ON A CLASS OF MINIMAX PROBLEMS IN THE CALCULUS OF VARIATIONS

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1. INTRODUCTION

We shall consider problems of the calculus of variations in which a functional is defined by a system of n ordinary differential equations containing "control" functions and parameters and satisfying prescribed boundary conditions. We wish to determine the minimax of this functional, the maximum being taken over permissible values of the parameters, and the minimum being taken over permissible choices of the control functions and of the boundary conditions. These minimax problems constitute a generalization of the classical problems of the calculus of variations such as the Bolza problem [1], [4], [5], [6] and of the more recently considered "control" problems [7], [2], [8], [9], [10], [11], in all of which the minimum (respectively, the maximum) rather than the minimax is the subject of the investigations. Our results, however, are restricted to those problems in which either the set of parameter choices is finite, or the "target" set of permissible endpoints has dimension n. Whether the results are valid without either of these two assumptions remains for us an open question.

Minimax (or maximin) problems of this type may arise, in particular, when there is only incomplete information about certain parameters of the problem, or when one of the players, in a zero-sum game, must disclose his strategy in advance. We may consider, as an example, a chemical reaction of fixed duration, used to produce a particular chemical substance. The reaction is described by a system of ordinary differential equations involving time-dependent concentrations of reactants and their derivatives, as well as the time-dependent fuel flow and certain fuel parameters. The reaction may be "controlled" by varying (with time) the flow of the fuel into the furnace. The fuel parameters are known within certain limits only. A maximin problem will arise if a decision is made to control the fuel flow in such a manner as to maximize the guaranteed yield (for all possible values of fuel parameters) of the desired chemical substance.

Our arguments are based on previously obtained results [9], [10], and the method of proof, resembling somewhat the approach of [11], is by passage to the limit, starting with the case where the set of parameter choices is finite. As in [9], [10], and [11], and in the spirit of the generalized-curve approach of L. C. Young [12], we replace the original problem by a "relaxed" (or generalized) problem. Our main results are contained in Theorem 3.1, which states that, subject to rather general assumptions, the relaxed problem has a solution, that this solution can be uniformly approximated with original controls, and that this solution satisfies "constructive" necessary conditions for minimax which generalize the classical Weierstrass E-condition and transversality conditions.

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2. DEFINITIONS AND ASSUMPTIONS

Let R be a compact Hausdorff space, E_n the euclidean n-space, T the closed interval $[t_0, t_1]$ of the real axis, V an open set in E_n, B_0 and B_1 compact sets in V, and P a compact set in some metric space. We are given the function

$$g(x, t, p, \rho) = (g^1(x, t, p, \rho), \cdots, g^n(x, t, p, \rho))$$

from $V \times T \times P \times R$ to E_n, and it is continuous on R for every $(x, t, p) \in V \times T \times P$.

Definition 2.1. We shall refer to a function $\rho(t)$ from T to R as an *original control,* and we shall say that it is an *admissible original control* if there exist a point b_0 in B_0 and a function $x(t, p)$ from $T \times P$ to V that is absolutely continuous on T for every p in P and such that, for every p in P,

(2.1.1) $\frac{dx(t, p)}{dt} = \dot{x}(t, p) = g(x(t, p), t, p, \rho(t))$ a.e. in T,

(2.1.2) $x(t_0, p) = b_0$,

(2.1.3) $x(t_1, p) \in B_1$

(throughout the paper, a.e. in T or measurable on T means with respect to Lebesgue measure).

We shall say that $\rho(t)$ is a *minimizing original control* if $x^0 = \mathrm{Max}_{p \in P}\, x^1(t_1, p)$ exists and $\rho(t)$ minimizes x^0 among all admissible original controls.

We next define relaxed (or generalized) controls. This concept, patterned after Young's definition of generalized curves [12, p. 231], is introduced to simulate "limits" of rapidly oscillating original controls.

