UNIVERSAL $\mathscr{P}$-LIKE COMPACTA

Michael C. McCord

1. INTRODUCTION

A *compactum* is a compact, metrizable space. A *continuum* is a connected compactum. By a *polyhedron* we mean a finitely triangulable space.

If $\mathscr{C}$ is a class of spaces, a *universal* member of $\mathscr{C}$ is a member of $\mathscr{C}$ in which every member of $\mathscr{C}$ can be imbedded. K. Menger [11] described an n-dimensional continuum which he conjectured (and proved in the case $n = 1$) to be a universal n-dimensional compactum. G. Nöbeling [12] produced a different space, which he showed to be a universal n-dimensional, separable, metrizable space. S. Lefschetz [8] (independently of [12]) verified Menger's conjecture. See Hurewicz and Wallman [7, p. 64] for a treatment of Nöbeling's theorem. R. M. Schori [13], [14], has shown that there exist universal snake-like continua (see R. H. Bing [1]).

We present a single, rather general theorem (Theorem 1) that implies (see Theorem 2) both Schori's result and the existence of universal n-dimensional compacta. The method of proof, involving inverse limit systems, is an extension of Schori's method. Lefschetz's proof of Menger's conjecture used a version of polyhedral inverse limit expansions. The main feature of the present approach is the additional use of polyhedral inverse limit systems to *define* the required universal spaces.

The framework needed for our theorems is the theory of $\mathscr{P}$-like compacta, where $\mathscr{P}$ is a class of polyhedra. See Mardešić and Segal [9]. If α is an open cover of the compactum X, a map f of X onto a compactum Y is called an *α-map* provided that for each y in Y, $f^{-1}(y)$ is contained in some member of α. Let $\mathscr{P}$ be a class of polyhedra. Following [9], we say a compactum X is *$\mathscr{P}$-like* if for each open cover α of X there exists an α-map of X onto some member of $\mathscr{P}$.

We are concerned with the following question: For which classes $\mathscr{P}$ is there a universal $\mathscr{P}$-like compactum? Theorems 1 and 2 are positive results; Theorems 3 and 4 are negative. Part of the results were announced in [10].

2. STATEMENT OF THEOREMS

Definition 1. The class $\mathscr{P}$ of polyhedra is called *amalgamable* if for each finite sequence $(P_1, \cdots, P_n)$ of members of $\mathscr{P}$ and maps $\phi_i: P_i \to Q$ $(1 \leq i \leq n)$, where $Q \in \mathscr{P}$, there exists a member P of $\mathscr{P}$ with imbeddings $\mu_i: P_i \to P$ and a map ϕ of P onto Q such that $\phi_i = \phi\mu_i$ for each i. We call $(P, \phi, \mu_1, \cdots, \mu_n)$ an *amalgamation* of $(\phi_1, \cdots, \phi_n)$.

THEOREM 1. *If $\mathscr{P}$ is an amalgamable class of polyhedra, then there exists a universal $\mathscr{P}$-like compactum.*

Received February 6, 1965.

This work was supported in part by the National Science Foundation, Contract G-11665.

THEOREM 2. *Let* $\mathscr{P}_a$ *be the class of all acyclic polyhedra,* $\mathscr{P}_c$ *the class of all contractible polyhedra, and* $\mathscr{P}_k$ *the class of all polyhedra of dimension at most* k. *Then each of the following classes is amalgamable:* (1) $\mathscr{P}_k$, (2) $\mathscr{P}_a$, (3) $\mathscr{P}_c$, (4) $\mathscr{P}_a \cap \mathscr{P}_k$, (5) $\mathscr{P}_c \cap \mathscr{P}_k$, (6) *the class of trees (the nondegenerate members of* $\mathscr{P}_a \cap \mathscr{P}_1 = \mathscr{P}_c \cap \mathscr{P}_1$), (7) *the class of all* k-*cells (for fixed* k). *Thus it follows from Theorem* 1 *that if* $\mathscr{P}$ *is any one of these classes, then there exists a universal* $\mathscr{P}$*-like compactum.*

Remark 1. A compactum X has dimension at most k if and only if X is $\mathscr{P}_k$-like. Hence (1) shows that there exists a universal k-dimensional compactum.

Remark 2. Part (7), with k = 1, generalizes Schori's result.

THEOREM 3. *Let* $\mathscr{M}^n$ *be a class of closed, connected, triangulable* n-*manifolds. Then there exists no universal* $\mathscr{M}^n$*-like continuum.*

THEOREM 4. *Let* $\mathscr{P}$ *be a class of* 1-*dimensional polyhedra whose* 1-*dimensional Betti numbers are bounded, but which contains at least one member with positive* 1-*dimensional Betti number. Then there exists no universal* $\mathscr{P}$*-like compactum.*

I wish to thank R. H. Bing for a suggestion that led to the formulation of Theorem 3.

There is another way of defining "$\mathscr{P}$-like compactum" that is more in accord with the original use of *snake-like* (= arc-like), *circle-like, tree-like* of Bing [1], [2] and others.

Definition 2. If $\mathscr{P}$ is a class of polyhedra, a compactum X is *weakly* $\mathscr{P}$*-like* if every open cover of X can be refined by a finite open cover whose nerve is homeomorphic to a member of $\mathscr{P}$.

Part or all of the following theorem may be known (see the remark in Example 1 of [9]), but to my knowledge it is not in the literature; I therefore include it for completeness.

THEOREM 5. *Every* $\mathscr{P}$*-like compactum is weakly* $\mathscr{P}$*-like. The converse is false; however, if* X *is a perfect, weakly* $\mathscr{P}$*-like compactum, then* X *is* $\mathscr{P}$*-like.*

The proof of the second part, for the case where X is connected, was supplied to me by the late M. K. Fort, Jr.

I am grateful to the referee for suggestions, in particular for an alternate proof of Lemma 8 in Section 6.

3. PROOF OF THEOREM 1

Let $\mathscr{P}$ be an amalgamable class of polyhedra. We may assume (and do it, for convenience) that $\mathscr{P}$ is closed under homeomorphism, in other words, that every polyhedron homeomorphic to a member of $\mathscr{P}$ is itself a member of $\mathscr{P}$. Since up to isomorphism there are only a countable number of finite abstract simplicial complexes, and since isomorphic abstract simplicial complexes have homeomorphic geometric realizations, there exists a sequence

$$(P_1, P_2, P_3, \cdots) \tag{3.1}$$

of members of $\mathscr{P}$ such that every member of $\mathscr{P}$ is homeomorphic to some member of this sequence.

