ON THE EXTENSION OF UNIFORMLY CONTINUOUS MAPPINGS

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The subject and the methods of this article belong to a system of ideas that originated some thirty years ago around the problems of extending Lipschitzian mappings (M. D. Kirszbraun [3], R. A. Valentine [12], [13], E. J. Mickle [4], I. J. Schoenberg [11]) and of imbedding a metric space isometrically into another (I. J. Schoenberg [6] to [10], J. von Neumann and I. J. Schoenberg [5]). To gain a proper perspective of the basic question, it is convenient to distinguish three successive levels of investigation:

Problem I. Under what conditions can a uniformly continuous mapping f: $\mathscr{S} \to \mathscr{T}$ of a metric space into another be extended from its domain of definition to the whole space $\mathscr{S}$ so that it remains uniformly continuous? (Throughout this paper, the notation f: $\mathscr{S} \to \mathscr{T}$ means that $\mathscr{D}(f) \subset \mathscr{S}$ and $\mathscr{R}(f) \subset \mathscr{T}$.)

The uniform continuity of a mapping f can be expressed by saying that its modulus of continuity $\delta_f(t)$, defined by

$$\delta_f(t) = \sup_{\|x_1 x_2\| \le t,\ x_1, x_2 \in \mathscr{D}(f)} \|f(x_1) f(x_2)\| \qquad \text{(bars denote distance in either space)}, \tag{1}$$

has the property $\lim_{t\to 0} \delta_f(t) = 0$. Clearly, $\delta_f(t)$ is the smallest function of a nonnegative variable such that

$$\|f(x_1) f(x_2)\| \le \delta_f(\|x_1 x_2\|). \tag{2}$$

In general, nothing can be said of $\delta_f(t)$ beyond the fact that it is a nonnegative, nondecreasing function of t ($0 \le t < \infty$); but if $\mathscr{D}(f)$ is metrically convex (that is, if for each pair $x_1, x_2 \in \mathscr{D}(f)$ and each choice of positive numbers s and t with $s + t = \|x_1 x_2\|$ there is a z such that $\|x_1 z\| = t$, $\|z x_2\| = s$), then $\delta_f(t)$ is in addition subadditive. Suppose now that $\mathscr{S}$ itself is metrically convex and that f admits an extension $\tilde{f}$ to $\mathscr{S}$. Then

$$\delta_f(t) \le \delta_{\tilde{f}}(t) \qquad (0 \le t < \infty), \tag{3}$$

and in particular $\delta_f(t)$ admits a subadditive majorant approaching zero as $t \to 0$. This is therefore a necessary condition in order that Problem I have a solution. It turns out that the condition is also sufficient if both $\mathscr{S}$ and $\mathscr{T}$ are Euclidean spaces (that is, complete real vector spaces with a scalar product). We remark that (3) is equivalent to

$$\|f(x_1) f(x_2)\| \le \delta_{\tilde{f}}(\|x_1 x_2\|) \qquad (x_1, x_2 \in \mathscr{D}(f)), \tag{4}$$

which in turn can be read as saying that $\tilde{f}$ is an extension of f to $\mathscr{S}$ that preserves (4). It is clear that we would know how to solve Problem I any time we can find for δ_f a majorant $\tilde{\delta}$ approaching zero as $t \to 0$ such that each mapping f satisfying $\|f(x_1) f(x_2)\| \le \tilde{\delta}(\|x_1 x_2\|)$ on $\mathscr{D}(f)$ can be extended to the whole space without violation of the condition. These considerations lead us naturally to the following question:

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Problem II. What are the functions $\delta(t)$ $(0 \leq t < \infty)$ such that each mapping f: $\mathscr{S} \to \mathscr{T}$ satisfying

$$\|f(x_1)f(x_2)\| \leq \delta(\|x_1 x_2\|) \qquad (x_1, x_2 \in \mathscr{D}(f)) \tag{5}$$

admits an extension to $\mathscr{S}$ that preserves (5)?

