of all sums $\sum_m \rho(f(z_m), f(z_{m-1}))$ taken for ordering monotone systems $(z_0, z_1, \cdots, z_n)$ on α. For a curve β on ∂D we define the generalized *length* as

$$\|\beta\| = V(f \mid \alpha).$$

If there is a nondegenerate prime end $\mathcal{W}_0$ in β, then $p\partial D$ oscillates nearby and thus accumulates infinite length for β.

For β to have finite length it is necessary but not sufficient that all its prime ends degenerate to points; the mapping pf on α is then the continuous continuation of the mapping f on K, and $\|\beta\|$ is the customary length of the path of the point pf(z) moving through the point set $pf\alpha$.

3. LENGTH AND DISTANCE OF A CURVE ON THE BOUNDARY

In this section, we state modified versions of three known results, and we introduce our first theorem.

LEMMA 1 (Fekete; Pommerenke [4, Satz 3]). *Let γ denote a curve in the finite plane whose projection $p\gamma$ is a compact point set. Let* D *denote the unbounded component of its complement, and let* f *denote a conformal mapping of* U: $|z| > 1$ *onto* D *with* $f(\infty) = \infty$. *Then the length of γ satisfies the inequality*

$$\|\gamma\| \geq 4 \cdot |f'(\infty)|;$$

the constant 4 *is best possible for the class of configurations.*

(Fekete's proof appeared in a rather inaccessible paper, and the author has not been able to find an exact reference.)

LEMMA 2 (Löwner [2, Satz VIII]). *Let* z^* *denote a point in the exterior* U: $|z| > 1$ *of the unit circle* C, *and let* f *denote a conformal mapping of* U *with* $f(\infty) = \infty$. *Then the distance between point and boundary in the image satisfies the inequality*

$$\text{dist}(f(z^*), fC) \leq |z^*| \cdot |f'(\infty)|;$$

the inequality is best possible for the class of mappings.

A conformal mapping f_0 of U with $f_0(\infty) = \infty$, $f_0(z^*) = 0$, and such that pf_0 maps C into C, is an example for equality. The proof of the inequality is an application of the maximum principle to the function $ff_0^{-1}(w)/w$.

LEMMA 3 (Milloux Problem; Nevanlinna [3]). *Let* D_0 *denote a simply connected domain lying outside of the unit circle* C *in the finite plane. Let* α *denote an open interval on the abstract boundary* ∂D_0 *whose projection* $p\alpha$ *lies on* C. *Then the harmonic measure* ω *of* α *at a point* z^* *in* D_0 *satisfies the inequality*

$$\tan^2 \frac{\pi\omega}{4} \leq \frac{1}{|z^*|};$$

the inequality is best possible for the class of configurations.

We shall now use the lemmas to obtain our first result.

THEOREM 1. *Let* β *denote an open interval on the abstract boundary of a simply connected domain* D *in the finite plane, and let* ω *denote the harmonic measure of* β *at a point* w^* *in* D. *For the class of these configurations, there exists a positive constant* $F(\omega)$ *such that the length of* β *satisfies the inequality*

$$\|\beta\| \geq F(\omega) \cdot \operatorname{dist}(w^*, \beta).$$

The best possible value $F_0(\omega)$ *of the constant satisfies the condition*

$$4 \cdot \tan^2 \frac{\pi\omega}{4} \leq F_0(\omega) \leq 8 \cdot \tan^{-1}\left(\tan^2 \frac{\pi\omega}{4}\right).$$

In the proof of the lower estimate for $F_0(\omega)$ we may assume β to be of finite length. The unbounded component $\tilde{D}$ of the complement of the closed set $\overline{p\beta}$ is a simply connected domain containing D. Let us take a conformal mapping f of U: $|z| > 1$ onto $\tilde{D}$ with $f(\infty) = \infty$, and put $D_0 = f^{-1} D$, $\alpha = f^{-1}\beta$, and $z^* = f^{-1}(w^*)$. Since the point set $p\partial\tilde{D}$ is equal to $\overline{p\beta}$, we have the relations

$$\operatorname{dist}(w^*, \beta) = \operatorname{dist}(w^*, \partial\tilde{D}) = \operatorname{dist}(f(z^*), fC).$$

Combining this with the lemmas, we find that

$$\|\beta\| = \|\overline{\beta}\| \geq 4 \cdot |f'(\infty)| \geq 4 \cdot \frac{1}{|z^*|} \cdot \operatorname{dist}(f(z^*), fC) \geq 4 \cdot \tan^2 \frac{\pi\omega}{4} \cdot \operatorname{dist}(w^*, \beta).$$

The following example establishes the upper estimate for $F_0(\omega)$.