For each pair (i, j) of positive integers, the space $M(P_j, P_i)$ of maps of P_j into P_i (with the compact-open topology) is separable. Hence there exists a sequence

$$\Phi_{ij} = (\phi_{ij}^1, \phi_{ij}^2, \phi_{ij}^3, \cdots)$$

such that for each n the tail

$$(\phi_{ij}^n, \phi_{ij}^{n+1}, \phi_{ij}^{n+2}, \cdots)$$

is dense in $M(P_j, P_i)$. (If $(f_1, f_2, f_3, \cdots)$ is dense in $M(P_j, P_i)$, then we can form the listing

$$(f_1, f_1, f_2, f_1, f_2, f_3, \cdots),$$

and then each tail is dense.)

LEMMA 1. *For each $\mathscr{P}$-like compactum* Y, *there exist sequences* $(n_1, n_2, \cdots)$ *and* $(m_1, m_2, \cdots)$ *of positive integers such that* Y *is homeomorphic to the inverse limit of the sequence*

$$P_{n_1} \xleftarrow{\phi_{n_1 n_2}^{m_1}} P_{n_2} \xleftarrow{\phi_{n_2 n_3}^{m_2}} P_{n_3} \xleftarrow{\phi_{n_3 n_4}^{m_3}} \cdots, \tag{3.2}$$

and such that, for each i,

$$m_i \geq n_i, \quad m_i \geq n_{i+1}, \quad m_{i+1} \geq m_i + 2. \tag{3.3}$$

Proof. By the proof of Theorem 1 in Mardešić and Segal [9] (see Remark 3, p. 154), Y is homeomorphic to the inverse limit of an inverse sequence of members of $\mathscr{P}$. Since every member of $\mathscr{P}$ is homeomoprhic to one of the P_i (in (3.1)), there exists a sequence $(n_1, n_2, \cdots)$ of positive integers such that Y is homeomorphic to an inverse limit

$$P_{n_1} \xleftarrow{f_1^2} P_{n_2} \xleftarrow{f_2^3} P_{n_3} \xleftarrow{f_3^4} \cdots. \tag{3.4}$$

Next we appeal to Theorem 2 in M. Brown [3]. This theorem essentially says the following. We still get the same (homeomorphic) limit in (3.4) if we modify the bonding maps f_i^{i+1} slightly. (However, the amount by which we are allowed to modify each f_i^{i+1} depends partly on how much we have already modified the previous maps f_j^{j+1} $(j < i)$.) In view of the fact that each tail of the sequence $\Phi_{n_i n_{i+1}}$ is dense in $M(P_{n_{i+1}}, P_{n_i})$, it is clear how to proceed and complete the proof of the lemma. (Note: The choice of the maps $\phi_{n_i n_{i+1}}^{m_i}$ is similar to the proof of Brown's Theorem 3, but we can not apply this theorem directly, because of the conditions (3.3) on the integers m_i.)

We shall construct an inverse limit sequence

$$X_1 \xleftarrow{g_1^2} X_2 \xleftarrow{g_2^3} X_3 \xleftarrow{g_3^4} \cdots$$

whose limit X_∞ will be our universal $\mathscr{P}$-like compactum. Each X_n will be a member of $\mathscr{P}$. The idea is outlined in the schematic diagram of Figure 1. The space X_1 will be simply P_1. We shall obtain X_2 and g_1^2: $X_2 \to X_1$ by taking an amalgamation (see Definition 1) of the maps

$$1: P_1 \to P_1, \quad \phi_{11}^1: P_1 \to P_1, \quad \phi_{12}^1: P_2 \to P_1.$$

Then g_2^3: $X_3 \to X_2$ is obtained by taking an amalgamation of a certain twelve-term sequence of maps. (See Figure 1.)

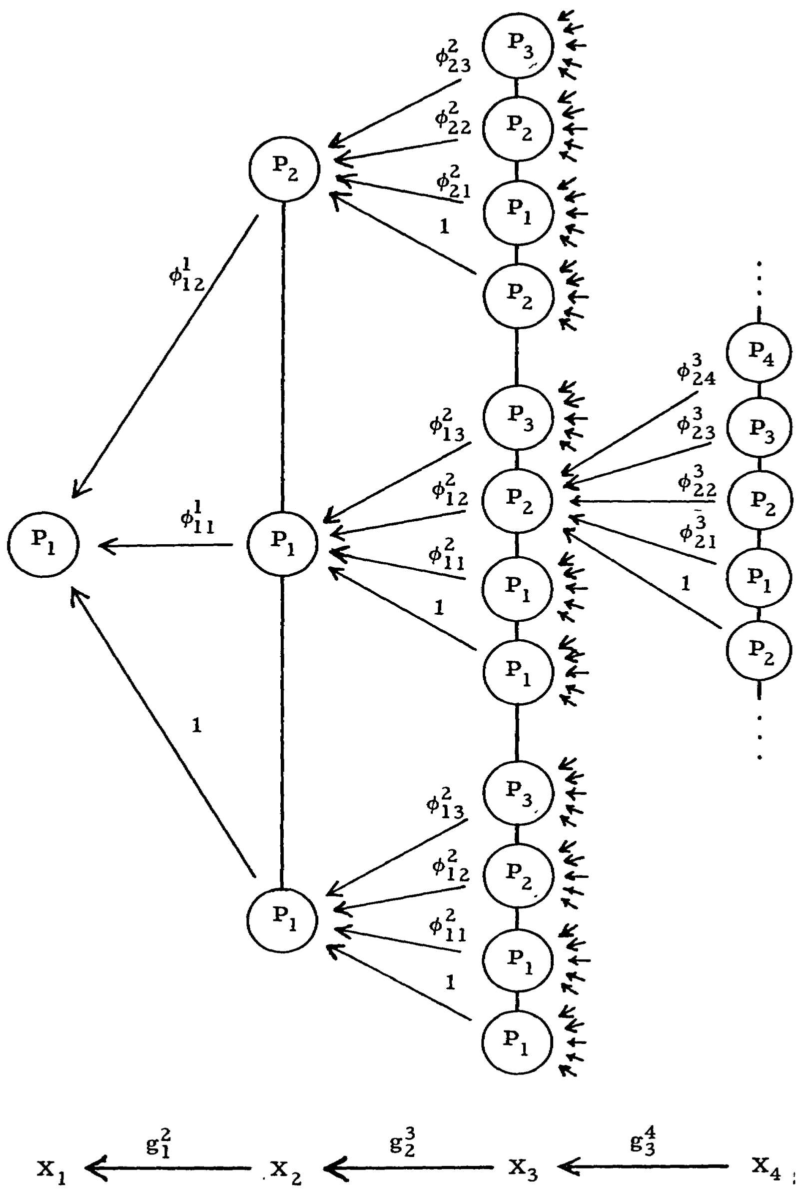

Figure 1.