Let measurable sets in R be the Borel sets, and let S be the class of probability measures over R. Then $\sigma \in S$ if σ is a completely additive, nonnegative set function defined on Borel sets, with $\sigma(R) = 1$. Let

$$f(x, t, p, \sigma) = \int_R g(x, t, p, \rho)\, d\sigma \qquad \text{for } (x, t, p, \sigma) \in V \times T \times P \times S.$$

Definition 2.2. We shall refer to a function $\sigma(t)$ from T to S as a *relaxed control,* and we shall say that it is an *admissible relaxed control* if there exist a point b_0 in B_0 and a function $x(t, p)$ from $T \times P$ to V that is absolutely continuous on T for every p in P and such that, for every p in P,

(2.2.1) $\frac{dx(t, p)}{dt} = \dot{x}(t, p) \doteq f(x(t, p), t, p, \sigma(t))$ a.e. in T,

(2.2.2) $x(t_0, p) = b_0$,

(2.2.3) $x(t_1, p) \in B_1$.

A function $\sigma(t)$ from T to S is a *minimizing relaxed control* if

$$x^0 = \mathop{\mathrm{Max}}_{p \in P} x^1(t_1, p)$$

exists and $\sigma(t)$ minimizes x^0 among all admissible relaxed controls.

We observe that every original control is also a relaxed control.

Assumption 2.3. There exist a finite or denumerable collection of disjoint measurable subsets T_r $(r = 1, 2, \cdots)$ of T such that $T' = \bigcup_r T_r$ has measure $t_1 - t_0$, a positive constant c, a function $\varepsilon(h)$ $(h > 0)$ converging to 0 as $h \to 0+$, and a compact set $D \subset V$ such that the following six conditions are satisfied.

(2.3.1). The functions $g^i(x, t, p, \rho)$ and $\partial g^i(x, t, p, \rho)/\partial x^j$ $(i, j = 1, \cdots, n)$ exist over $V \times T' \times P \times R$, and over that set they are continuous functions of (x, t, p) uniformly in ρ, are uniformly continuous in p, and are continuous in ρ for each (x, t, p). Furthermore, $|g(x, t, p, \rho) - g(x, t', p, \rho)| \leq \varepsilon(|t - t'|)$, provided t and t' belong to the same set T_r $(r = 1, 2, \cdots)$ ($|g|$ represents the euclidean length of g).

(2.3.2) $|g(x, t, p, \rho)| \leq c$ and $|g_x(x, t, p, \rho)| \leq c$ on $V \times T' \times P \times R$ (here g_x is the matrix $(\partial g^i/\partial x^j)$ $(i, j = 1, \cdots, n)$, and $|g_x| = \sum_{i,j=1}^{n} |\partial g^i/\partial x^j|$).

(2.3.3) There exists at least one admissible relaxed control.

(2.3.4) If x(t) is an absolutely continuous function from T to V such that

$$\dot{x}(t) = f(x(t), t, p, \sigma(t)) \quad \text{a.e. in } T, \qquad x(t_0) \in B_0, \qquad x(t_1) \in B_1,$$

for some $\sigma(t)$ from T to S and some p in P, then $x(t) \in D$ for $t \in T$.

(2.3.5) $B_0 = c_0(C_0)$ and $B_1 = c_1(C_1)$, where C_0 and C_1 are compact, convex euclidean sets and $c_0(\xi_0)$ and $c_1(\xi_1)$ are continuously differentiable homeomorphic mappings of C_0 and C_1 onto B_0 and B_1, respectively.

(2.3.6) The set C_1 (hence also B_1) is of dimension n, and the matrix $c_{1,\xi} = (\partial c_1^i/\partial \xi^j)$ $(i, j = 1, \cdots, n)$ is nonsingular over C_1.

3. EXISTENCE OF MINIMAX. APPROXIMATIONS WITH ORIGINAL CONTROLS. NECESSARY CONDITIONS

We now state our principal results.