For $\delta(t) = t$, (5) becomes a Lipschitz condition, and the problem of finding out whether the function belongs to the class in question coincides with the problem, originally treated by Kirszbraun [3], of extending Lipschitz functions to the whole space without violating the individual Lipschitz condition they satisfy. Kirszbraun proved that if both $\mathscr{S}$ and $\mathscr{T}$ are Euclidean spaces, such an extension is always possible, a conclusion also reached independently a few years later by Valentine [12], [13]. It was also Valentine who first posed Problem II for real-valued functions, and who solved it by showing that each nonnegative, nondecreasing, subadditive $\delta(t)$ is a solution [12, p. 107]. N. Aronszajn and P. Panitchpakdi [1] have shown that this result characterizes metrically convex spaces $\mathscr{T}$ having the property that the intersection of any family of pairwise intersecting balls is not empty, in the sense that only for such spaces the solution of Problem II is the class of all nondecreasing subadditive functions δ regardless of the metric space $\mathscr{S}$. Among Euclidean spaces, only those of dimension one have this property. The problem in the more general context considered here was formulated by Mickle, who gave the question a new turn by relating it technically and conceptually to the following problem.

Problem III. For which functions $\delta(t)$ $(0 \leq t < \infty)$ can the space $\mathscr{S}$, remetrized with the distance $\delta(\|x_1 x_2\|)$, be imbedded isometrically into $\mathscr{T}$?

This problem is equivalent (at least if $\mathscr{T}$ is Euclidean) to the problem of extending a mapping f: $\mathscr{S} \to \mathscr{T}$ under preservation of the equality $\|f(x_1)f(x_2)\| = \delta(\|x_1 x_2\|)$. It is the subject of a series of papers by Schoenberg ([6] to [10]) and by von Neumann and Schoenberg [5], who, in a theory of classical elegance solved the question entirely for Euclidean spaces by tying it to the theory of positive definite and completely monotonic functions. Let us assume from now on that $\mathscr{T}$ is a Euclidean space, and, as usual, let us write $\|y_1 - y_2\|$ in place of $\|y_1 y_2\|$; a Euclidean space of dimension m shall be denoted by $\mathscr{H}_m$. By Schoenberg's results, δ is a solution of Problem III if and only if

$$\sum_{i,j \neq 0} [\delta^2(\|x_i x_0\|) + \delta^2(\|x_j x_0\|) - \delta^2(\|x_i x_j\|)]\xi_i \xi_j \geq 0 \tag{6}$$

for all finite sets $x_0, x_1, \cdots, x_n$ in $\mathscr{S}$ and all sets of real numbers $\xi_1, \xi_2, \cdots, \xi_n$. Schoenberg further showed that if $\mathscr{S} = \mathscr{T} = \mathscr{H}_\infty$, the class of functions δ satisfying (6) and hence solving Problem III is the class of all functions of the form

$$\delta(t) = \left\{ \int_0^{t^2} \psi(s)\,ds \right\}^{1/2},$$

with $\psi(s)$ integrable and completely monotonic in $0 < t < \infty$ (that is, $(-1)^n d^n \psi(t)/dt^n \geq 0$ for $n = 0, 1, 2, \cdots$ and $0 < t < \infty$). This class includes functions such as t^ν $(0 < \nu \leq 1)$, $t/\sqrt{1+t^2}$, $(1 - e^{-\lambda t^2})/t$.

The picture becomes much blurred but remains still recognizable when the condition $\|f(x_1)f(x_2)\| = \delta(\|x_1 x_2\|)$ is relaxed to the inequality (5) in Problem II. The connection with quadratic forms (6) subsists in a weakened form: Any δ such that

(6) holds for nonnegative numbers ξ_i is a solution of Problem II. It is not known whether the converse holds, but it is unlikely. Moreover, as counterparts to integrals of completely monotonic functions we have here nonnegative concave functions. More precisely, any $\delta(t)$ such that $\delta^2(\sqrt{t})$ is concave over $0 < t < \infty$ satisfies (6) for nonnegative numbers ξ_i, and hence it solves Problem II. It is by no means clear that such functions δ exhaust the class defined by (6) with nonnegative numbers ξ_i, even when both $\mathscr{S}$ and $\mathscr{T}$ are the Hilbert space. This article is devoted to the proofs of the above-stated facts concerning Problem II; the necessary and sufficient condition that a uniformly continuous mapping be extendable to the whole space mentioned in the discussion of Problem I follows then as a corollary to the remark that any nonnegative, subadditive δ admits a concave majorant.