In order to be precise, we shall now construct an *auxiliary* inverse limit sequence

$$S_1 \overset{\lambda^2}{\leftarrow} S_2 \overset{\lambda^3}{\leftarrow} S_3 \overset{\lambda^4}{\leftarrow} \cdots \tag{3.5}$$

which exactly expresses the combinatorial scheme of Figure 1. (For convenience, we write λ^n instead of the usual λ^n_{n-1}.) Each S_n will consist of a certain finite set of triples.

To construct (3.5) inductively, we let S_1 consist of a single element $s_1 = (P_1, -, -)$. (It does not matter what the second and third coordinates are.) Then we let

$$S_2 = \{(P_1, 1_{P_1}, s_1), (P_1, \phi^1_{11}, s_1), (P_2, \phi^1_{12}, s_1)\}$$

(please refer to Figure 1), and we let λ^2 map each member of S_2 onto s_1.

Suppose we have constructed

$$S_1 \overset{\lambda^2}{\leftarrow} S_2 \overset{\lambda^3}{\leftarrow} \cdots \overset{\lambda^n}{\leftarrow} S_n,$$

where $n \geq 2$. We now construct $\lambda^{n+1}: S_{n+1} \to S_n$. Let $s = (P_i, -, -)$ be a member of S_n, and let $(\lambda^{n+1})^{-1}(s)$ consist of the triples

$$(P_i, 1_{P_i}, s), (P_1, \phi^n_{i1}, s), (P_2, \phi^n_{i2}, s), \cdots, (P_{n+1}, \phi^n_{in+1}, s).$$

Then let $S_{n+1} = \bigcup_{s \in S_n} (\lambda^{n+1})^{-1}(s)$. This determines λ^{n+1} also.

For each n and each s in S_n, we let $K(s)$ denote the first coordinate of s (thus $K(s) = P_j$ for some $j \leq n$), and we let $\phi(s)$ denote the second coordinate of s. Thus, for $n \geq 2$, $\phi(s): K(s) \to K(\lambda^n(s))$; and if $K(s) = P_j$ and $K(\lambda^n(s)) = P_i$, then

$$\phi(s) = \phi^{n-1}_{ij} \text{ if } i \neq j \quad \text{and} \quad \phi(s) = \phi^{n-1}_{ii} \text{ or } 1_{P_i} \text{ if } i = j.$$

LEMMA 2. *Let the sequences of integers* $(n_1, n_2, \cdots)$ *and* $(m_1, m_2, \cdots)$ *satisfy the conditions* (3.3) *of Lemma* 1. *Then there is an element* $(s_1, s_2, \cdots)$ *of the inverse limit* S_∞ *of* (3.5) *such that for each* $i \geq 1$ *the following conditions hold:*

$$K(s_{m_i}) = P_{n_i}, \tag{3.6$_i$}$$

$$\phi(s_{m_i+1}) = \phi^{m_i}_{n_i n_{i+1}}, \tag{3.7$_i$}$$

$$K(s_j) = P_{n_{i+1}} \quad \textit{for all } j \textit{ satisfying } m_i < j \leq m_{i+1}, \tag{3.8$_i$}$$

$$\phi(s_j) = 1 = \textit{identity map on } P_{n_{i+1}} \tag{3.9$_i$}$$

for all j *satisfying* $m_i + 1 < j \leq m_{i+1}$.

Proof. Since $m_1 \geq n_1$, S_{m_1} contains triples s with $K(s) = P_{n_1}$. Choose any one of them and call it s_{m_1}. Then $(3.6)_1$ holds. Since also $m_1 \geq n_2$,

$$(P_{n_2}, \phi^{m_1}_{n_1 n_2}, s_{m_1})$$

is one of the inverse images of s_{m_1} under λ^{m_1+1}. Let this be s_{m_1+1}. Thus $(3.7)_1$ holds. Then we can take succeeding inverse images s_j ($j = m_1 + 2, m_1 + 3, \cdots, m_2$) (recall that $m_2 \geq m_1 + 2$) with $K(s_j) = P_{n_2}$ and $\phi(s_j)$ equal to the identity map on P_{n_2}. Thus $(3.8)_1$ and $(3.9)_1$ hold, as well as $(3.6)_2$ (which is included in $(3.8)_1$). But since $m_2 \geq n_3$, we can continue the process, obtaining s_j for all $j \geq m_1$, with $\lambda^{j+1}(s_{j+1}) = s_j$. The initial segment $(s_1, \cdots, s_{m_1-1})$ is then uniquely determined by the requirement that $(s_1, s_2, \cdots)$ lie in S_∞. This completes the proof.

The next lemma asserts the existence of an inverse limit sequence (X, g) with properties that will turn out to guarantee that its limit X_∞ is a universal $\mathscr{P}$-like compactum.

LEMMA 3. *There exists an inverse limit sequence*

$$X_1 \xleftarrow{g_1^2} X_2 \xleftarrow{g_2^3} X_3 \xleftarrow{g_3^4} \cdots$$

satisfying the following conditions: Each X_n *is a member of* $\mathscr{P}$. *Each bonding map* g_n^{n+1} *is onto. For each* n, *there exist imbeddings*

$$\mu(s)\colon K(s) \to X_n \qquad (\textit{all } s \in S_n) \tag{3.10}$$

such that for each $n \geq 2$ *and each* $s \in S_n$ *the following diagram commutes:*

$$(3.11)_n \qquad \begin{array}{ccc} K(\lambda^n(s)) & \xleftarrow{\ \phi(s)\ } & K(s) \\ \downarrow{\scriptstyle \mu(\lambda^n(s))} & & \downarrow{\scriptstyle \mu(s)} \\ X_{n-1} & \xleftarrow{\ g_{n-1}^n\ } & X_n \end{array}$$

Proof. Let $X_1 = P_1$, and let $\mu(s)\colon P_1 \to X_1$ be the identity, where s is the only element of S_1.

