SPLITTING AND DECOMPOSITION BY REGRESSIVE SETS

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1. INTRODUCTION

In [5] there appears a very simple proof of the fact that indecomposable number sets need not be cohesive. In the present paper we show, with a bit more effort but along the same general lines, that some noncohesive sets of numbers cannot be decomposed (in the generous sense defined in Section 2) by any pair of *regressive* sets. We further show that some *cohesive* sets can be split by regressive sets but cannot be decomposed (in the sense of Section 2) by pairs of regressive sets, and that there exist sets decomposable by pairs of r. e. sets but not by any pair of *retraceable* sets. As to notation, we follow [5]; special terminology not explicitly defined below (such as "cohesive", "retraceable", "regressive") has the same meaning as in [2], [3], and [5].

2. DEFINITIONS AND PRELIMINARY LEMMAS

LEMMA 1. *Let* α *be an infinite set of numbers. Then* α *has at most* $\aleph_0$ *regressive supersets.*

Proof. If $\alpha \subset \beta$ and β is regressive, then p_n regresses β for some n, where $\{p_n\}$ is some enumeration of the partial recursive functions of one variable. It is easily seen (and has been noted in [3] and [7] for the special case of *retracing* functions) that if p_n regresses two *distinct* infinite sets γ_1 and γ_2, then γ_1 and γ_2 must have finite intersection. Thus for any n, p_n regresses at most one superset of α, and the lemma follows.

Definition 1. A set α of numbers is *supercohesive* if and only if it is infinite and no regressive set β splits α.

Remark. The existence of retraceable sets with immune complements ([6]) implies that not all cohesive sets are supercohesive.

LEMMA 2. *Every infinite set of numbers has a supercohesive subset.*

Proof. Let α be an infinite set of numbers; with the aid of the axiom of choice, we shall extract a supercohesive subset from α. If γ is any set of numbers, let $I(\gamma, p_n)$ be the collection of all infinite subsets β of γ such that

$$(\exists\, \alpha)(p_n \text{ regresses } \alpha \text{ and } \beta = \gamma \cap \alpha).$$

Let $\mathscr{I}$ be the class of all *nonempty* $I(\gamma, p_n)$ as γ and n vary. Let F be a choice function for $\mathscr{I}$. We define a sequence of subsets of α as follows:

$$\alpha_0 = \alpha,$$

$$\alpha_{n+1} = \begin{cases} \alpha_n \cap F(I(\alpha_n, p_n)) & \text{if } I(\alpha_n, p_n) \neq \Box, \\ \alpha_n \text{ otherwise}. \end{cases}$$

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Then $\{\alpha_j\}$ is a nested sequence of infinite subsets of α. Let $t_0, t_1, t_2, \cdots$ be a nonrepeating sequence of numbers such that $t_j \in \alpha_j$ for every j; then $\tau = \{t_0, t_1, t_2, \cdots\}$ is a supercohesive subset of α, as we leave for the reader to verify in detail.

COROLLARY 1. *There are continuum many permutations* p *of the natural numbers such that both* p *and* p^{-1} *permute the regressive sets among themselves, preserving recursiveness, recursive enumerability, and retraceability.*

LEMMA 3 ([2]). *Let* α *be an infinite set of numbers; then* α *has an infinite regressive subset if and only if it has an infinite retraceable subset.*

LEMMA 4 ([7]). *An* r.e. *set* β *is hyper-hypersimple if and only if* $\bar{\beta}$ *is infinite and has no infinite retraceable subset.*

LEMMA 5 (see [1, Theorem 4]). *Let* $\alpha_1, \alpha_2, \cdots, \alpha_k$ *be regressive sets such that* $\bigcap_{j=1}^{k} \alpha_j$ *is infinite. Then* $\bigcap_{j=1}^{k} \alpha_j$ *has an infinite regressive subset.*