THEOREM 3.1. *Let Assumption* 2.3 *be satisfied, and let* $f(x, t, p, \sigma)$ *be defined as in Section* 2. *Then the following conclusions hold.*

(3.1.1) *There exist a minimizing relaxed control* $\sigma(t)$, *an associated point* $b_0 \in B_0$, *and a function* $x(t, p)$ *satisfying Definition* 2.2. *The vector function* $f(x, t, p, \sigma(t))$ *and the matrix function* $f_x(x, t, p, \sigma(t))$ *are measurable on* T *for every* $(x, p) \in V \times P$.

(3.1.2) *There exist a sequence* $\rho_1(t), \rho_2(t), \cdots$ *of original controls and a sequence of functions* $x_1(t, p), x_2(t, p), \cdots$ *from* $T \times P$ *to* V, *absolutely continuous on* T *for every* p *in* P, *such that*

$$\frac{dx_s(t, p)}{dt} = g(x_s(t, p), t, p, \rho_s(t)) \ \textit{a.e. in } T \qquad (p \in P;\ s = 1, 2, \cdots),$$

$$\lim_{s \to \infty} x_s(t, p) = x(t, p) \ \textit{uniformly on } T \times P, \textit{ and}$$

$g(x, t, p, \rho_s(t))$ *is a measurable function of* t *for all* $(x, p) \in V \times P$ $(s = 1, 2, \cdots)$.

(3.1.3) *There exist*

a) *a measurable subset* T^* *of* T *of measure* $t_1 - t_0$,

b) *a nonnegative regular measure* ω *defined on Borel subsets of* P,

c) ω*-integrable functions* $\psi^0(p)$ *and* $\psi(p) = (\psi^1(p), \cdots, \psi^n(p))$,

d) *a continuous function* $\bar{\xi}(p)$ *from* P *to* C_1,

e) *a point* $\tilde{\xi} \in C_0$,

f) *continuous functions* $h_j(t, p)$ $(j = 1, \cdots, n)$ *from* $T \times P$ *to* E_n, *absolutely continuous on* T *for each* p *in* P,

such that

(3.1.3.1) $\omega(P) > 0$; $\omega(U) = 0$, where

$$U = \{p \in P \mid x(t_1, p) \in \text{Interior of } B_1 \text{ and } x^1(t_1, p) < \operatorname*{Max}_{p' \in P} x^1(t_1, p')\};$$

$\sum_{j=0}^{n} |\psi^j(p)| = 1$ a.e. with respect to ω; $\psi^0(p) \geq 0$ on P and $\psi^0(p) = 0$ a.e. in Z with respect to ω, where

$$Z = \{p \in P \mid x^1(t_1, p) < \operatorname*{Max}_{p' \in P} x^1(t_1, p')\};$$

(3.1.3.2)
$$\frac{dx(t, p)}{dt} = f(x(t, p), t, p, \sigma(t)) \quad \text{on } T^* \times P,$$

$$\frac{dh_j(t, p)}{dt} = -f_x^T(x(t, p), t, p, \sigma(t))h_j(t, p) \quad \text{on } T^* \times P \ (j = 1, \cdots, n),$$

$$x(t_0, p) = c_0(\tilde{\xi}) = b_0 \in B_0, \quad x(t_1, p) = c_1(\bar{\xi}(p)) \in B_1 \quad \text{for } p \in P,$$

$$h_j(t_0, p) = \delta_j \quad (p \in P, \ j = 1, \cdots, n),$$

where f_x^T is the transpose of the matrix $f_x = \left(\frac{\partial f^i}{\partial x^j}\right)$ $(i, j = 1, \cdots, n)$ and δ_j is the j-th column of the unit matrix of order n;

(3.1.3.3) (Weierstrass E-condition)

$$\sum_{j=1}^{n} \int_P \psi^j(p) h_j(t, p) \cdot f(x(t, p), t, p, \sigma(t))\, d\omega$$

$$= \operatorname*{Min}_{\sigma \in S} \sum_{j=1}^{n} \int_P \psi^j(p) h_j(t, p) \cdot f(x(t, p), t, p, \sigma)\, d\omega \quad \text{on } T^*;$$