Suppose we have constructed

$$X_1 \xleftarrow{g_1^2} X_2 \xleftarrow{g_2^3} \cdots \xleftarrow{g_{m-1}^m} X_m$$

and the imbeddings (3.10) for $n \leq m$, where for $n \leq m$, $X_n \in \mathscr{P}$ and $(3.11)_n$ commutes. Consider the finite system of maps

$$(\mu(\lambda^{m+1}(s)) \circ \phi(s)\colon K(s) \to X_m)_{s \in S_{m+1}}.$$

Since $\mathscr{P}$ is amalgamable, there exist a member X_{m+1} of $\mathscr{P}$, a map g_m^{m+1} of X_{m+1} onto X_m, and imbeddings

$$\mu(s)\colon K(s) \to X_{m+1} \qquad (s \in S_{m+1})$$

such that $g_m^{m+1} \circ \mu(s) = \mu(\lambda^{m+1}(s)) \circ \phi(s)$ for all s in S_{m+1}. In other words, the diagram $(3.11)_{m+1}$ commutes. This completes the proof of Lemma 3.

Completion of Proof of Theorem 1. Let (X, g) be an inverse limit sequence with the properties listed in Lemma 3. We claim that its limit X_∞ is a universal $\mathscr{P}$-like compactum. First of all, since each $X_n \in \mathscr{P}$ and the bonding maps g_n^{n+1} are onto, X_∞ is itself $\mathscr{P}$-like, by Lemma 1 of Mardešić and Segal [9].

Now let Y be any $\mathscr{P}$-like compactum. Choose sequences of integers $(n_1, n_2, \cdots)$ and $(m_1, m_2, \cdots)$ in accordance with Lemma 1. We may suppose that Y is equal to the limit of the sequence (3.2) of Lemma 1. Then choose an element $(s_1, s_2, \cdots)$ of S_∞ in accordance with Lemma 2. Since $\lambda^n(s_n) = s_{n-1}$ for each $n \geq 2$, (3.11) implies that the diagram

$$\begin{array}{ccccccc} K(s_1) & \xleftarrow{\phi(s_2)} & K(s_2) & \xleftarrow{\phi(s_3)} & K(s_3) & \xleftarrow{\phi(s_4)} & \cdots \\ \downarrow{\scriptstyle \mu(s_1)} & & \downarrow{\scriptstyle \mu(s_2)} & & \downarrow{\scriptstyle \mu(s_3)} & & \\ X_1 & \xleftarrow{g_1^2} & X_2 & \xleftarrow{g_2^3} & X_3 & \xleftarrow{g_3^4} & \cdots \end{array} \tag{3.12}$$

is commutative. By conditions (3.6) to (3.9), the top row of this diagram starting with level m_1, is equal to

$$P_{n_1} \xleftarrow{\phi_{n_1 n_2}^{m_1}} P_{n_2} \xleftarrow{1} P_{n_2} \xleftarrow{1} \cdots \xleftarrow{1} P_{n_2} \xleftarrow{\phi_{n_2 n_3}^{m_2}} P_{n_3} \xleftarrow{1} \cdots .$$

Taking the obvious subsequence of this sequence, we see that its limit is homeomorphic to Y (see [4, Corollary 3.16, p. 220]). From the commutativity of (3.12) we see that the imbeddings $\mu(s_n)$ then induce an imbedding of Y into X_∞. This completes the proof.

4. PROOF OF THEOREM 2

The proofs of parts (2) to (6) can essentially be given simultaneously; that is, we describe a construction that works for all of these cases. Let $(P_1, \cdots, P_n)$ and $\phi_i\colon P_i \to Q$ be given. Let P' be a disjoint union $P_1' \cup \cdots \cup P_n'$, where P_i' is a homeomorphic copy of P_i. Let T be an n-od; that is, let T be the union of arcs $A_1, \cdots, A_n$ with a common endpoint p such that $A_i \cap A_j = \{p\}$ whenever $i \neq j$. Let the other endpoint of A_i be a_i. For each i, choose a vertex a_i' of P_i', and define a map $f\colon \{a_1, \cdots, a_n\} \to P'$ by $f(a_i) = a_i'$. Finally, let P be the adjunction space

$$P = T \cup_f P'.$$

We shall identify P' with its image in P, and T with its image in P. Let $\mu_i\colon P_i \to P$ be an imbedding such that $\mu_i(P_i) = P_i'$. Now define $g_0\colon P' \to Q$ by

$$g_0(x) = \phi_i \mu_i^{-1}(x) \qquad \text{for } x \in P_i'.$$

We can extend g_0 to a map g of P onto Q, provided Q is connected, as it is in each of the cases (2) to (6). To do this, we first take a segment S in the interior of, say, the arc A_1, and a map g_1 of S *onto* Q (which exists, by the Hahn-Mazurkiewicz Theorem). Since Q is arcwise connected, we can extend $g_0 \cup g_1$ to a map $g: P \to Q$. Since g extends g_0, $g\mu_i = \phi_i$ for all i.

If each P_i is acyclic, it is easy to see from the Mayer-Vietoris sequence that P is acyclic. If each P_i is contractible, it is easy to see that P is contractible. If in addition to one of these two cases each P_i is of dimension k (at most k), then P is of dimension k (at most k), except in the case k = 0, which is trivial. From these statements, parts (2) to (6) follow.

To take care of case (1), we modify the above construction as follows: As before, we take the disjoint union $P' = P_1' \cup \cdots \cup P_n'$. But instead of adjoining the n-od T, we obtain P by adding on as many disjoint k-cells as there are components of Q, and we use these to get the map g of P onto Q. Actually, looking at the *proof* of Theorem 1, we see that is is unnecessary to have g onto, since every limit of an inverse sequence of polyhedra of dimension at most k is a compactum of dimension at most k. It might also be added that we can use the proof of Theorem 1 and essentially the same construction as in the preceding paragraph to get a *connected* universal k-dimensional compactum.