Proof. The case k = 2 is Theorem 4 of [1]. In order to establish the general case, we only need rewrite the proof of [1, Theorem 4], using the following definition of loop, which applies to any $k \geq 2$: let $p_{n_1}, \cdots, p_{n_k}$ be the regressing functions of $\alpha_1, \cdots, \alpha_k$, respectively, that have been chosen for use in the proof; then a *loop* is a pair (x_0, y_0) of numbers such that x_0 is in the domain of each p_{n_j} $(1 \leq j \leq k)$, each p_{n_j} leads x_0 to y_0 in a finite number of iterations, and there is no z for which $x_0 \neq z$, $z \neq y_0$, and all p_{n_j} lead z to y_0 in a finite number of iterations and lead x_0 to z in a finite number of iterations. For the remainder of the proof, the reader is referred to the proof of Theorem 4 in [1].

LEMMA 6 (see [1, Theorem 1]). *The union of any two immune regressive sets is immune.*

Proof. Since regressiveness is preserved both by recursive equivalence and by intersection with recursive sets ([2]), it suffices to show that the set of all natural numbers is not the union of two immune regressive sets. But for this we may use, with only trivial changes, the alternate proof of [1, Theorem 1] appearing in Section 4 of [1].

LEMMA 7 ([5]). *Let* β *be a nonrecursive set of numbers; then* β *has an infinite subset* α *such that if* $\gamma \subset \beta$ *and* γ *is recursive, then* $\gamma \cap \alpha$ *is finite.*

Remark. Lemma 7 is easily generalized; however, we shall need only the simple assertion above, concerning recursive subsets of a nonrecursive set.

The next lemma is an extension of a theorem of Friedberg. It is due to J. P. Cleave and C. E. M. Yates, but the proof (a modification of the proof of Theorem 1 in [4]) has not yet been published.

LEMMA 8 (Cleave, Yates). *Let* β *be any* r.e., *nonrecursive set of numbers; then there exist a recursive function* f(x) *and a two-place recursive function* g(x, y) *such that, for all numbers* j *and* k,

(i) $j \neq k \Rightarrow W_{f(j)} \cap W_{f(k)} = \square$,

(ii) $\bigcup_k W_{f(k)} = \beta$,

(iii) $W_j \cap W_{f(k)} = \square \Rightarrow W_{g(j,k)} = (W_j - \beta) \cup$ *(a finite set)*.

We remark that the modifications required in the proof of [4, Theorem 1] can be carried out in such a way that condition (iii) of Lemma 8 is replaced by the stronger condition

(iii') $W_j \cap W_{f(k)}$ is finite $\Rightarrow W_{g(j,k)} = (W_j - \beta) \cup$ (a finite set).

The notion of *decomposability*, standard in the literature, is this: a set α of numbers is said to be *decomposable* if and only if there are two r.e. sets W_i and W_j such that $W_i \cap W_j = \square$, $\alpha \subset W_i \cup W_j$, and both $\alpha \cap W_i$ and $\alpha \cap W_j$ are infinite. But it is easily seen that an equivalent definition would read: $\alpha \subset W_i \cup W_j$, and both W_i and W_j split α. This observation accounts for our next definition:

Definition 2. α is *regressively decomposable* if and only if there exist regressive sets β and γ such that $\alpha \subset \beta \cup \gamma$ and both β and γ split α.

3. THREE THEOREMS ON SPLITTING AND DECOMPOSITION

THEOREM 1. *There exist disjoint supercohesive sets α and β such that*

$$(\exists j)(\alpha \subset W_j \subset \overline{\beta}), \qquad (\forall \gamma)(\gamma \text{ regressive} \Rightarrow \beta \not\subset \gamma \vee \gamma \not\subset \overline{\alpha}).$$