(3.1.3.4) *(Support (transversality) conditions)*

$$\int_P \psi(p)\, d\omega \cdot c_{0,\xi}(\tilde{\xi})\tilde{\xi} = \operatorname*{Min}_{\xi \in C_0} \int_P \psi(p)\, d\omega \cdot c_{0,\xi}(\tilde{\xi})\, \xi,$$

and

$$\left(\psi^0(p)\,\delta_1 - \sum_{j=1}^{n} \psi^j(p)\,h_j(t_1, p)\right)\cdot c_{1,\xi}(\bar\xi(p))\,\bar\xi(p)$$

$$= \min_{\xi\in C_1}\left(\psi^0(p)\,\delta_1 - \sum_{j=1}^{n} \psi^j(p)\,h_j(t_1, p)\right)\cdot c_{1,\xi}(\bar\xi(p))\,\xi \quad \textit{a.e. with respect to } \omega,$$

where $c_{0,\xi}$ *is the matrix* $\left(\frac{\partial c_0^i}{\partial \xi_0^j}\right)$ $(i = 1, \cdots, n;\ j = 1, \cdots, \ell;$ ℓ *is the dimension of* C_0) *and* $c_{1,\xi} = \left(\frac{\partial c_1^i}{\partial \xi^j}\right)$ $(i, j = 1, \cdots, n)$.

The above theorem, which we shall prove in Sections 4 to 8, remains valid when Assumption (2.3.6) is replaced by the assumption that the set P is finite. We can easily deduce this modified theorem from Lemma 4.1 and from the proof of statement (3.1.2) in Section 8.

We are unable to make any assertions about the validity of Theorem 3.1 when the set P is infinite and the set B_1 is of dimension $\ell' < n$. The proof of Lemma 6.1 breaks down when $\ell' < n$, and the possibility remains that the measure ω is identically 0. In that case, the necessary conditions for minimax, described in (3.1.3), become trivial.

Neither can we shed any light on the following question: under what conditions does there exist a finite subset P' of P such that a relaxed solution of our minimax problem with P' replacing P is also a relaxed solution of the original problem? We can easily verify that P' exists in some rather simple situations, and it seems that the existence of P' characterizes a large class of minimax problems; but we have been unable to reach any definite conclusions.

Finally, we observe that several related minimax problems can either be reduced to the problem described in Definition 2.2 or can be handled by the same methods. Consider, as an example, the case when condition (2.2.2) is replaced by the condition $x(t_0, p) = b_0(p)$ $(p \in P)$, where $b_0(p)$ is a given function from P to B_0. We may reduce this new problem to the old one by setting $t_0' = t_0 - 1$,

$$g(x, t, p, \rho) \equiv b_0(p) \qquad (t_0' \le t < t_0;\ (x, p, \rho) \in V \times P \times R),$$

$B_0' = \{(0, \cdots, 0)\}$, and replacing t_0 and B_0 by t_0' and B_0', respectively. If condition (2.2.2) is replaced by the condition $x(t_0, p) \in B_0$ $(p \in P)$, then a slight modification of the arguments in Sections 4 to 8 leads to the conclusion that there exists a continuous function $\tilde\xi(p)$ from P to C_0 such that $x(t_0, p) = c_0(\tilde\xi(p))$ and

$$\psi(p)\cdot c_{0,\xi}(\tilde\xi(p))\,\tilde\xi(p) = \min_{\xi\in C_0} \psi(p)\cdot c_{0,\xi}(\tilde\xi(p))\,\xi \qquad \text{a.e. with respect to } \omega,$$

and these relations replace the first support condition in (3.1.3.4). Similar remarks apply to problems of variable duration and to other related problems.

4. FINITE SETS OF PARAMETER CHOICES

We shall devote the remaining sections of the paper to the proof of Theorem 3.1. Our approach will be to consider first a minimax problem in which the set P of parameters is replaced by a finite subset, and then to pass to the limit, letting the finite subset increase monotonically to a dense subset of P.