There remains only part (7). Let I^{k-1} be the standard (k - 1)-cube. Let P be the k-cell $[1, 2n] \times I^{k-1}$, and let $P_i' \subset P$ be the k-cell $[2i - 1, 2i] \times I^{k-1}$. Choose imbeddings $\mu_i: P_i \to P$ with $\mu_i(P_i) = P_i'$. Define a map $g_0: \bigcup P_i' \to Q$ by

$$g_0(x) = \phi_i \mu_i^{-1}(x) \qquad (x \in P_i').$$

Using the Tietze Extension Theorem, we easily see that we can extend g_0 to a map g of P *onto* Q. This completes the proof.

5. PROOF OF THEOREM 5

Theorem 5 consists of Lemma 4, Example 1, and Lemma 7 of this section. It is obvious that if a compactum X is given a specific metric, then X is $\mathscr{P}$-like if and only if for each $\varepsilon > 0$ there exists an ε-*map* of X onto some member of $\mathscr{P}$ (the diameters of all point-inverses are less than ε); and X is weakly $\mathscr{P}$-like if and only if for every $\varepsilon > 0$ there exists an open cover of mesh less than ε whose nerve is homeomorphic to a member of $\mathscr{P}$. Here we are thinking of open covers as collections of subsets rather than indexed systems of subsets as in [4]. If α is a finite open cover, let $N(\alpha)$ denote the nerve of α, and let $|N(\alpha)|$ denote the standard underlying polyhedron.

LEMMA 4. *For any class $\mathscr{P}$ of polyhedra, every $\mathscr{P}$-like compactum* X *is weakly $\mathscr{P}$-like.*

Proof. Let the $\mathscr{P}$-like compactum X be given a metric d. Let ε be a positive number. Choose an $(\varepsilon/2)$-map $f: X \to |K|$, where K is a rectilinear Euclidean complex and $|K|$ is homeomorphic to a member of $\mathscr{P}$. It is easy to see from the compactness of X and $|K|$ that there exists a $\delta > 0$ such that whenever $U \subset |K|$ and $\text{diam}(U) < \delta$, then $\text{diam } f^{-1}(U) < \varepsilon$.

Let $K^{(n)}$ denote the n^{th} barycentric subdivision of K, and let β_n be the corresponding cover of $|K^{(n)}| = |K|$ by open stars of vertices. We know that $N(\beta_n)$ is isomorphic to $K^{(n)}$, so that $|N(\beta_n)| \equiv |K|$. By taking n large enough, we can make the diameters of the members of β_n smaller than δ. Let $\alpha_n = f^{-1}(\beta_n)$. Since f is onto, α_n is an open cover of X such that $N(\alpha_n)$ is isomorphic to $N(\beta_n)$. Thus $|N(\alpha_n)| \equiv |K|$. By the choice of δ, mesh $\alpha_n < \varepsilon$. This completes the proof.

Example 1. Let the class $\mathscr{P}$ be the same as in Example 5 of Mardešić and Segal [9] (used there for a different purpose): $\mathscr{P} = \{P_1, P_2, \cdots\}$, where

$$P_n = [0, 1] \cup \{2, 3, 4, \cdots, n+1\}.$$

Let $X = \{0, 1, 1/2, 1/3, \cdots\}$. (These are all subspaces of the real line.) Obviously, X is not $\mathscr{P}$-like, for since X is countable and each P_n is uncountable, X cannot be mapped *onto* any P_n. However, X is weakly $\mathscr{P}$-like: The open cover

$$\alpha_n = \left\{ \left\{0, \frac{1}{n+1}, \frac{1}{n+2}, \cdots\right\}, \{1\}, \left\{\frac{1}{2}\right\}, \cdots, \left\{\frac{1}{n+1}\right\}\right\}$$

has nerve homeomorphic to P_n, and mesh $(\alpha_n) \to 0$. Notice that this X is not perfect (it has isolated points).

LEMMA 5. *Let* X *be a perfect compactum, and* σ *a simplex. Then for every nonempty open subset* U *of* X *there exist a compact set* C *contained in* U *and a map* ϕ *of* C *onto* σ.

Proof. Let V be a nonempty open subset of X such that $\overline{V} \subset U$. Then $\overline{V}$ is a perfect compactum. If $\overline{V}$ is totally disconnected, then $\overline{V}$ is a Cantor set (see [6, p. 100]); hence $\overline{V}$ can be mapped onto σ (see [6, p. 127]). Otherwise, let C be a nondegenerate component of $\overline{V}$. Then C can be mapped onto a closed interval of the real line, which in turn can be mapped onto σ by the Hahn-Mazurkiewicz Theorem.

If X is a space and α is a finite open cover of X, a map $f: X \to |N(\alpha)|$ is called *canonical* if $f(x) \in \Delta_\alpha(x)$ for each x in X, where $\Delta_\alpha(x)$ is the simplex of $N(\alpha)$ spanned by all $U \in \alpha$ such that $x \in U$. In case X is normal, there always exist such canonical maps [4, p. 286].

LEMMA 6. *If* X *is a perfect compactum and* α *is a finite open cover of* X, *then there exists a canonical map of* X *onto* $|N(\alpha)|$.

Proof. First we prove that if $f: X \to |N(\alpha)|$ is a canonical map and σ is any simplex of $N(\alpha)$, then there exists a canonical map $f': X \to |N(\alpha)|$ such that $f'(X) \supset f(X) \cup \sigma$. We may suppose that σ is a principal simplex. Let its vertices be $\{U_0, \cdots, U_m\}$, and let $U = U_0 \cap \cdots \cap U_m$. Observe that $\sigma = \Delta_\alpha(x)$ for each x in U, so that $f(U) \subset \sigma$. Hence also $f(\mathrm{Bd}\, U) \subset f(\overline{U}) \subset \sigma$.

Since U is a nonempty open subset of X, there exist (by Lemma 5) a compact subset C of U and a map ϕ of C onto σ. Now C and Bd U are disjoint closed sets in $\overline{U}$, and by the Tietze Extension Theorem, there exists a map f_0 of $\overline{U}$ onto σ that agrees with ϕ on C and with f on Bd U.

Define $f': X \to |N(\alpha)|$ by

$$f'(x) = f(x) \quad \text{if } x \in X - U \qquad \text{and} \qquad f'(x) = f_0(x) \quad \text{if } x \in \overline{U}.$$

Notice that f' is well-defined and continuous, since f and f_0 agree on $(X - U) \cap \overline{U} = \mathrm{Bd}\, U$. Now $f'(X) \supset \sigma$, since f' agrees with ϕ on C. Thus

$f'(X) \supset f(X) \cup \sigma$, for if $y \in f(X) - \sigma$, say $y = f(x)$, then $x \notin U$; hence $y = f'(x)$. Finally, f' is canonical. For if $x \notin U$, then $f'(x) = f(x) \in \triangle_\alpha(x)$; and if $x \in U$, then $f'(x) = f_0(x) \in \sigma = \triangle_\alpha(x)$.