Example 1. For a constant a with $0 < a < \pi$, let D be the plane slit along the real axis from $-\infty$ to -1 and along the unit circle C from -1 to e^{ia} and to e^{-ia}, let β correspond to the inner edge of the slit on C without the endpoints, and let w^* be 0.

4. LENGTH ON THE BOUNDARY AND DISTORTION IN THE INTERIOR

THEOREM 2a. *Let* α *denote an open arc on the unit circle* C, *and let* ω *denote the harmonic measure of* α *at a point* z^* *in the unit disk* K. *For the class of conformal mappings* f *of* K *into the finite plane, there exists a positive constant* $E(z^*, \omega)$ *such that the length of* $f\alpha$ *satisfies the inequality*

$$\|f\alpha\| \geq E(z^*, \omega) \cdot |f'(z^*)|.$$

The best possible value $E_0(z^*, \omega)$ *of the constant satisfies the condition*

$$(1 - |z^*|^2) \cdot \tan^2 \frac{\pi\omega}{4} \leq E_0(z^*, \omega) \leq \frac{1}{2} \cdot (1 - |z^*|^2) \cdot \tan^2 \frac{\pi\omega}{2}.$$

Our result gives a relation between the length of the interval on the boundary after the mapping, and the *distortion* $|dw|/|dz| = |f'(z^*)|$ under the mapping *at the interior point* z^* of K. Since there is a conformal mapping f_0 of K onto K with $f_0(0) = z^*$ and $f_0'(0) = 1 - |z^*|^2$, we see that $E_0(z^*, \omega) = (1 - |z^*|^2) \cdot E_0(0, \omega)$.

The upper estimate for $E_0(0, \omega)$ is a consequence of the conformal mapping of the configuration (K, α, 0) onto the configuration in the following example.

Example 2. Let D be the plane slit along the real axis from $-\infty$ to 0 and along the imaginary axis from $-i$ to i, let β correspond to the right edge of the vertical slit without the endpoints, and let w^* be positive.

To prove the lower estimate for $E_0(0, \omega)$, we use Theorem 1 with $D = fK$, $\beta = f\alpha$, and $w^* = f(0)$, together with tne 1/4-Theorem of Koebe, and we get the inequalities

$$\|f\alpha\| = \|\beta\| \geq F_0(\omega)\cdot \mathrm{dist}(w^*, \beta) \geq F_0(\omega)\cdot \mathrm{dist}(w^*, \partial D)$$

$$= F_0(\omega)\cdot \mathrm{dist}(f(0), fC) \geq F_0(\omega)\cdot \frac{1}{4}\cdot |f'(z^*)| \geq \tan^2 \frac{\pi\omega}{4}\cdot |f'(z^*)| .$$

5. LOCAL LENGTH DISTORTION ON THE BOUNDARY

We now formulate Piranian's conjecture for a wider class of curves.

Generalized Piranian Problem. Let α denote an open arc on the unit circle C, and let α^* denote the circular arc in the unit disk K on which the harmonic measure of α has the constant value ω. For the class of conformal mappings f of K into the finite plane, does there exist a finite constant $A(\omega)$ such that the lengths of $f\alpha^*$ and $f\alpha$ satisfy the inequality

$$\|f\alpha^*\| \leq A(\omega)\cdot \|f\alpha\| \ ?$$

We use the notation $D = fK$, $\beta = f\alpha$, and $\beta^* = f\alpha^*$. In the case $\|\beta\| < \infty$ we intend to estimate

$$\|\beta^*\| = \|f\alpha^*\| = \int_{\alpha^*} |f'(z^*)|\cdot|dz^*| .$$

Therefore we need an upper estimate for the distortion $|f'(z^*)|$ at the points z^* of α^*. It seems natural to apply Theorem 2a: $|f'(z^*)| \leq \|\beta\|/E_0(z^*, \omega)$; but unfortunately the integral of $1/(1 - |z^*|^2)$ over α^* is infinite. For the conjecture to be true, it is necessary that $|f'(z^*)|$ be considerably smaller than Theorem 2a tells us.

We try to explain this situation in a second way. Let f_0 be the conformal mapping of (K, α, z^*) onto the following configuration.