Consider a finite set $Q = \{p_1, p_2, \cdots, p_q\} \subset P$. We shall call a set of absolutely continuous functions $x_i(t)$ $(i = 1, \cdots, q)$ from T to V a *Q-admissible sheaf* if there exist points $\tilde{\xi} \in C_0$ and $\bar{\xi}_i \in C_1$ $(i = 1, \cdots, q)$ and a function $\sigma(t)$ from T to S such that

$$\text{(4.0.1)} \qquad \begin{cases} \dot{x}_i(t) = f(x_i(t), t, p_i, \sigma(t)) \quad \text{a.e. in } T \quad (i = 1, \cdots, q), \\ x_i(t_0) = c_0(\tilde{\xi}), \quad x_i(t_1) = c_1(\bar{\xi}_i) \quad (i = 1, \cdots, q). \end{cases}$$

A *Q-minimizing sheaf* is a Q-admissible sheaf that minimizes, among all such sheaves, the value $\underset{1 \leq i \leq q}{\text{Max}} \; x_i^1(t_1)$.

We shall now show that a Q-minimizing sheaf exists for every finite subset Q of P, and we shall describe some of its properties.

LEMMA 4.1. *There exist a Q-minimizing sheaf* $x_i(t)$ $(i = 1, \cdots, q)$, *points* $\tilde{\xi}$ *and* $\bar{\xi}_i$ $(i = 1, \cdots, q)$, *and a function* $\sigma(t)$ *satisfying relations* (4.0.1). *Furthermore, there exist nonnegative numbers* ζ_i^0 $(i = 1, \cdots, q)$, *vectors* ζ_i $(i = 1, \cdots, q)$ *in* E_n, *and absolutely continuous functions* $h_{i,j}(t)$ $(i = 1, \cdots, q;\ j = 1, \cdots, n)$ *from* T *to* E_n *such that*

$$\text{(4.1.1)} \qquad \dot{h}_{i,j}(t) = -f_x^T(x_i(t), t, p_i, \sigma(t))\, h_{i,j}(t) \quad \text{a.e. } \textit{in } T$$

$$(i = 1, \cdots, q;\ j = 1, \cdots, n),$$

where f_x^T *is the transpose of the matrix* $f_x = \left(\dfrac{\partial f^i}{\partial x^j}\right)$ $(i, j = 1, \cdots, n)$,

$$\text{(4.1.2)} \qquad h_{i,j}(t_0) = \delta_j \quad (j = 1, \cdots, n;\ i = 1, \cdots, q),$$

where δ_j *is the* j*-th column of the unit matrix of order* n,

$$\text{(4.1.3)} \qquad \zeta_i^0 \geq 0 \quad \textit{and} \quad \zeta_i^0 = 0 \textit{ if } x_i^1(t_1) < x^0 \quad (i = 1, \cdots, q),$$

where $x^0 = \underset{1 \leq i \leq q}{\text{Max}} \; x_i^1(t_1)$,

$$\text{(4.1.4)} \qquad \sum_{i=1}^{q} \zeta_i \cdot c_{0,\xi}(\tilde{\xi})\, \tilde{\xi} = \underset{\xi \in C_0}{\text{Min}} \sum_{i=1}^{q} \zeta_i \cdot c_{0,\xi}(\tilde{\xi})\, \xi,$$

where $c_{0,\xi}$ *is the matrix* $\left(\dfrac{\partial c_0^i}{\partial \xi_0^j}\right)$ $(i = 1, \cdots, n;\ j = 1, \cdots, \ell;\ \ell = \textit{dimension of } C_0)$

(4.1.5)
$$\left(\zeta_i^0\delta_1 - \sum_{j=1}^{n}\zeta_i^j h_{i,j}(t_1)\right)\cdot c_{1,\xi}(\bar{\xi}_i)\,\bar{\xi}_i$$
$$= \operatorname*{Min}_{\xi\in C_1}\left(\zeta_i^0\delta_1 - \sum_{j=1}^{n}\zeta_i^j h_{i,j}(t_1)\right)\cdot c_{1,\xi}(\bar{\xi}_i)\,\xi \qquad (i = 1, \cdots, q),$$

where $c_{1,\xi} = \left(\dfrac{\partial c_1^i}{\partial \xi^j}\right)$ $(i, j = 1, \cdots, n)$.