LEMMA 1. *In order that a mapping* f: $\mathscr{S} \to \mathscr{T}$ *satisfying*

$$\|f(x_1)f(x_2)\| \leq \delta(\|x_1 x_2\|) \qquad (x_1, x_2 \in \mathscr{D}(f)) \tag{7}$$

admit an extension $\tilde{f}$ *to* $\mathscr{S}$ *with the same property, it is sufficient that any finitely defined mapping* g *be so extendable to* $\mathscr{D}(g) \cup \{x_0\}$ *for each* $x_0 \notin \mathscr{D}(g)$.

Proof. Assume the condition is satisfied, and let x_0 be a point not in $\mathscr{D}(f)$. Denote by E any finite subset of $\mathscr{D}(f)$. By hypothesis, the closed bounded set $F_E \subset \mathscr{T}$ of all points y such that

$$\|f(x) - y\| \leq \delta(\|xx_0\|) \qquad \text{for } x \in E$$

is not empty. Since $F_{E_1} \cap F_{E_2} = F_{E_1 \cup E_2}$ and $\mathscr{T}$ is Euclidean, $\{F_E\}$ is a family of weakly compact sets with the finite-intersection property. Hence there is at least one point y_0 belonging to all sets F_E. The function

$$\tilde{f}(x) = \begin{cases} f(x) & (x \in \mathscr{D}(f)), \\ y_0 & (x = x_0) \end{cases}$$

is an extension of f to $\mathscr{D}(f) \cup \{x_0\}$ that preserves (7). Therefore it is possible to add any point to $\mathscr{D}(f)$, and by Zermelo's Axiom (or if one prefers, by Zorn's Lemma), we can extend f to the whole space without losing condition (7). (This is an adaptation of an argument used by F. Browder [2, Theorem 2].)

Notation. For any metric space $\mathscr{S}$, we denote by $\mathscr{Q}(\mathscr{S})$ the class of all nonnegative functions $\delta(t)$ of a nonnegative real variable satisfying

$$\sum_{i,j \neq 0} [\delta^2(\|x_i x_0\|) + \delta^2(\|x_j x_0\|) - \delta^2(\|x_i x_j\|)]\xi_i \xi_j \geq 0, \tag{8}$$

for any choice of the finite set $\{x_i\}_0^m \subset \mathscr{S}$ and of the nonnegative reals $\{\xi_i\}_1^m$.

We can obtain significant indications of the meaning of relations (8) by considering a few special cases. When written for two points x_0, x_1 and nonvanishing ξ_1, (8) says

$$2\delta^2(\|x_1 x_0\|) \geq \delta^2(0). \tag{9}$$

Moreover, for three points x_0, x_1, x_2, it can be written in the form

$$2[\delta(\|x_1x_0\|)\xi_1 - \delta(\|x_2x_0\|)\xi_2]^2 + 2\{[\delta(\|x_1x_0\|) + \delta(\|x_2x_0\|)]^2 - \delta^2(\|x_1x_2\|)\}\xi_1\xi_2 \geq \delta^2(0)(\xi_1^2 + \xi_2^2). \tag{10}$$

Since the function $\delta(t) = \text{const.} \neq 0$ satisfies (8), $\delta(0)$ does not vanish in general, but—as we see from (9)—if $\delta(t)$ vanishes at some t in the range of distances, then it vanishes at $t = 0$ also. The equation $\delta(\|x_1 x_2\|) = 0$ determines a relation between pairs of points which, if it ever takes place, is an equivalence relation. Indeed, it is reflexive, because, as we have seen, $\delta(\|x_1 x_1\|) = \delta(0) = 0$; it is reciprocal, by the symmetry of the distance; and it is transitive, by virtue of (10), since if

$$\delta(\|x_1 x_0\|) = \delta(\|x_2 x_0\|) = 0,$$

then $-2\delta^2(\|x_1 x_2\|)\xi_1\xi_2 \geq 0$, and $\delta(\|x_1 x_2\|) = 0$. Moreover, the value of δ at $\|x_1 x_2\|$ depends only on the equivalence classes to which the points belong. In fact, if $\delta(\|x_2 x_0\|) = 0$, we get from (10) the inequality

$$2\delta^2(\|x_1 x_2\|)\xi_1^2 + 2[\delta^2(\|x_1 x_0\|) - \delta^2(\|x_1 x_2\|)]\xi_1\xi_2 \geq 0.$$