Proof. Let W_e be a hyper-hypersimple set ([4]). By Lemmas 3 and 4, $\overline{W}_e$ is devoid of infinite regressive subsets. Applying Lemma 2, let β be a supercohesive subset of $\overline{W}_e$. Let

$$\Gamma = \{\gamma \mid \beta \subset \gamma \text{ and } \gamma \text{ is regressive}\};$$

from Lemma 1 we see that Γ must be countably infinite. Enumerating Γ as $\gamma_0, \gamma_1, \gamma_2, \cdots$, we define a sequence of sets λ_j as follows: for $j \geq 0$, $\lambda_j = \bigcap_{k=0}^{j} \gamma_k$. Each λ_j is infinite, since it includes β; hence, by Lemma 5, each λ_j possesses an infinite regressive subset. But therefore, in view of Lemma 4, each λ_j must have infinite intersection with W_e. Let $t_0, t_1, t_2, \cdots$ be a nonrepeating sequence of numbers such that, for every j, $t_j \in \lambda_j \cap W_e$, and set $\tau = \{t_j \mid j \geq 0\}$. Let α be a supercohesive subset of τ. Then clearly $\alpha \subset W_e \subset \overline{\beta}$, whereas

$$\gamma \text{ regressive} \Rightarrow \beta \not\subset \gamma \vee \gamma \not\subset \alpha;$$

this finishes the proof of Theorem 1.

COROLLARY 2. *There exists a noncohesive number set τ that is not regressively decomposable.*

Proof. If α, β have the properties described in Theorem 1, then plainly we may take $\tau = \alpha \cup \beta$.

THEOREM 2. *There exists a cohesive set that can be split by a retraceable set but is not regressively decomposable.*

Proof. Let τ be a retraceable set with an immune complement ([6]). We begin by obtaining a cohesive set that is split by τ. To this end, let $\alpha_0, \alpha_1, \alpha_2, \cdots$ be the sequence of sets defined by

$$\alpha_0 = \text{the set of all natural numbers},$$

$$\alpha_{n+1} = \begin{cases} \alpha_n \cap W_n & \text{if } \alpha_n \cap W_n \text{ is infinite}, \\ \alpha_n & \text{otherwise}. \end{cases}$$

Each set α_n is an infinite r.e. set; thus, $\alpha_0, \alpha_1, \alpha_2, \cdots$ is a nest of infinite sets, each split by τ. Let $t_0, t_1, t_2, \cdots$ be a nonrepeating sequence of numbers such that $t_j \in \alpha_j \cap \tau$ for even j and $t_j \in \alpha_j \cap \bar{\tau}$ for odd j; then it is easy to see that by setting $\hat{\tau} = \{t_j \mid j \geq 0\}$ we obtain a cohesive set split by τ. In particular, $\hat{\tau} \cap \tau$ is infinite; let β be a supercohesive subset of $\hat{\tau} \cap \tau$. The set α, whose existence is asserted in Theorem 2, is an extension of β. Let $\gamma_0, \gamma_1, \gamma_2, \cdots$ be an enumeration of all the regressive supersets of β, and let $\{p_n\}$ be defined as in the proof of Lemma 1. We shall deal with the set of all pairs of the form (γ_j, p_k); let $P_0, P_1, P_2, \cdots$ be an enumeration of these pairs. Recall, from the proof of Lemma 1, that if p_n regresses two distinct infinite sets λ_1 and λ_2, then $\lambda_1 \cap \lambda_2$ is finite: from this it follows that if p_n regresses the two infinite sets λ_1 and λ_2 $(\lambda_1 \neq \lambda_2)$, then there exist numbers n_1 and n_2 such that if $m_1 \geq n_1$, $m_2 \geq n_2$, $m_1 \in \lambda_1$, and $m_2 \in \lambda_2$, then m_1 and m_2 are not connected by p_n.