(4.1.6)
$$\sum_{i=1}^{q}\sum_{j=1}^{n}\zeta_i^j h_{i,j}(t)\cdot f(x_i(t), t, p_i, \sigma(t))$$
$$= \operatorname*{Min}_{\sigma\in S}\sum_{i=1}^{q}\sum_{j=1}^{n}\zeta_i^j h_{i,j}(t)\cdot f(x_i(t), t, p_i, \sigma) \qquad \text{a.e. } in\ T,$$

(4.1.7)
$$\sum_{i=1}^{q}\left(\zeta_i^0 + \sum_{j=1}^{n}|\zeta_i^j|\right) \neq 0,$$

(4.1.8) *for all* $(x, p) \in V \times P$ *and almost all* t *in* T, $f(x, t, p, \sigma(\tau))$ *is a measurable function of* τ *on* T.

Proof. We observe that absolutely continuous functions $x_i(t)$ $(i = 1, \cdots, q)$ form a Q-minimizing sheaf if and only if there exist absolutely continuous scalar functions $x^0(t)$, $y^i(t)$ $(i = 1, \cdots, q)$ on T such that $x^0(t_1)$ is minimum subject to conditions (4.0.1) and

(4.1.9)
$$\begin{cases} \dot{x}^0(t) = 0 & \text{a.e. in } T,\\ \dot{y}^i(t) = f^1(x_i(t), t, p, \sigma(t)) & \text{a.e. in } T\ (i = 1, \cdots, q),\\ y^i(t_1) \le x^0(t_1) & (i = 1, \cdots, q),\\ y^i(t_0) = c_0^1(\tilde{\xi}) & (i = 1, \cdots, q). \end{cases}$$

The problem of minimizing $x^0(t_1)$ subject to (4.0.1) and (4.1.9) is a relaxed variational problem [9, p. 111] in $(qn + 1)$-dimensional euclidean space. By Assumption (2.3.3), there exists at least one function

$$(x^0(t), y^1(t), \cdots, y^q(t), x_1(t), \cdots, x_q(t))$$

from T to $E_1 \times \cdots \times E_1 \times V \times \cdots \times V$ satisfying relations (4.0.1) and (4.1.9), and by Assumption (2.3.4) every such function must be contained in the compact set $J \times J \times \cdots \times J \times D \times \cdots \times D$, where $I = \{x^1 \mid x \in D\}$ and J is a compact interval containing I in its interior. It follows then from [9, Theorem 3.3, p. 123] that there exists a minimizing function, hence there exist absolutely continuous functions $x_i(t)$ $(i = 1, \cdots, q)$, points $\tilde{\xi} \in C_0$ and $\bar{\xi}_i \in C_1$ $(i = 1, \cdots, q)$, and a function $\sigma(t)$ satisfying relation (4.0.1) and minimizing $x^0(t_1)$.

We next consider statement (4.1.8) that $f(x, t, p, \sigma(\tau))$ is a measurable function on T for all $(x, p) \in V \times P$ and almost all t in T. The proof of this statement is identical with that of [9, Theorem 4.1, p. 124] except that the Banach space $\mathscr{B}$ defined in this reference includes now the functions $g^i(x, t, p, \rho)$ for

$$(x, t, p) \in V \times T' \times P$$

(where T' is as defined in Assumption 2.3).

By [9, Theorem 4.1, p. 124], the function

$$(0, f^1(x_1, t, p_1, \sigma), \cdots, f^1(x_q, t, p_q, \sigma), f(x_1, t, p_1, \sigma), \cdots, f(x_q, t, p_q, \sigma))$$

from $V \times V \times \cdots \times V \times T \times S$ to E_{nq+q+1} is a "proper representation," and we may therefore apply Theorem 6.1 of [10, p. 142].