Example 3. Let D_0 be the parallel strip $0 < x < 1$, let α_0 be the line $x = 1$, and put $z_0^* = \omega + i\eta$.

For the conformal mapping $\tilde{f} = ff_0^{-1}$ of D_0 onto D, the assertion of Theorem 2a has the following equivalent formulation

$$\|\tilde{f}\alpha_0\| \geq \tilde{E}_0(z_0^*, \omega)\cdot |\tilde{f}'(z_0^*)| ,$$

with

$$\tilde{E}_0(z_0^*, \omega) \geq \frac{2}{\pi}\cdot \sin \pi\omega \cdot \tan^2 \frac{\pi\omega}{4} .$$

The translational invariance of the configuration causes the constant to be independent of z_0^* in the sense that $\tilde{E}_0(z_0^*, \omega) = \tilde{E}_0(\omega, \omega)$. Now let us try to estimate

$$\|\beta^*\| = \|\tilde{f}f_0\,\alpha^*\| = \int_{-\infty}^{+\infty} |\tilde{f}'(\omega + i\eta)| \cdot d\eta .$$

The reformulated Theorem 2a gives for $|\tilde{f}'(\omega + i\eta)|$ the upper estimate $\|\beta\|/\tilde{E}_0(\omega, \omega)$, which is a constant and therefore useless for the integration along the infinite line. Instead of this, we should show that $|\tilde{f}'(\omega + i\eta)|$ goes to 0 rapidly as $\eta \to \pm\infty$.

Since the infinite line α_0 is mapped onto a curve β of finite length, the parts $\tilde{\alpha}$ of α_0 lying near ∞ must have very small *length distortion* $\|\tilde{f}\tilde{\alpha}\|/\|\tilde{\alpha}\|$. What remains to be done is a comparison of the distortion $|\tilde{f}'(z_0^*)|$ at an interior point z_0^* with the local length distortion on a nearby part $\tilde{\alpha}$ of the boundary. It turns out that the result we need is merely another conformally equivalent version of Theorem 2a.

THEOREM 2b. *Let* D_0 *denote the parallel strip* $0 < x < 1$, *let* α_0 *denote the line* $x = 1$ *on its boundary, and let* α *denote the open interval on* α_0 *between the points* $1 + i(\eta - k)$ *and* $1 + i(\eta + k)$. *The harmonic measure of* α *at the point* $z^* = x^* + i\eta$ *in* D_0 *has the value*

$$\omega = \frac{2}{\pi} \cdot \tan^{-1}\left(\tanh\frac{\pi k}{2} \cdot \tan\frac{\pi x^*}{2}\right).$$

For the class of conformal mappings f *of* D_0 *into the finite plane, there exists a positive constant* $B(x^*, k)$ *such that the length of* $f\alpha$ *satisfies the inequality*

$$\|f\alpha\| \geq B(x^*, k) \cdot |f'(z^*)| .$$

The best possible value $B_0(x^*, k)$ *of the constant satisfies the conditions*

$$B_0(x^*, k) \leq \sqrt{2} \cdot k$$

and

$$\frac{2}{\pi} \cdot \sin \pi x^* \cdot \tan^2\frac{\pi\omega}{4} \leq B_0(x^*, k) \leq \frac{1}{\pi} \cdot \sin \pi x^* \cdot \tan^2\frac{\pi\omega}{2}$$

The assertion $B_0(x^*, k) \leq 2k/\sqrt{2}$ means that for each choice of the point z^* and of the length 2k of the corresponding interval α on the line $x = 1$, there exists such a mapping f with the distortion $|f'(z^*)|$ at z^* being at least $\sqrt{2}$ times as large as the local length distortion $\|f\alpha\|/\|\alpha\|$ on the boundary. This follows from the conformal mapping onto the configuration of Example 2.

To prove the remaining assertions, we consider the conformal mapping f_0 of the unit disk K onto D_0 with $f_0(0) = z^*$. Explicit calculations show that $|f_0'(0)| = (2/\pi) \cdot \sin \pi x^*$ and that an arc of length $2\pi\omega$ is mapped onto α. We apply Theorem 2a to the mapping ff_0 and get the inequalities

$$\tan^2\frac{\pi\omega}{4} \leq E_0(0, \omega) \leq \frac{1}{2} \cdot \tan^2\frac{\pi\omega}{2}$$

and

$$\|f\alpha\| \geq E_0(0, \omega) \cdot \frac{2}{\pi} \cdot \sin \pi x^* \cdot |f'(z^*)| .$$

6. LENGTH OF A LEVEL CURVE OF HARMONIC MEASURE

Omitting the conformal mapping f of the configuration (K, α, α^*) mentioned in the Generalized Piranian Problem, we give our main result in a conformally invariant formulation.

