PSEUDO-ISOTOPIES AND CELLULAR SETS

Ross L. Finney

In this paper we study some of the relationships between cellular decompositions of manifolds and pseudo-isotopies on manifolds. Our manifolds have no boundary.

A pseudo-isotopy on a manifold N is a homotopy $h: I \times N \to N$ for which each restriction $h_t = h \mid t \times N$ is onto and each h_t is a homeomorphism for $t < 1$. We think of each h_t as a map of N onto N. The map h_1 is said to be the *end* of h, and h is said to end in h_1.

A pseudo-isotopy h *shrinks* the elements of a decomposition D of N if

1. h_0 is the identity,
2. for each point x of N the set $h_1^{-1}(x)$ is an element of D.

Thus h does not collapse subsets of N with reckless abandon. It keeps elements of D separate at all times, and when the shrinking has been carried out at $t = 1$, we find that the pre-image $h_1^{-1}(x)$ of each point is no more and no less than an element of D.

Let f be a map of N onto itself, and let D_f denote $\{f^{-1}(x) \mid x \in N\}$, the decomposition of N induced by f. A pseudo-isotopy on N that shrinks the elements of D_f to points does not necessarily end in f. For example, if f were an orientation-preserving map of a sphere onto itself, there would be no homotopy of f with the identity. One can however prove the following equivalence:

PROPOSITION. *Let* X *be a compact Hausdorff space. A map* f *of* X *onto itself is the end of a pseudo-isotopy on* X *if and only if there exists some pseudo-isotopy on* X *that shrinks the elements of* D_f *to points.*

The proof one way is easy, because if f is the end of a pseudo-isotopy h on X, then $h_0^{-1}h$ is a pseudo-isotopy that shrinks the elements of D_f to points.

The converse follows from the next lemma.

LEMMA. *If* f *and* h_1 *are maps of a compact Hausdorff space onto itself with* $D_f = D_{h_1}$, *then there exists a homeomorphism* g *such that* $gh_1 = f$.

More general forms of this lemma are known, but this one is sufficient here and for the proof of Theorem 2.

A subset A of an n-manifold N is *cellular* if each open subset that contains A also contains a closed n-cell B with $A \subset \text{Int } B$. A cellular set need not be locally connected, nor need it be contractible on itself to a point, or have the fixed-point property. Yet each cellular subset A of N can be shrunk to a point by a pseudo-isotopy on N. That is, there exists a pseudo-isotopy h on N such that h_0 is the identity and A is the only nondegenerate element of D_{h_1}. It is sometimes useful to know that this pseudo-isotopy may be chosen to be the identity outside an arbitrary neighborhood of A. In general, a decomposition of N is called cellular if each of its elements is cellular, and a mapping f defined on N is cellular if D_f is cellular.

Received January 6, 1966.

It follows from results of Youngs [5] and of Floyd and Fort [3] that a map of the 2-sphere onto itself is monotone if and only if f is the end of a pseudo-isotopy on the sphere. Since it is a consequence of R. L. Moore's decomposition theorem [4] that the properties of being *monotone* and *cellular* are equivalent for subsets of S^2, we know that a map of S^2 onto itself is cellular if and only if it is the end of a pseudo-isotopy on the sphere.

The properties of being cellular and monotone are not equivalent for mappings of S^3 onto itself. R. H. Bing [1] has a decomposition of S^3 into points and topological figure-eights whose quotient space is again homeomorphic to S^3. The projection map is thus a map of S^3 onto itself that is monotone but not cellular. This projection mapping is not the end of a pseudo-isotopy on the sphere:

THEOREM 1. *If* f *is the end of a pseudo-isotopy on a manifold, then each compact element of* D_f *is cellular.*

COROLLARY. *The end of a pseudo-isotopy on a compact manifold is cellular.*

The proof of the corollary is trivial. To prove the theorem, let f be the end of the pseudo-isotopy $h\colon I \times N \to N$ on the n-manifold N, and let x be a point of N for which $A = f^{-1}(x)$ is compact. Let U be an open set containing A. We begin our search for a closed n-cell in U whose interior contains A by choosing first some open set V with $A \subset V \subseteq U$ and with $\overline{V}$ compact. Then routine arguments with sequences and compactness show that

1. diameter $(h_t(A) \cup \{x\}) \to 0$ as $t \to 1$,
2. there exists an $\varepsilon_0 > 0$ such that $\rho(h_t(A), h_t(\partial\overline{V})) \geq \varepsilon_0$ for all t (ρ is the distance given by the metric in N).

Let B be a closed n-cell in N whose interior contains x and whose diameter is less than ε_0. For some $t_0 < 1$, $h_{t_0}(A) \subset \operatorname{Int} B$. For this t_0, $h_{t_0}(\partial\overline{V})$ and B are disjoint. Therefore $B \subset h_{t_0}(V)$, so that $h_{t_0}^{-1}(B)$ is the required n-cell. I would like to thank the referee for suggesting this argument, which shortens my original proof.