Let

$$B_0' = \{(b_0, b_0, \cdots, b_0) \in E_{qn} \mid b_0 = c_0(\tilde{\xi}) + c_{0,\xi}(\tilde{\xi})\cdot(\xi - \tilde{\xi}) \text{ for some } \xi \in C_0\}$$

and

$$B_{1,i}' = \{c_1(\bar{\xi}_i) + c_{1,\xi}(\bar{\xi}_i)\cdot(\xi - \bar{\xi}_i) \mid \xi \in C_1\} \qquad (i = 1, \cdots, q).$$

Then we can easily verify that the set $J \times J \times \cdots \times J \times B_0'$ is bounded and convex, contains the point

$$\mathfrak{x}(t_0) = (x^0(t_0), y^1(t_0), \cdots, y^q(t_0), x_1(t_0), \cdots, x_q(t_0)),$$

and can be mapped, by a continuously differentiable transformation with fixed point $\mathfrak{x}(t_0)$ and a Jacobian equal to the unit matrix at $\mathfrak{x}(t_0)$, into the set of permissible initial conditions. Similarly, the set $J \times J \times \cdots \times J \times B_{1,1}' \times \cdots \times B_{1,q}'$ is bounded and convex, contains $\mathfrak{x}(t_1) = (x^0(t_1), y^1(t_1), \cdots, x_q(t_1))$, and can be mapped, by a continuously differentiable transformation with fixed point $\mathfrak{x}(t_1)$ and a Jacobian equal to the unit matrix at $\mathfrak{x}(t_1)$, into the set of permissible end conditions. Thus

$$B' = (J \times J \times \cdots \times J \times B_0') \times (J \times J \times \cdots \times J \times B_{1,1}' \times \cdots \times B_{1,q}')$$

adheres at $(\mathfrak{x}(t_0), \mathfrak{x}(t_1))$ [10, Definition 2.2, p. 131] to the set of permissible boundary conditions.

Since

$$x^0(t_1) = \max_{1 \leq i \leq q} x_i^1(t_1) \qquad \text{and} \qquad x(t_1) \in D$$

if $x(t)$ is the minimizing curve, we see that $x^0(t_1) \in I$. Thus $x^0(t_1) \neq \text{Min } J$. It follows then from [10, Theorem 6.1, p. 142] that there exists an absolutely continuous function $(\gamma(t), \gamma^1(t), \cdots, \gamma^q(t), z_1(t), \cdots, z_q(t))$ (corresponding to the function $z(t)$ referred to in [10, Theorem 6.1] such that

$$(4.1.10)\quad \begin{cases} \dot\gamma(t) = 0 \quad \text{a.e. in } T, \\ \gamma^i(t) = 0 \quad \text{a.e. in } T \quad (i = 1, \cdots, q), \\ \dot z_i(t) = -f_x^1(x_i(t), t, p_i, \sigma(t))\gamma^i(t) - f_x^T(x_i(t), t, p_i, \sigma(t)) z_i(t) \\ \qquad\qquad \text{a.e. in } T \quad (i = 1, \cdots, q), \end{cases}$$

$$(4.1.11)\quad \begin{cases} \sum_{i=1}^{q} (\gamma^i(t)\,\delta_1 + z_i(t))\cdot f(x_i(t), t, p_i, \sigma(t)) \\ \qquad = \operatorname*{Min}_{\sigma \in S} \sum_{i=1}^{q} (\gamma^i(t)\,\delta_1 + z_i(t))\cdot f(x_i(t), t, p_i, \sigma) \quad \text{a.e. in } T, \end{cases}$$

$$(4.1.12)\qquad \gamma(t_0)\,x^0(t_0) = \operatorname*{Min}_{\xi^0 \in J} \gamma(t_0)\,\xi^0 ,$$