Dividing by ξ_1 (assumed to be different from zero) and letting ξ_1 approach zero, one obtains the inequality $\delta(\|x_1 x_2\|) \leq \delta(\|x_1 x_0\|)$, and since x_0 and x_2 play symmetrical roles, $\delta(\|x_1 x_2\|) = \delta(\|x_1 x_0\|)$. Thus if in $\delta(\|x_1 x_0\|)$ we replace x_0 by any other point in its class, the value does not change. Incidentally, we have proved

$$\delta(\|x_1 x_2\|) \leq \delta(\|x_1 x_0\|) + \delta(\|x_2 x_0\|), \tag{11}$$

in the case when at least one of the three quantities involved is zero; if none vanishes, the same result follows from (10) if we take for ξ_1 and ξ_2 nonvanishing values proportional to $\delta(\|x_2 x_0\|)$ and $\delta(\|x_1 x_0\|)$, respectively. Therefore, if $\delta(0) = 0$, $\delta(\|x_1 x_2\|)$ is a distance in the space of equivalence classes.

We remark, however, that the vanishing of a $\delta \in \mathscr{Q}(\mathscr{S})$ at points other than zero without vanishing identically is an exception possible only in the presence of a certain pathology in $\mathscr{S}$. Such a situation cannot occur if some sphere

$$S_\rho(x_0) = \{x \in \mathscr{S},\ \|xx_0\| = \rho\}$$

of positive radius is not totally disconnected; for, in this case, if $\delta(\rho) = 0$ $(\rho > 0)$, the mutual distances of the points in the sphere fill out a closed interval $(0, t_0)$ of nonvanishing length, and in such an interval (and hence everywhere) δ vanishes identically, by (11). Of all Euclidean spaces $\mathscr{H}_m$, $\mathscr{H}_1$ is the only one with totally disconnected spheres and the only one where δ's with the above behavior exist ($\delta(t) = |\sin t|$, for example).

LEMMA 2. *If* $\delta(t) \in \mathscr{Q}(\mathscr{S})$, *then each mapping* $f\colon \mathscr{S} \to \mathscr{T}$ *satisfying* (7) *admits an extension to* $\mathscr{S}$ *meeting the same requirements.*

Proof. In view of Lemma 1, it suffices to show that if $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ are points in $\mathscr{S}$ and $\mathscr{T}$ such that

$$\|y_i - y_j\| \leq \delta(\|x_i x_j\|) \quad (i, j = 1, 2, \cdots, n), \tag{12}$$

and if $x_0 \neq x_i$ $(i = 1, 2, \cdots, n)$, then there is a $y_0 \in \mathscr{T}$ for which

$$\|y_i - y_0\| \leq \delta(\|x_i x_0\|) \quad (i = 1, 2, \cdots, n). \tag{13}$$

To prove this, assume first that $\delta(t)$ vanishes at no point other than 0, and consider the real-valued function

$$p(y) = \sup_{i=1,\cdots,n} \frac{\|y_i - y\|}{\delta(\|x_i x_0\|)}.$$

This is a continuous function tending to infinity with y. Moreover, its infimum is not smaller than that of its restriction to the finite-dimensional linear space spanned by $y_1, y_2, \cdots, y_n$, and hence is attained at a point $y_0 \in \mathscr{T}$. Thus

$$\mu = \inf_y \sup_{i=1,\cdots,n} \frac{\|y_i - y\|}{\delta(\|x_i x_0\|)} = \sup_{i=1,\cdots,n} \frac{\|y_i - y_0\|}{\delta(\|x_i x_0\|)}.$$

The point is to prove that $\mu \leq 1$. Assume the y_i are numbered so that the supremum of $\|y_i - y_0\|/\delta(\|x_i x_0\|)$ is attained for $i = 1, 2, \cdots, m$ $(m \leq n)$. Then

$$\|y_i - y_0\| = \mu\delta(\|x_i x_0\|) \quad (i = 1, 2, \cdots, m), \tag{14}$$

$$\|y_i - y_0\| < \mu\delta(\|x_i x_0\|) \quad (i = m+1, m+2, \cdots, n). \tag{15}$$