THEOREM 3. *Let* β *denote an open interval on the abstract boundary of a simply connected domain* D *in the finite plane, and let* β^* *denote the level curve in* D *on which the harmonic measure of* β *has the constant value* ω. *For the class of these configurations, there exists a finite constant* $A(\omega)$ *such that the lengths of* β^* *and* β *satisfy the inequality*

$$\|\beta^*\| \leq A(\omega)\cdot\|\beta\| .$$

The best possible value $A_0(\omega)$ *of the constant satisfies the conditions*

$$A_0(\omega) > 1$$

and

$$\frac{1}{2\omega^2} \leq A_0(\omega) \leq \frac{6}{\omega^3};$$

for $\omega = \frac{1}{2}$, *we have the stronger inequalities*

$$4.56 \leq A_0\left(\frac{1}{2}\right) \leq 17.45 .$$

The proof of the lower estimate will be given in Section 7. To prove the upper estimate for $A_0(\omega_0)$, we use a conformal mapping f of the configuration (D_0, α_0) in Example 3 onto (D, β). Let us take the notation of Theorem 2b, and put $x^* = \omega_0$. At each point $z_0^* = \omega_0 + i\eta$ on the line α_0^*: $x = \omega_0$, the harmonic measure of the corresponding interval α has the value ω, and the harmonic measure of α_0 has the value ω_0. The variation of f on α_0 is a measure μ_y. Using Theorem 2b and Fubini's theorem, we get the relations

$$B_0(\omega_0, k)\cdot\|\beta^*\| = B_0(\omega_0, k)\cdot\|f\alpha^*\| = \int_{-\infty}^{+\infty} B_0(\omega_0, k)\cdot|f'(\omega_0 + i\eta)|\cdot d\eta$$

$$\leq \int_{-\infty}^{+\infty}\|f\alpha\|\cdot d\eta = \int_{-\infty}^{+\infty}\int_{\eta-k}^{\eta+k} d\mu_y\cdot d\eta = \int_{-\infty}^{+\infty}\int_{y-k}^{y+k} d\eta\cdot d\mu_y$$

$$= \int_{-\infty}^{+\infty} 2k\cdot d\mu_y = 2k\cdot\|f\alpha_0\| = 2k\cdot\|\beta\| .$$

Therefore $A_0(\omega_0) \leq 2k/B_0(\omega_0, k)$ for each $k > 0$. For the fairly good choices

$$k = \sqrt{3} \qquad \text{for } 0 < \omega_0 \leq \frac{1}{2}$$

and

$$k = \sqrt{3}\cdot\sin\pi\omega_0 \qquad \text{for } \frac{1}{2} < \omega_0 < 1,$$

laborious calculations lead to the upper estimates stated above.

The ratio of the interior distortion $|f'(\omega_0 + i\eta)|$ and the local length distortion $\|f\alpha\|/\|\alpha\|$ on the boundary can be large only for a set of values η that is small compared with α_0, but the application of Theorem 2b replaces this ratio with the upper estimate $2\sqrt{3}/B_0(\omega_0, \sqrt{3}) = O(\omega_0^{-3})$ for all η. This explains why our method yields only $A_0(\omega_0) = O(\omega_0^{-3})$ as $\omega_0 \to 0$ instead of the expected magnitude $O(\omega_0^{-2})$.

7. EXAMPLES AND CONJECTURES

We look for configurations (D, β, β^*) with large values of the ratio $\|\beta^*\|/\|\beta\|$ in Theorem 3. The following is a modified form of the example used by Gehring and Hayman [1, p. 354].