The corollary was discovered independently by Morton Brown. His formulation is: $f\colon N \to N$ is cellular if it is a uniform limit of homeomorphisms of a compact manifold N onto itself. Brown's proof (unpublished) is easier than our proof of Theorem 1, but it depends on the compactness of N and does not seem to lead to the more general result.

That there exist pseudo-isotopies on manifolds that shrink noncompact sets to points is shown by an example of James Munkres. In the first quadrant of E^2, let $A = \{(x, y) \mid y \geq x\}$ and $B = \{(x, y) \mid 0 \leq y \leq x\}$. Then, for each t in I, let

$$H_t(x, y) = \begin{cases} (1 - t)(x, y) + t(0, x) & \text{on } A, \\ (x - ty, y) & \text{on } B. \end{cases}$$

Extend H to the second quadrant by reflection in the y-axis, and then to the rest of E^2 by reflection in the x-axis. Then H_1 is not cellular, because $H_1^{-1}(0, 0)$ is the y-axis. Also, H_1 is not monotone, because $H_1^{-1}(0, y)$ is the union of two rays if $y \neq 0$.

Is each cellular map f of a compact manifold onto itself the end of a pseudo-isotopy? If D_f has only finitely many nondegenerate elements, yes. When else?

In [2] we proved that if f is a simplicial, cellular map of a triangulated, compact three-manifold onto a triangulated space T, then T is homeomorphic to M, so that f is a cellular map of M onto itself. It follows from the proof, in a way we shall sketch below, that f is the end of a pseudo-isotopy on M.

THEOREM 2. *Let* f *be a cellular map of a triangulated, compact* 3-*manifold* M *onto itself. If* f *is simplicial, then* f *is the end of a pseudo-isotopy on* M.

COROLLARY. *A simplicial map of a compact* 3-*manifold onto itself is the end of a pseudo-isotopy if and only if it is cellular.*

The corollary is an immediate consequence of Theorems 1 and 2.

In what follows, all complexes are locally-finite and simplicial, and simplices are closed. Whenever a symbol is used to denote a complex, it is also used to denote the underlying space.

To prove Theorem 2, we begin with a cellular, simplicial map f of a triangulated compact 3-manifold M onto itself. The union of the nondegenerate elements of D_f is a proper subcomplex K_1 of M (see [2] for details). Following the reduction described in Lemma 4.5 of [2], one can obtain a finite sequence $\{D_i\}_{i=1}^n$ of cellular decompositions of M and a corresponding sequence $\{K_i\}_{i=1}^n$ of subcomplexes of M such that

1. $D_1 = D_f$,
2. D_i is obtained from D_{i-1}, and K_i from K_{i-1}, by deletion of a simplex τ_{i-1} from K_{i-1},
3. K_i is the closure of the union of the nondegenerate elements of D_i,
4. D_n has only finitely many nondegenerate elements.

It is easy to make a pseudo-isotopy that shrinks the nondegenerate elements of D_n to points, because they are isolated cellular sets. Each can be shrunk to a point by a pseudo-isotopy that starts with the identity and remains the identity outside of a small neighborhood of the element. The separate shrinkings fit together to give a pseudo-isotopy (call it H^n) that shrinks the nondegenerate elements of D_n simultaneously.

Starting with H^n, one can prove Theorem 2 by induction. Finding H^n was the first step. The next is to show that if there exists a pseudo-isotopy H^i on M shrinking the nondegenerate elements of D_i to points, then there exists a pseudo-isotopy H^{i-1} on M that shrinks the nondegenerate elements of D_{i-1} to points. The way to prove this is to construct a function $G: I \times K_{i-1} \to K_i$ that (a) moves no point of K_i and (b) satisfies the condition $G_1(d) \subset d$ for each nondegenerate element d of D_{i-1}. That is, G collapses each nondegenerate element that protrudes from K_{i-1} to its intersection with ∂K_i. Because of (a) and (b), G will in general fail to be continuous on certain faces of τ_{i-1} (the example below is typical). But G can be chosen so that G_0 is the identity, G is continuous except on $1 \times \dot{\tau}_{i-1}$, and an extension of G to $I \times (M - K_i)$ fits with H^i to give the required pseudo-isotopy H^{i-1}.

For example, suppose that τ_{i-1} is a 2-simplex, sticking out from K_i, whose interior meets nondegenerate elements of D_{i-1} in line segments parallel to $\overline{ab}$. If $\overline{ab}$ is not free in K_{i-1}, one can choose G to collapse τ_{i-1} within itself to $\overline{ab} \cup \overline{bc}$ by leaving $\overline{ab} \cup \overline{bc}$ fixed and by moving each point of $\tau_{i-1} - (\overline{ab} \cup \overline{bc})$ parallel to $\overline{ab}$ until it comes to rest in $\overline{bc}$.