These relations make it plain that y_0 must belong to the convex hull of $y_1, y_2, \cdots, y_m$, for otherwise we could bring it simultaneously closer to all these points without affecting inequalities (15), contrary to the minimum property of μ. Hence there exist nonnegative numbers $\xi_1, \xi_2, \cdots, \xi_m$ such that

$$y_0 = \sum_{i=1}^{m} \xi_i y_i, \quad 1 = \sum_{i=1}^{m} \xi_i. \tag{16}$$

From (12) and (14), we get for $i, j = 1, 2, \cdots, m$ the estimates

$$\delta^2(\|x_i x_j\|) \geq \|y_i - y_j\|^2 = \|y_i - y_0\|^2 + \|y_j - y_0\|^2 - 2(y_i - y_0, y_j - y_0)$$
$$= \mu^2(\delta^2(\|x_i x_0\|) + \delta^2(\|x_j x_0\|)) - 2(y_i - y_0, y_j - y_0),$$

which multiplied by $\xi_i \xi_j$ and added together yield the inequality

$$\sum_{i,j=1}^{m} \delta^2(\|x_i x_j\|)\xi_i\xi_j \geq \mu^2\left(\sum_{i,j=1}^{m} (\delta^2(\|x_i x_0\|) + \delta^2(\|x_j x_0\|))\xi_i\xi_j\right) - 2\left\|\sum_{i=1}^{m} \xi_i(y_i - y_0)\right\|^2.$$

The last term vanishes, by (16), and the rest says that $\mu \leq 1$ by virtue of (8). If $\delta(t)$ vanishes at other points besides $t = 0$, we pass to the space $\tilde{\mathscr{S}}$ of equivalence classes modulo the relation $\delta(\|x_1 x_2\|) = 0$ under the semimetric

$$\|\tilde{x}_1 \tilde{x}_2\| = \inf_{x_1 \in \tilde{x}_1, x_2 \in \tilde{x}_2} \|x_1 x_2\|,$$

and apply the above result.

LEMMA 3. *If* $\delta(t)$ *belongs to* $\mathscr{Q}(\mathscr{S})$, *then so does* $\delta_\lambda(t) = \min(\delta(t), \lambda)$, *for each* $\lambda > 0$.

Proof. Assume the condition is fulfilled, and write

$$\sum_{i,j\neq 0} [\delta_\lambda^2(\|x_i x_0\|) + \delta_\lambda^2(\|x_j x_0\|) - \delta_\lambda^2(\|x_i x_j\|)]\xi_i\xi_j$$
$$= \left(\sum{}' + \sum{}''\right) [\delta_\lambda^2(\|x_i x_0\|) + \delta_\lambda^2(\|x_j x_0\|) - \delta_\lambda^2(\|x_i x_j\|)]\xi_i\xi_j,$$

where the sum $\sum'$ extends over the indices i, j with

$$\max(\delta(\|x_i x_0\|),\ \delta(\|x_j x_0\|)) \le \lambda,$$

and the sum $\sum''$ over the indices with $\max(\delta(\|x_i x_0\|), \delta(\|x_j x_0\|)) > \lambda$. We can put the first sum into the form

$$\sum{}' [\delta^2(\|x_i x_0\|) + \delta^2(\|x_j x_0\|) - \delta^2(\|x_i x_j\|)]\xi_i\,\xi_j$$
$$+ \sum{}' [\delta^2(\|x_i x_j\|) - \delta_\lambda^2(\|x_i x_j\|)]\xi_i\,\xi_j$$

and conclude that it is nonnegative by hypothesis. The second sum is also nonnegative, because all its terms are nonnegative. The proof is complete.

LEMMA 4. *If* $\delta(t)$ *belongs to* $\mathcal{Q}(\mathcal{S})$, *then so does any function of the form* $\delta^{(\kappa)}(t) = (\kappa(\delta^2(t)))^{1/2}$, *where* $\kappa(u)$ *is a nonnegative concave function on* $(0, \infty)$.

Proof. Let $\kappa(u)$ be any nonnegative concave function on $(0, \infty)$, and assume for a moment that it is twice continuously differentiable on the open positive real axis. Such a function is necessarily nondecreasing, and both the limits

$$\kappa(0+) = \lim_{u \downarrow 0} \kappa(u) \qquad \text{and} \qquad \kappa'(+\infty) = \lim_{u\to\infty} \kappa'(u)$$

exist and are finite and nonnegative. By integration by parts,

$$\kappa(u) = \kappa(0+) + \int_0^u \kappa'(s)\,ds = \kappa(0+) + s\kappa'(s)\Big|_0^u - \int_0^u s\kappa''(s)\,ds$$

$$= \kappa(0+) + u\,\kappa'(u) - \int_0^u s\,\kappa''(s)\,ds$$

$$= \kappa(0+) + u\,\kappa'(+\infty) - \int_0^u s\,\kappa''(s)\,ds - \int_u^\infty u\,\kappa''(s)\,ds$$

$$= \kappa(0+) + u\,\kappa'(+\infty) - \int_0^\infty \inf(u, s)\,\kappa''(s)\,ds\,.$$