We are now ready to proceed with the main section of the proof. The set α is to be obtained by a stage-by-stage buildup; at stage n we work with P_n, and for convenience we designate the first member of P_n by P_n^1, the second by P_n^2. Our procedure at stage n breaks down into three cases, as follows:

Case 1. P_n^2 regresses two different immune sets λ_1 and λ_2, each of which has infinite intersection with $\alpha_n - P_n^1$. In this case, there exist numbers m and $\hat{m}$ in $\alpha_n - P_n^1$ such that m and $\hat{m}$ are not connected by P_n^2. Let n_1 and n_2 be the two smallest such number (minimizing first m, then $\hat{m}$), and place both n_1 and n_2 in α. (Naturally, throughout stage n, no numbers are placed in α except as we explicitly stipulate.) Next, since $\alpha_n \cap \bar{\tau}$ is infinite, it must contain a number m not yet placed in α; we place the smallest such m in α. This finishes Case 1. (Thus, in Case 1, exactly three numbers are placed in α.) The first part of our procedure in Case 1 guarantees that α will not be decomposed by P_n^1 and a set regressed by P_n^2; the second part contributes to the splitting of α by τ (since α is to be an extension of β).

Case 2. P_n^1 is immune and P_n^2 regresses exactly one set λ such that λ is immune and $\lambda \cap (\alpha_n - P_n^1)$ is infinite. Here it follows from Lemma 6, since α_n is an infinite r.e. set, that $\alpha_n - (P_n^1 \cup \lambda)$ is infinite. Let n_1 be the smallest member of $\alpha_n - (P_n^1 \cup \lambda)$ that has not yet been placed in α, let n_2 be the smallest member of $(\alpha_n \cap \bar{\tau}) - \{n_1\}$ not yet placed in α, and place both n_1 and n_2 in α. This finishes Case 2.

Case 3. Neither Case 1 nor Case 2 obtains. Both $\alpha_n \cap \tau$ and $\alpha_n \cap \bar{\tau}$ are infinite; we place n_1 and n_2 in α, where n_1 is the smallest number in $\alpha_n \cap \tau$ not previously placed in α, while n_2 is the smallest number in $(\alpha_n \cap \bar{\tau})$ not previously placed in α.

Let μ be the set of all numbers n for which there exists a stage s such that n is placed in α at stage s; we define the set α by $\alpha = \beta \cup \mu$. We must verify that α is cohesive, is split by a retraceable set, and is not regressively decomposable. First, suppose $W_e \cap \alpha$ is infinite. It is easy to see that this implies the existence of an n such that $\alpha_{n+1} = \alpha_n \cap W_e$. But therefore $\alpha - W_e$ must be finite. Thus, α is cohesive. It is evident from the definition of α that α is split by the retraceable set τ. Finally, suppose that γ and δ are two regressive sets that decompose α. Since α is cohesive, γ and δ must both be immune. Since β is supercohesive, either $\beta - \gamma$ or $\beta - \delta$ is finite, say, $\beta - \gamma$ is finite. We may suppose, without loss of generality, that $\beta \subset \gamma$. Letting p_t regress δ and supposing that $P_n = (\gamma, p_t)$, we see that, at stage n of the construction of α, either Case 1 or Case 2 holds. But in each of these cases we provided that α contain a number not in $P_n^1 \cup \delta$ $(= \gamma \cup \delta)$. From this contradiction we conclude that α is, in fact, not regressively decomposable, which completes the proof of Theorem 2.

For convenience in stating Theorem 3, we introduce at this point some special notation: let E denote the class of recursive sets, F the class of r.e. sets, R the class of regressive sets.

THEOREM 3. *There exists a number set* α *such that* (i) α *is not regressively decomposable by a pair of sets each belonging to* R - (F - E), *and* (ii) α *is sequentially decomposable; that is, there exists a recursive function* f(x) *such that*

$\alpha \subset \bigcup_n W_{f(n)}$, $j \neq k \Rightarrow W_{f(j)} \cap W_{f(k)} = \square$, *and, for all* n, $\alpha \cap W_{f(n)} \neq \square$.