One can observe immediately that such a collapsing G is not continuous along the edge $\overline{ab}$ at $t = 1$. This is where it becomes necessary to fit G with H^i. Because of the way τ_{i-1} is decomposed, the edge $\overline{ab}$ is a subset of a nondegenerate

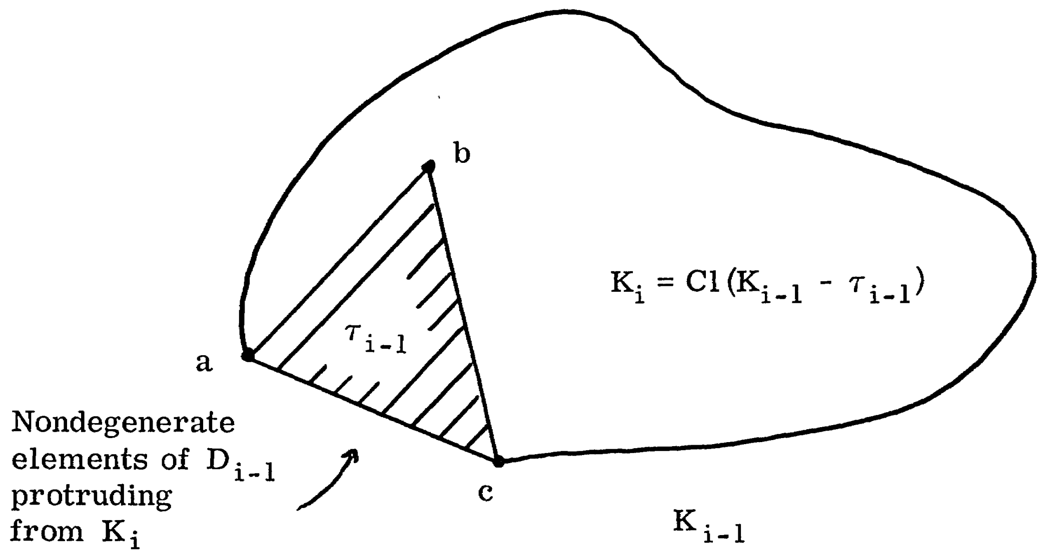

Figure 1.

element of D_i, so that H^i collapses $\overline{ab}$ to a point. Then, if G and H^i are carried out simultaneously, the result is continuous at $t = 1$, even though G is not.

If $\overline{ab}$ is free in K_{i-1}, we can collapse it with the segments to get a continuous G.

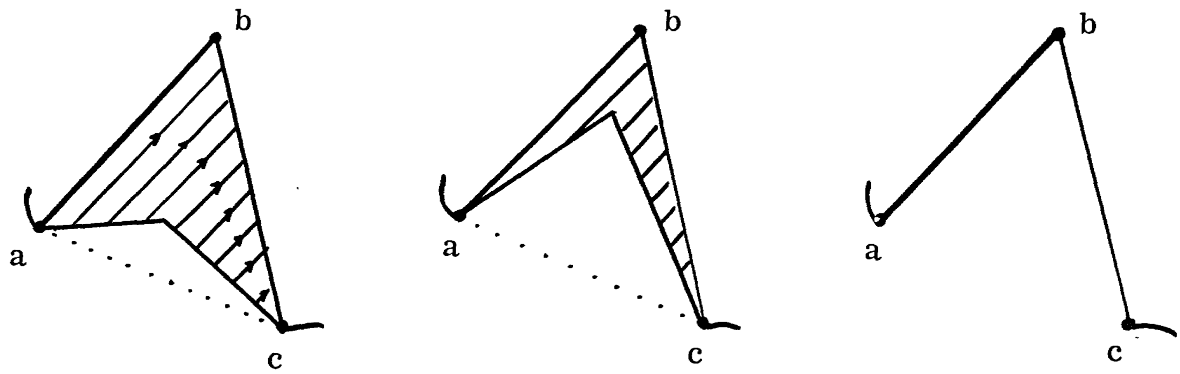

Figure 2.

Of course, the τ_{i-1} we have just imagined are special. The other possibilities for τ_{i-1} are described on page 595 of [2]. For each it is possible to find a function, like the G we have found here, to shrink the nondegenerate elements of D_{i-1} into elements of D_i without disturbing the nondegenerate elements of D_i. In each case, the pseudo-isotopy can be combined with H^i to give H^{i-1}.

By induction we then arrive at a pseudo-isotopy H that shrinks the nondegenerate elements of D_f. Of course, H does not necessarily end in f, but the Lemma says that there exists a homeomorphism g of M onto itself such that gH ends in f. The pseudo-isotopy gH satisfies the conclusion of Theorem 2.

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University of Illinois