Therefore

$$(\delta^{(\kappa)}(t))^2 = \kappa(\delta^2(t)) = \kappa(0+) + \kappa'(+\infty)\,\delta^2(t) - \int_0^\infty (\delta_{\sqrt{s}}(t))^2\,\kappa''(s)\,ds\,,$$

and consequently

$$\sum_{i,j\neq 0} [(\delta^{(\kappa)}(\|x_i x_0\|))^2 + (\delta^{(\kappa)}(\|x_j x_0\|))^2 - (\delta^{(\kappa)}(\|x_i x_j\|))^2]\xi_i \xi_j$$

$$= \kappa(0+)\left(\sum_i \xi_i\right)^2 + \kappa'(+\infty)\sum_{i,j\neq 0} [\delta^2(\|x_i x_0\|) + \delta^2(\|x_j x_0\|) - \delta^2(\|x_i x_j\|)]\xi_i \xi_j$$

$$+ \int_0^\infty \sum_{i,j\neq 0} [\delta^2_{\sqrt{s}}(\|x_i x_0\|) + \delta^2_{\sqrt{s}}(\|x_j x_0\|) - \delta^2_{\sqrt{s}}(\|x_i x_j\|)]\xi_i \xi_j (-\kappa''(s))\,ds\,.$$

A simple inspection and an application of Lemma 3 show that all terms on the right are nonnegative, and therefore the sum on the left is nonnegative. Thus the lemma has been proved under the additional condition that $\kappa(u)$ be of class $C^2(0+, \infty)$. This restriction, however, can be removed at once by a limiting process, upon observing that the function

$$\kappa_\varepsilon(u) = \int_0^\infty \kappa(u/v)\,\phi_\varepsilon(\log v)\,\frac{dv}{v} \qquad (\varepsilon > 0)$$

constructed from any nonnegative concave function $\kappa(u)$ with a nonnegative mollifier $\phi_\varepsilon(u)$ is again nonnegative and concave, belongs to $C^\infty(0+, \infty)$, and converges to $\kappa(u)$ uniformly over any finite interval away from zero, as $\varepsilon \to 0$.

We can obtain specific results from Lemma 4 by applying it to Euclidean spaces ($\mathscr{S} = \mathscr{H}_m$). A typical element of $\mathscr{Q}(\mathscr{H}_m)$ is the function $\delta(t) = t$, since clearly

$$(17)\qquad \sum_{i,j\neq 0} [\|x_i - x_0\|^2 + \|x_j - x_0\|^2 - \|x_i - x_j\|^2]\xi_i \xi_j = \left\|\sum_{i\neq 0} \xi_i (x_i - x_0)\right\|^2 \geq 0\,.$$

Therefore all functions of the form $\delta(t) = (\kappa(t^2))^{1/2}$ with κ nonnegative and concave belong to $\mathscr{Q}(\mathscr{H}_m)$ $(m = 1, 2, \cdots, \infty)$. This class of functions will be denoted by $\mathscr{K}$.

Other elements of $\mathscr{Q}(\mathscr{H}_m)$ already at hand are the functions

$$(18)\qquad \delta(t) = \int_0^\infty (1 - \Omega_m(tu))\,d\alpha(u) \qquad (1 \leq m < \infty),$$

where $\alpha(u)$ is a nondecreasing bounded function and

$$\Omega_m(r) = \Gamma\left(\frac{m}{2}\right) J_{\frac{m-2}{2}}(r)\cdot\left(\frac{r}{2}\right)^{-\frac{m-2}{2}}$$