Proof. Let W_e be a simple set, and let f(x) and g(x, y) be recursive functions related to W_e as in Lemma 8 (with W_e in place of β). Applying Lemma 7, let β be an infinite subset of $\overline{W}_{f(0)}$ whose intersection with any recursive subset of $\overline{W}_{f(0)}$ is finite. (Condition (iii) of Lemma 8 clearly implies that the sets $W_{f(n)}$ are pairwise recursively inseparable, and hence individually nonrecursive.) Let γ be a supercohesive subset of β. Let the set of all immune regressive supersets of γ be enumerated in a list $\gamma_0, \gamma_1, \gamma_2, \cdots$, and let $\rho_0, \rho_1, \rho_2, \cdots$ be a listing of all of the *recursive* supersets of γ. We construct α by stages, as follows.

Stage 2n $(n \geq 0)$. We first set

$$\alpha_n = W_e \cap \bigcap_{j \leq n} \rho_j .$$

Now, $\bigcap_{j \leq n} \rho_j$ is a recursive superset of γ and must therefore have infinite intersection with $\overline{W}_{f(0)}$. In fact, $\left(\bigcap_{j \leq n} \rho_j\right) \cap \overline{W}_e$ must be infinite; for, as is easily seen, there would otherwise be an effective test for membership in $\overline{W}_{f(0)} \cap \bigcap_{j \leq n} \rho_j$. Since W_e is simple, it follows from Lemma 8 that $\left(\bigcap_{j \leq n} \rho_j\right) \cap W_{f(k)}$ is infinite for all k. For each $j \leq n$, let n_j be the smallest member of $\left(\bigcap_{j \leq n} \rho_j\right) \cap W_{f(j)}$ not yet placed in α; place each of these numbers n_j in α and proceed to Stage 2n + 1.

Stage 2n + 1. We work inside the r.e. set α_n defined at the preceding stage. As in the proof of Theorem 2, there are three cases; we assume that $P_0, P_1, P_2, \cdots$ is a listing of all pairs of form (γ_j, p_k), and we consider the pair P_n (the notations P_n^1, P_n^2 will be used as in the proof of Theorem 2).

Case 1. P_n^2 regresses two distinct immune sets λ_1 and λ_2 such that $\lambda_i \cap (\alpha_n - P_n^1)$ is infinite for i = 1, 2. Just as in the proof of Theorem 2, this implies the existence of a pair of numbers $n_1, n_2 \in \alpha_n - P_n^1$ that are not connected by P_n^2; minimizing first n_1, then n_2, we may suppose n_1 and n_2 to be the smallest such pair. Place both n_1 and n_2 in α, and go on to Stage 2(n + 1).

Case 2. P_n^2 regresses *exactly one* immune set λ such that $\lambda \cap (\alpha_n - P_n^1)$ is infinite. By Lemma 6, there exist numbers m such that $m \in \alpha_n$ and $m \notin P_n^1 \cup \lambda$. Letting $\hat{m}$ be the smallest such m, place $\hat{m}$ in α and proceed to Stage 2(n + 1).

Case 3. Neither Case 1 nor Case 2 obtains. Proceed directly to Stage 2(n + 1).

We define α by $\alpha = \gamma \cup \mu$, where μ is the set of all numbers that are placed in α at some stage s.

It is evident from the procedure at Stage 2n that α has property (ii) of the theorem. Suppose there were sets β and δ in R - (F - E) such that β and δ decompose α. Since γ is supercohesive, one of $\gamma - \beta$ and $\gamma - \delta$ is finite; let us assume, without loss of generality, that in fact $\gamma \subset \beta$. Now, β cannot be recursive. For suppose $\beta = \rho_n$ for some n. Then we see from the construction of α that all numbers placed in α at or after stage 2n are members of β, whence all but finitely many members of α are in β. Similarly, δ cannot be recursive, for if it were, we could simply repeat the above argument, using $\overline{\delta}$ in place of β. Thus, β and δ are immune. But then either Case 1 or Case 2 must arise at Stage 2n + 1, where $\beta = \gamma_j$, δ is regressed by p_k, and $P_n = (\gamma_j, p_k)$. But, at Stage 2n + 1, in both Case 1 and Case 2 we have ensured that α is not a subset of $\beta \cup \delta$ $(= \gamma_j \cup \delta)$. It follows that α is not regressively decomposed by β and δ, and the proof of Theorem 3 is complete.