Now it is clear how to complete the proof of the lemma. We begin with a canonical map of X into $|N(\alpha)|$. Since $N(\alpha)$ has only a finite number of simplexes, we can use finite induction, with the preceding argument for the induction step, to get a canonical map of X onto $|N(\alpha)|$.

LEMMA 7. *If* X *is a perfect, weakly* $\mathscr{P}$*-like compactum, then* X *is* $\mathscr{P}$*-like.*

Proof. Let α be an open cover of X. Since X is weakly $\mathscr{P}$-like, there exists a refinement β of α such that $|N(\beta)|$ is homeomorphic to a member Y of $\mathscr{P}$. By the preceding lemma, there exists a canonical map f of X onto $|N(\beta)|$. Therefore f is a β-map. (If $y \in |N(\beta)|$, then $y \in \cap \{\triangle_\beta(x): x \in f^{-1}(y)\}$, so that the latter set is a nonempty simplex, of which we take $U \in \beta$ to be a vertex. Then $f^{-1}(y) \subset U$.) Hence f is also an α-map, so that there exists an α-map of X onto Y. This completes the proof.

6. PROOF OF THEOREM 3

In this section, let $\mathscr{M}^n$ be a class of closed, connected, triangulable n-manifolds. $H_q(X; Z_2)$ will denote the q-dimensional Čech homology group of the space X, with coefficients in Z_2 (integers mod 2).

LEMMA 8. *If* X *is an* $\mathscr{M}^n$*-like continuum and* A *is a proper closed subset of* X, *then* $H_n(A; Z_2) = 0$.

Remark. A result similar to this is in [5], for the case where X is an ANR.

Proof. Choose a metric d for X. Let ε be a positive number. Choose a point x in X - A, and let $\delta = d(x, A) > 0$. Since X is $\mathscr{M}^n$-like, there exists by Lemma 4 an open cover α of X with mesh $\alpha < \min(\varepsilon, \delta)$ such that $|N(\alpha)|$ is homeomorphic to a member of $\mathscr{M}^n$. Now there exists a member U_x of α such that $x \in U_x$. Since diam $U_x < \delta$, $U_x \cap A = \emptyset$.

The collection $\alpha' = \{U \cap A: U \in \alpha, U \cap A \neq \emptyset\}$ is an open cover of A of mesh less than ε. We claim that $N(\alpha')$ is isomorphic to a proper subcomplex of $N(\alpha)$. Select a function $\phi: \alpha' \to \alpha$ such that for each $V \in \alpha'$, $V = \phi(V) \cap A$. Obviously, ϕ is one-to-one. It is easy to see then that ϕ defines a one-to-one simplicial map of $N(\alpha')$ into $N(\alpha)$, which is not onto, since U_x is not in the image of ϕ. Since $|N(\alpha)|$ is a connected n-manifold and $|N(\alpha')|$ is homeomorphic to a proper subpolyhedron, $H_n(|N(\alpha')|; Z_2) = 0$. Since ε was arbitrary, it follows from the definition of Čech homology that $H_n(A; Z_2) = 0$.

Alternate proof. By [9] we may assume that X is the limit of an inverse sequence (X_i, f_i^{i+1}), where each X_i is in $\mathscr{M}^n$. For each i, let $A_i = f_i(A) \subset X_i$. It is easy to see from the compactness of A that A is the inverse limit of the restricted inverse sequence $(A_i, f_i^{i+1} \mid A_{i+1})$. Since A is a proper subset of X, we may assume that each A_i is a proper (closed) subset of X_i. Since X_i is a closed, connected (triangulable) n-manifold, $H_n(A_i; Z_2) = 0$. (This follows by taking sufficiently fine triangulations of M^n.) Hence by the continuity theorem for Čech homology [4, p. 261], $H_n(A; Z_2) = 0$.

To complete the proof of Theorem 3, we shall assume the following theorem (to be proved in Section 7).

THEOREM 6. *If* M^n *is a closed, connected* n*-manifold, then there exists an* M^n*-like continuum* M_0^n *that is not locally connected, but such that*

$$H_*(M_0^n; Z_2) \approx H_*(M^n; Z_2).$$

Now suppose there exists a universal $\mathscr{M}^n$-like continuum X. Choose a member M^n of $\mathscr{M}^n$, and let M_0^n be a continuum with the properties described in Theorem 6. Obviously, M^n and M_0^n are $\mathscr{M}^n$-like. By universality, X contains subsets A and A_0 homeomorphic to M^n and M_0^n, respectively. Since

$$H_n(M_0^n; Z_2) \approx H_n(M^n; Z_2) \approx Z_2,$$

it follows from Lemma 8 that $A = A_0 = X$, so that $M^n = M_0^n$. But this is impossible, since M_0^n is not locally connected.

7. PROOF OF THEOREM 6

By a *relative* n*-cell* we shall mean a pair consisting of an n-cell and its boundary.

LEMMA 9. *There exists a pair* (D^*, S^*) *such that* D^* *is a continuum which is not locally connected at some of the points of* $D^* - S^*$ *and which has the following property: for any relative* n*-cell* (D, S) *there is a homeomorphism* $\phi: S^* \to S$ *such that for each open cover* α *of* D^* *there is an* α*-map* f *of* D^* *onto* D *with* $f \mid S^* = \phi$. *(In particular,* D^* *is* n*-cell-like.)*

Proof. For each positive integer q, let R^q denote Euclidean q-space with the usual norm. If $x_0 \in R^n$ and $r > 0$, let $D^n(x_0, r)$ be the n-disk

$$\{x \in R^n: \|x - x_0\| \leq r\}.$$

Let $D^n = D^n(O, 1)$, and let $S^{n-1} = \mathrm{Bd}\, D^n$. We shall specify a certain sequence $(D_0, D_1, D_2, \cdots)$ of disjoint n-disks in the interior of D^n, with respective boundary (n - 1)-spheres $(S_0, S_1, S_2, \cdots)$. Let $D_0 = D^n((-1/4, 0, \cdots, 0), 1/4)$, and for each $i \geq 1$ let

$$D_i = D^n((2^{-i}, 0, \cdots, 0), 2^{-(i+2)}).$$

Note that the origin O belongs to S_0, and that the sequence $(D_1, D_2, \cdots)$ converges to $\{O\}$.