(Γ indicates the Gamma function, and J_ν the Bessel function of order ν). Schoenberg [9] has shown that quadratic forms (8) associated with these functions are nonnegative definite (no restriction on the signs of the ξ_i) and that therefore the functions $\delta(\|x_1 x_2\|)$ are new metrics for $\mathscr{H}_m$, turning it into a new space isometrically imbeddable in $\mathscr{H}_\infty$. For $\mathscr{H}_\infty$, we have in place of (18) the functions

$$(19)\qquad \delta(t) = \left\{\int_0^{t^2} \psi(s)\,ds\right\}^{1/2},$$

where ψ is any completely monotonic function in $(0+, \infty)$ (see the introductory discussion); these are all the functions with associated nonnegative forms (8) over $\mathscr{H}_\infty$. It is not yet clear what classes of functions we obtain by composing the above classes with nonnegative concave functions according to the prescription of Lemma 4. We can easily see that for $\mathscr{H}_\infty$ we do not get out of the previously found class of functions $(\kappa(t^2))^{1/2}$ (with κ nonnegative and concave); but this is not the case for $\mathscr{H}_1$, where composition with functions such as $|\sin \lambda t|$ adds fresh material to our knowledge.

We note that for Euclidean spaces, (11) implies

$$\max[\delta(|t_1 - t_2|), \delta(t_1 + t_2)] \leq \delta(t_1) + \delta(t_2), \tag{20}$$

a condition a bit stronger than subadditivity but not quite as strong as subadditivity and monotonicity; let $\mathscr{G}$ denote the class of functions δ satisfying (20). Since (8) becomes more restrictive in larger spaces, part of our discussion can be summarized as follows:

$$\mathscr{G}' \supset \mathscr{Q}(\mathscr{H}_1) \supset \mathscr{Q}(\mathscr{H}_2) \supset \cdots \supset \mathscr{Q}(\mathscr{H}_\infty) \supset \mathscr{K}. \tag{21}$$

We know no alternate way of characterizing classes $\mathscr{Q}(\mathscr{H}_m)$, nor do we know whether all inclusions in (21) are proper. The combination of (21) with Lemma 2 leads to the following theorem.

THEOREM 1. *Let* $\delta(t) = (\kappa(t^2))^{1/2}$, *with* $\kappa(t)$ *nonnegative and concave over* $(0+, \infty)$, *and let* $f: \mathscr{S} \to \mathscr{T}$ *be a mapping of a Euclidean space into another, satisfying the condition*

$$\|f(x_1) - f(x_2)\| \leq \delta(\|x_1 - x_2\|) \qquad (x_1, x_2 \in \mathscr{D}(f)). \tag{22}$$

Then f *admits an extension* $\tilde{f}$ *to the whole space* $\mathscr{S}$ *which also satisfies* (22).

Now we turn to the problem of extending uniformly continuous functions, which we discussed at the beginning of this article; we shall assume that both $\mathscr{S}$ and $\mathscr{T}$ are Euclidean spaces. As we have seen, in order that a uniformly continuous mapping $f: \mathscr{S} \to \mathscr{T}$ be extendable it is necessary that its modulus of continuity $\delta_f(t)$ admit a subadditive majorant $\tilde{\delta}(t)$ with $\lim_{t \to 0} \tilde{\delta}(t) = 0$. Let us show that this condition is also sufficient. To this end we need a lemma.

LEMMA 5. *For each nonnegative, subadditive function* $\delta(t)$ $(0 \leq t < \infty)$ *there is a nonnegative, concave* $\hat{\delta}(t)$ *such that*

$$\sup_{0 \leq s \leq t} \delta(s) \leq \hat{\delta}(t) \leq 2 \sup_{0 \leq s \leq t} \delta(s). \tag{23}$$

Proof. Let $\delta^+(t) = \sup_{0 \leq s \leq t} \delta(s)$. Clearly, $\delta^+(t)$ is nonnegative and does not decrease, and since

$$\delta^+(t_1 + t_2) = \sup_{0 \leq s \leq t_1 + t_2} \delta(s) = \sup_{\substack{0 \leq s_1 \leq t_1 \\ 0 \leq s_2 \leq t_2}} \delta(s_1 + s_2) \leq \sup_{\substack{0 \leq s_1 \leq t_1 \\ 0 \leq s_2 \leq t_2}} \{\delta(s_1) + \delta(s_2)\}$$
$$= \delta^+(t_1) + \delta^+(t_2),$$

it is subadditive. Define $\hat{\delta}(t)$ as the smallest concave majorant of $\delta(t)$; that is, set