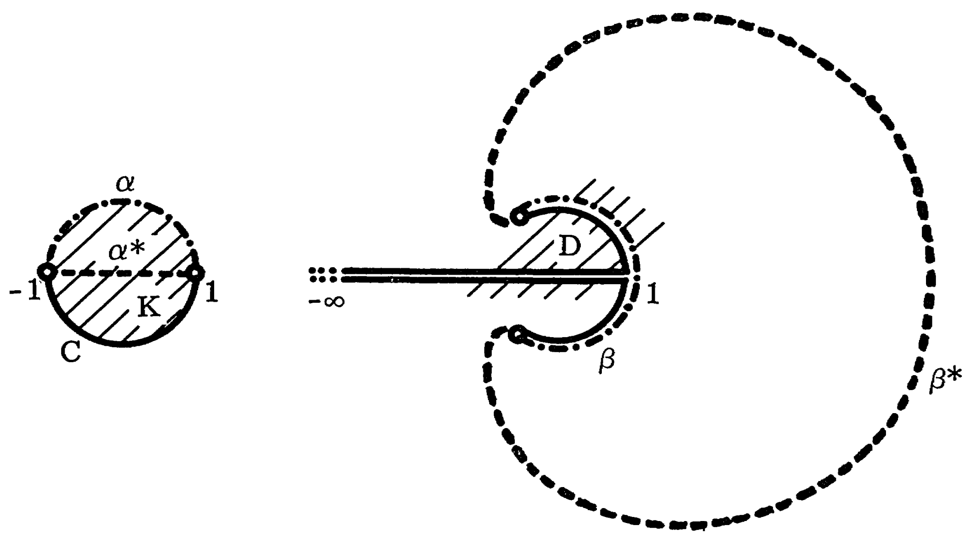
Example 4. Let D be the plane slit along the real axis from $-\infty$ to 1, and let β correspond to both edges of the slit from -1 to 1 without the endpoints at -1.

In the case $\omega = 1/2$, one finds that β^* is the unit circle without the point -1, and thus $\|\beta^*\|/\|\beta\| = \pi$. The conformal mapping $w = z + (z^2 - 1)^{1/2}$ leads to the following example.

Example 5. Let D be the exterior of the unit circle C slit along the real axis from $-\infty$ to -1, and let β correspond to C without the point -1.

A simple calculation gives the lower estimate of Theorem 3: $\|\beta^*\| \geq \|\beta\|/(2\omega^2)$. We omit a detailed discussion of this example, because it is merely a limiting case of the following.

Example 6. For a constant a with $0 < a < \pi$, let D be the plane slit along the real axis from $-\infty$ to 1 and along the unit circle C from 1 to e^{ia} and to e^{-ia}, and let β correspond to the outer edge of the slit on C without the endpoints (see the picture).



We have performed the following numerical computations on the *ZUSE* 23 *Elektronische Rechenanlage* at the University of Giessen in Germany. A certain sequence of linear, square, and square-root mappings leads from the unit disk K, half-circle α, and diameter α^* to the configuration of Example 6. Using N = 21 suitably spaced points of α^*, we obtained points of β^* and hence an approximation L(a) for $\|\beta^*\|$. Arc length was computed with a formula based on circular three-point-approximation, which has relative error $O(N^{-4})$ for the curves under consideration. With nine choices of a, we arrived at a value a_9 not more than 0.001 distant from a value where L(a)/a has a local maximum. Now, using N = 201 points of β^* for $a_9 = 2.21658\cdots$ ($\approx 127°$) we found that $\|\beta^*\| \geq 20.22447$ and thus $A_0(1/2) \geq \|\beta^*\|/(2a_9) \geq 4.562\cdots$. This completes the proof of Theorem 3.

It seems that the *extremal configuration* with $\|\beta^*\| = A_0(\omega)\cdot\|\beta\|$ should have the following properties:

(i) If we move along β, then the tangent turns away from D.

(ii) The complement of $\overline{D}$ is enclosed by $\overline{\beta}$.

If (i) is not satisfied, we pull a part of β inward and thus make β shorter, make D smaller, and push β^* away from the old β. If (ii) is not satisfied, we push a part of ∂D not belonging to β outward, and thus make D larger and pull β^* away from β. Since β^* moves away from β, we expect β^* to become longer; this would mean an increase of the ratio $\|\beta^*\|/\|\beta\|$.

We conjecture the extremal configuration for Theorem 3 to be qualitatively like Example 6; β will consist of a symmetrical pair of analytic arcs, starting with right angles at the endpoint of the half-line slit, with tangent turning toward the half-line, and curvature increasing. We estimate that $A_0(1/2)$ is near 5 and less than 2π; in general, the *extremal ratio* $\|\beta^*\|/\|\beta\| = A_0(\omega)$ will be $O(\omega^{-2})$ as $\omega \to 0$.

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