For each ε $(0 < \varepsilon < 1/2)$, let D_ε be the disk $D^n((\varepsilon, 0, \cdots, 0), \varepsilon)$. If $0 < \delta < \varepsilon$, it is easy to see that there exists a map ψ_δ^ε of D_ε onto itself which is the identity on Bd D_ε but which maps D_δ onto $\{O\}$. Let Ψ_δ^ε be the map of D^n onto itself that agrees with ψ_δ^ε on D_ε and is the identity on $D^n - D_\varepsilon$. Note that Ψ_δ^ε is a 2ε-map.

For each $i \geq 1$, choose ε and δ such that $0 < \delta < \varepsilon < 1/2$ and

$$\bigcup_{j=i+1}^{\infty} D_j \subset D_\delta \subset D_\varepsilon \subset D^n - \bigcup_{j=1}^{i} D_j,$$

and let $g_i = \Psi_\delta^\varepsilon$. Then g_i is a 2^{-i}-map of D^n onto itself.

We shall construct D^* as a subset of $R^{n+1} = R^n \times R^1$. Let us identify R^n with $R^n \times \{0\}$ in R^{n+1}. For each $i \geq 0$, let C_i be the "inverted tin can" (picturing the case $n = 2$)

$$C_i = (S_i \times [0, 1]) \cup (D_i \times \{1\});$$

for each $i \geq 1$, let

$$D_i^* = \left(D^n - \bigcup_{j=0}^{i} D_j\right) \cup \bigcup_{j=0}^{i} C_j;$$

and finally, let

$$D^* = \left(D^n - \bigcup_{j=0}^{\infty} D_j\right) \cup \bigcup_{j=0}^{\infty} C_j.$$

We let S^* be simply S^{n-1}. It is easy to see that D_i^* is an n-cell with boundary S^* and that D^* is not locally connected at any point of the segment $\{O\} \times [0, 1] \subset C_0$.

To complete the proof of the lemma, let (D, S) be any relative n-cell, and choose a homeomorphism $\phi\colon S^* \to S$. For each $i \geq 1$, there clearly exists a homeomorphism $\phi_i\colon (D_i^*, S^*) \to (D, S)$ such that $\phi_i \mid S^* = \phi$. Now let $\varepsilon > 0$. Choose i so that $2^{-i} < \varepsilon$, and let $g_i\colon D^n \to D^n$ be as above. Define $g_i^*\colon (D^*, S^*) \to (D_i^*, S^*)$ as follows: If $(x, t) \in D^*$, where $x \in D^n$ and $t \in [0, 1]$, let $g_i^*(x, t) = (g_i(x), t)$. It is easy to see from the choice of g_i that g_i^* maps D^* onto D_i^* and that g_i^* is the identity on S^*. Furthermore, g_i^* is an ε-map. For if $g_i^*(x, t) = g_i^*(x', t')$, then $g_i(x) = g_i(x')$ and $t = t'$, so that $\|x - x'\| < \varepsilon$. Hence

$$\|(x, t) - (x', t')\| = \|x - x'\| < \varepsilon.$$

Finally, let $f = \phi_i g_i^*\colon (D^*, S^*) \to (D, S)$. Since ϕ_i is a homeomorphism, f is an ε-map. Moreover, if $x \in S^*$, then $f(x) = \phi_i g_i^*(x) = \phi_i(x) = \phi(x)$. This completes the proof of the lemma.

To prove Theorem 6, let M^n be a closed, connected n-manifold. Select a relative n-cell (D, S) in M^n, and let (D^*, S^*), $\phi\colon S^* \to S$ have the properties described in Lemma 9. Let $E = M^n - \text{int } D$, and let

$$M_0^n = D^* \cup_\phi E,$$

(this is the continuum obtained by attaching D^* to E by the homeomorphism ϕ; briefly speaking, we have replaced D by D^*). Let $p\colon (D^* \cup E) \to M_0^n$ be the identification map, and let

$$D_0^* = p(D^*), \qquad E_0 = p(E), \qquad S_0 = p(S^*) = p(S) = D_0^* \cap E_0.$$

Obviously, M_0^n is not locally connected at some points of D_0^*. It is easy to see that for every open cover α of M_0^n there exists an α-map $F\colon M_0^n \to M^n$ that maps the triad $(M_0^n; D_0^*, E_0)$ onto the triad $(M^n; D, E)$, carrying the pair (E_0, S_0) homeomorphically onto the pair (E, S). (Define F by

$$F(p(x)) = x \ (x \in E) \quad \text{and} \quad F(p(x)) = f(x) \ (x \in D^*),$$

where f: $D^* \to D$ is obtained from Lemma 9.) In particular, M_0^n is M^n-like.

To complete the proof, choose F as above. (It does not matter how fine α is.) In the remainder of this section, we write $H_q(X)$ for the reduced group $\tilde{H}_q(X; Z_2)$. By [4, p. 41], F induces a map of the reduced Mayer-Vietoris sequence of the triad $(M_0^n; D_0^*, E_0)$ into that of the triad $(M^n; D, E)$. (These sequences are exact, by [4, pp. 248 and 266].) Thus, letting $F_1 = F \mid D_0^*$, $F_2 = F \mid E_0$, and $F_3 = F \mid S_0$, we have for each integer q the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} H_{q-1}(D_0^*) \oplus H_{q-1}(E_0) & \leftarrow & H_{q-1}(S_0) & \leftarrow & H_q(M_0^n) & \leftarrow & H_q(D_0^*) \oplus H_q(E_0) & \leftarrow & H_q(S_0) \\ \downarrow F_{1*} \oplus F_{2*} & & \downarrow F_{3*} & & \downarrow F_* & & \downarrow F_{1*} \oplus F_{2*} & & \downarrow F_{3*} \\ H_{q-1}(D) \oplus H_{q-1}(E) & \leftarrow & H_{q-1}(S) & \leftarrow & H_q(M^n) & \leftarrow & H_q(D) \oplus H_q(E) & \leftarrow & H_q(S) \end{array}$$

Now all the vertical arrows except possibly the middle one represent isomorphisms, for the following reasons. Since D_0^* is D^n-like, $H_*(D_0^*) = 0$. Since $D \equiv D^n$, $H_*(D) = 0$. The maps F_2 and F_3 are homeomorphisms. Thus by the "five lemma," the middle arrow also represents an isomorphism. This completes the proof.