COROLLARY 3. *There exist number sets that can be decomposed by pairs of* r.e. *sets, but not by a pair of* retraceable *sets.*

4. REMARKS

1. Each of Theorems 1, 2, and 3 made use of Lemma 2. Since we have no proof of Lemma 2 not requiring the axiom of choice, we must state that as far as we now know, the three results of this section are dependent upon the axiom of choice (which is not needed, however, except perhaps in the application of Lemma 2).

2. Since the set α of Theorem 3 is sequentially decomposable, it follows from a slightly stronger form of Lemma 4 (pointed out in a footnote in [7]) that α has an infinite retraceable subset. At present we do not know an example of a set α of numbers that is decomposable, or even splittable, by r.e. sets but cannot be split by a retraceable set; in fact, we do not know of any sets that can be split by a regressive set but not by a retraceable set; we conjecture, however, that sets of all these types exist. Sets that can be decomposed by a pair of retraceable sets but not split by an r.e. set are known from [6].

3. Theorem 3 is stronger than Theorem 2 of [5], which states that there exist sequentially decomposable sets not splittable by recursive sets. Indeed, Theorem 3 of the present paper is provably stronger than [5, Theorem 2], in that we can easily give examples of number sets α that are sequentially decomposable, not splittable by a recursive set, and decomposable by a pair of (separably disjoint) retraceable sets (for example, any infinite retraceable subset of the set constructed in the proof of Theorem 3).

4. Let R* denote the class of all (nonvoid) finite unions of regressive sets. Suppose we say that a number set α is R*-*decomposable* if and only if there exists a finite collection $\beta_1, \cdots, \beta_k$ $(k \geq 2)$ of members of R* such that each β_j splits α, $\alpha \subset \bigcup_{j=1}^{k} \beta_j$, and for each set β_ℓ $(1 \leq \ell \leq k)$, $\left(\bigcup_{j=1}^{k} \beta_j\right) - \beta_\ell$ splits α. It is easy to see that Theorem 1 implies the existence of a noncohesive set that is not R*-decomposable. Further, Proposition D of [6] can easily be strengthened

somewhat so as to read: for every cardinal number κ $(2 \leq \kappa \leq \aleph_0)$, there exists a cohesive set α such that (i) α is decomposable by a disjoint family of κ retraceable sets, (ii) α is not R*-decomposable by a finite family of more than κ elements of R*, and (iii) α is R*-decomposable by a family of $\hat{\kappa}$ elements of R* for $2 \leq \hat{\kappa} < \kappa$, provided κ is finite. Theorems 2 and 3 of the present paper, however, cannot be generalized at once to the level of R*-decompositions. The main difficulty seems to be the restricted form of our Lemma 6, which we are unable at present to extend even to the case of *three* immune regressive sets.

5. Note that Theorem 1 proves more, in still another direction, than that regressive indecomposability does not imply cohesion. It shows that α is not necessarily cohesive, even if there exists no pair β, γ of regressive sets that gives a "branched splitting" of α (that is, has the property that both β and γ split α and both $(\beta \cap \alpha) - \gamma$ and $(\gamma \cap \alpha) - \beta$ are infinite). Theorem 2 is not as strong in this regard. However, we can slightly modify the proof of Theorem 2 to show that existence of such a branched splitting of a cohesive set does not imply regressive decomposability.

Added October 25, 1965. We are now able to prove Theorems 2 and 3 without recourse to the axiom of choice. Furthermore, in the case of Theorem 2 we now have R*-indecomposability. The latter improvement, however, carries with it a certain loss (at least, in terms of the proof that is known to us): in the above proof of Theorem 2 (which can be modified to avoid use of the axiom of choice, with no change in the retraceable splitting set τ), the set τ can be shown (without appeal to the axiom of choice) to lie in $\Sigma_2^0 \cap \Pi_2^0$; but in order to insure R*-indecomposability, we have made use of *nonarithmetical* retraceable splitting sets.

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