$$\hat{\delta}(t) = \sup_{\alpha_i, t_i} \sum_i \alpha_i \delta(t_i), \tag{24}$$

the sup being taken with regard to all pairs of finite sets of nonnegative reals $\alpha_1, \alpha_2, \cdots, \alpha_k$ and $t_1, t_2, \cdots, t_k$ such that

$$\sum_i \alpha_i = 1 \quad \text{and} \quad \sum_i \alpha_i t_i \leq t.$$

It is obvious that $\hat{\delta}(t)$ is nonnegative and concave and that it satisfies the first inequality (23). To prove that it also satisfies the second, we avail ourselves of the fact that it is indifferent whether on the left of (24) one uses $\delta(t)$ or $\delta^+(t)$. Since $\delta^+(t)$ is nondecreasing,

$$\sum_i \alpha_i \delta^+(t_i) = \sum_i \alpha_i \delta^+\left(\frac{t_i}{\sum_j \alpha_j t_j} \sum_j \alpha_j t_j\right) \leq \sum_i \alpha_i \delta^+\left(\left(\left[\frac{t_i}{\sum_j \alpha_j t_j}\right] + 1\right) \sum_j \alpha_j t_j\right),$$

where the square brackets indicate "integral part of". By the subadditivity of $\delta^+(t)$ we see that if $\sum \alpha_i = 1$ and $\sum \alpha_i t_i \leq t$, then

$$\sum_i \alpha_i \delta^+(t_i) \leq \sum_i \alpha_i \left(\left[\frac{t_i}{\sum_j \alpha_j t_j}\right] + 1\right) \delta^+\left(\sum_j \alpha_j t_j\right)$$

$$\leq \sum_i \alpha_i \left(\frac{t_i}{\sum_j \alpha_j t_j} + 1\right) \delta^+\left(\sum_j \alpha_j t_j\right) = 2\,\delta^+\left(\sum_j \alpha_j t_j\right) \leq 2\,\delta^+(t).$$

which is another form of the inequality we seek.

Resuming the interrupted argument, we suppose that the modulus of continuity $\delta_f(t)$ of a uniformly continuous mapping f has a subadditive majorant $\tilde{\delta}(t)$ approaching zero as $t \to 0$. Then, by the lemma above, $\hat{\tilde{\delta}}(t)$ also tends to zero as $t \to 0$, and

$$\|f(x_1) - f(x_2)\| \leq \hat{\tilde{\delta}}(\|x_1 - x_2\|) \quad (x_1, x_2 \in \mathscr{D}(f)). \tag{25}$$

A simple calculation shows that the property of being nonnegative and concave is transferred from $\hat{\tilde{\delta}}(t)$ to $(\hat{\tilde{\delta}}(t^{1/2}))^2$, and that in consequence $\hat{\tilde{\delta}}(t)$ belongs to the class $\mathscr{K}$. Hence, by Theorem 1, there exists an $\tilde{f}(x)$ defined all over $\mathscr{S}$; coinciding with f(x) on $\mathscr{D}(f)$, and such that

$$\|\tilde{f}(x_1) - \tilde{f}(x_2)\| \leq \hat{\tilde{\delta}}(\|x_1 - x_2\|) \quad (x_1, x_2 \in \mathscr{S}).$$

Clearly, $\tilde{f}$ is a uniformly continuous extension of f. Summing up:

THEOREM 2. *In order that a uniformly continuous mapping* f: $\mathscr{S} \to \mathscr{T}$ *of a Euclidean space into another have a uniformly continuous extension to the whole space* $\mathscr{S}$, *it is necessary and sufficient that its modulus of continuity* $\delta_f(t)$ *admit a subadditive majorant* $\hat{\delta}(t)$ *approaching zero as* $t \to 0$.

A useful corollary can be derived from the observation that $\delta_f(t)$ is subadditive if $\mathscr{D}(f)$ is convex:

COROLLARY. *Any uniformly continuous mapping* f: $\mathcal{S} \to \mathcal{T}$ *of a Euclidean space into another, defined over a convex set, admits a uniformly continuous extension to the whole space, with a modulus of continuity not larger than twice the modulus of continuity of the original function.*

For finite-dimensional $\mathcal{T}$'s, Theorem 2 may be considered as implicitly contained in the results of Valentine [12] that we mentioned in the introduction. It is instructive to observe that the majorizing condition of Theorem 2 is equivalent to $\limsup_{t \to \infty} \delta_f(t)/t < \infty$ [1, Theorem 1].

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