8. PROOF OF THEOREM 4

LEMMA 10. *Let* (G, ϕ) *be a direct system of abelian groups over a directed set* M, *with limit group* G^∞ (see [4, p. 220]). (1) *If* G^m *is torsion-free for each* $m \in M$, *then* G^∞ *is torsion-free.* (2) *If there exists an integer* k *such that* rank $G^m \leq k$ *for each* $m \in M$, *then* rank $G^\infty \leq k$.

The proof is a straightforward application of the definitions, and we omit it. For the use of *rank*, see [4, p. 52].

In the following, let $H^q(X)$ denote the q-dimensional Čech cohomology group of X with integer coefficients (Z). Also, if X is a polyhedron, let $R_q(X)$ be the q-dimensional Betti number of X ($R_q(X)$ = rank $H_q(X)$ = rank $H^q(X)$). For each integer $q \geq 0$, let Z^q denote the direct sum of the group Z, taken q times.

Now let $\mathscr{P}$ be a class that satisfies the conditions assumed in Theorem 4. Let $k > 0$ be the maximum value of $R_1(X)$ ($X \in \mathscr{P}$), and choose a fixed $P \in \mathscr{P}$ with $R_1(P) = k$.

LEMMA 11. *If* X *is any* $\mathscr{P}$*-like compactum, then* $H^1(X)$ *is a torsion-free group of rank at most* k.

Proof. Applying Lemma 4, the definition of Čech cohomology, and [4, Corollary 4.14, p. 224] (on cofinal subsystems) we see that $H^1(X)$ is isomorphic to the limit of a direct sequence $H^1(P_1) \to H^1(P_2) \to \cdots$, where each P_n belongs to $\mathscr{P}$. Now Lemma 10 is applicable.

LEMMA 12. *If* X *is a* $\mathscr{P}$*-like compactum and* A *is a compactum that can be imbedded in* X, *then* $H^1(A)$ *is isomorphic to a quotient group of* $H^1(X)$.

Proof. We may suppose that $A \subset X$. Since X is 1-dimensional, $H^2(X, A) = 0$. Hence, by the cohomology exact sequence of (X, A), the homomorphism $H^1(X) \to H^1(A)$ induced by inclusion is *onto*.

Now suppose there does exist a universal $\mathscr{P}$-like compactum Y.

LEMMA 13. $H^1(Y) \approx Z^k$.

Proof. P itself is $\mathscr{P}$-like. Hence P can be imbedded in Y. By Lemma 12, $Z^k \approx H^1(P) \approx H^1(Y)/L$, where L is a subgroup of $H^1(Y)$. By Lemma 11, rank $H^1(Y) \leq k$. Hence

$$k \geq \text{rank } H^1(Y) = \text{rank}(H^1(Y)/L) + \text{rank } L = k + \text{rank } L \geq k.$$

Hence rank L = 0. Since $H^1(Y)$ is torsion-free, this implies L = 0. Hence $Z^k \approx H^1(Y)$.

Since $R_1(P) > 0$, P contains a simple closed curve S. It is easy to see that there exists a map f of P onto P which is the identity on $\overline{P - S}$ and which maps S onto S in such a way that $f \mid S$ is of degree 2. Let P_∞ be the limit of the inverse sequence

$$P \xleftarrow{f} P \xleftarrow{f} P \xleftarrow{f} \cdots. \tag{8.1}$$

Since f is onto, P_∞ is P-like [9, Lemma 1], hence $\mathscr{P}$-like. Hence P_∞ can be imbedded in Y. Now the sequence

$$S \xleftarrow{f|S} S \xleftarrow{f|S} S \xleftarrow{f|S} \cdots \tag{8.2}$$

imbeds in the sequence (8.1), so that its limit S_∞ imbeds in P_∞, therefore also imbeds in Y. Then, by Lemmas 12 and 13, $H^1(S_\infty)$ is isomorphic to a quotient group of Z^k, hence is finitely generated. On the other hand, since degree$(f \mid S) = 2$, we see, by applying the continuity theorem for Čech cohomology to (8.2) that $H^1(S_\infty)$ is isomorphic to the group of dyadic rationals, which is not finitely generated. This contradiction completes the proof.

REFERENCES

1. R. H. Bing, *Snake-like continua,* Duke Math. J. 18 (1951), 653-663.
2. ———, *Embedding circle-like continua in the plane,* Canad. J. Math. 14 (1962), 113-128.
3. M. Brown, *Some applications of an approximation theorem for inverse limits,* Proc. Amer. Math. Soc. 11 (1960), 478-483.
4. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology,* Princeton Univ. Press, Princeton, N. J., 1952.
5. T. Ganea, *On ε-maps onto manifolds,* Fund. Math. 47 (1959), 35-44.
6. J. G. Hocking and G. S. Young, *Topology,* Addison-Wesley, Reading, Mass., 1961.
7. W. Hurewicz and H. Wallman, *Dimension theory,* Princeton Univ. Press, Princeton, N. J., 1941.
8. S. Lefschetz, *On compact spaces,* Ann. of Math. (2) 32 (1931), 521-538.
9. S. Mardešić and J. Segal, *ε-mappings onto polyhedra,* Trans. Amer. Math. Soc. 109 (1963), 146-164.
10. M. C. McCord, *Universal* P-*like compacta,* Notices Amer. Math. Soc. 11 (1964), 593, Abstract 64T-396.

11. K. Menger, *Über umfassendste* n-*dimensionale Mengen*, Proc. Akad. Wetensch. Amsterdam 29 (1926), 1125-1128.

12. G. Nöbeling, *Über eine* n-*dimensionale Universalmenge im* R_{2n+1}, Math. Ann. 104 (1931), 71-80.

13. R. M. Schori, *A universal snake-like continuum*, Notices Amer. Math. Soc. 11 (1964), 351, Abstract 612-5.

14. ———, *A universal snake-like continuum*, Proc. Amer. Math. Soc. (to appear).

The University of Wisconsin
Madison, Wisconsin

The University of Georgia
Athens, Georgia