$$(4.1.13)\qquad \sum_{i=1}^{q} (\gamma^i(t_0)\,\delta_1 + z_i(t_0))\cdot c_{0,\xi}(\tilde\xi)\,\tilde\xi = \operatorname*{Min}_{\xi \in C_0} \sum_{i=1}^{q} (\gamma^i(t_0)\,\delta_1 + z_i(t_0))\cdot c_{0,\xi}(\tilde\xi)\,\xi ,$$

$$(4.1.14)\quad \begin{cases} (\zeta^* - \gamma(t_1))\,x^0(t_1) - \sum_{i=1}^{q} \gamma^i(t_1)\,y^i(t_1) \\ \qquad = \operatorname*{Min}_{\substack{\eta^i \le \xi^0 \\ \eta^i \in I,\ \xi^0 \in J}} \left\{ (\zeta^* - \gamma(t_1))\,\xi^0 - \sum_{i=1}^{q} \gamma^i(t_1)\,\eta^i \right\} \quad \text{for some } \zeta^* \geq 0, \end{cases}$$

$$(4.1.15)\qquad -\sum_{i=1}^{q} z_i(t_1)\,c_{1,\xi}(\tilde\xi_i)\cdot\tilde\xi_i = \operatorname*{Min}_{\xi_i \in C_1} \left\{ -\sum_{i=1}^{q} z_i(t_1)\cdot c_{1,\xi}(\tilde\xi_i)\cdot\xi_i \right\},$$

$$(4.1.16)\qquad |\gamma(t)| + \sum_{i=1}^{q} \left(|\gamma^i(t)| + \sum_{j=1}^{n} |z_i^j(t)| \right) \neq 0 \qquad (t \in T).$$

Since $x^0(t_1)$, being in I, is in the interior of J, (4.1.10) and (4.1.12) yield the identity $\gamma(t) \equiv 0$ $(t \in T)$. Also, by (4.1.10),

$$\gamma^i(t) \equiv \gamma^i(t_0) \equiv \gamma^i(t_1) \qquad (t \in T;\ i = 1, \cdots, q).$$

Let now $\zeta_i^0 = \gamma^i(t_0)$ and $\zeta_i = \zeta_i^0\,\delta_1 + z_i(t_0)$ $(i = 1, \cdots, q)$, and let absolutely continuous functions $h_{i,j}(t)$ $(i = 1, \cdots, q;\ j = 1, \cdots, n)$ from T to E_n be solutions of equations (4.1.1) and (4.1.2). The latter system is now defined, since we have proved the existence of $x_i(t)$ $(i = 1, \cdots, q)$ and $\sigma(t)$ and have shown that $x_i(t) \in D \subset V$. It has a unique solution, since by Assumption (2.3.2) and by the definition of $f(x, t, p, \sigma)$ the matrix $f_x(x_i(t), t, p_i, \sigma(t))$ has a norm not exceeding c and, as an easy consequence of (4.1.8) and of Assumption (2.3.1), this matrix is measurable.

We easily verify that, by (4.1.10), (4.1.1), and (4.1.2),

$$z_i(t) + \zeta_i^0 \delta_1 = \sum_{j=1}^{n} \zeta_i^j h_{i,j}(t) \qquad (t \in T;\ i = 1, \cdots, q),$$

and that relation (4.1.13) implies (4.1.4). Furthermore, by (4.0.1) and (4.1.9), $y^i(t) = x_i^1(t)$ $(t \in T;\ i = 1, \cdots, q)$, and thus, by (4.1.14),

$$\sum_{i=1}^{q} \zeta_i^0(x^0(t_1) - x_i^1(t_1)) = \operatorname*{Min}_{\substack{\eta^i \leq x^0(t_1) \\ \eta^i \in J}} \sum_{i=1}^{q} \zeta_i^0(x^0(t_1) - \eta^i),$$

which yields relation (4.1.3). Since, in (4.1.15), $\xi_1, \xi_2, \cdots, \xi_q$ are independent, relation (4.1.5) follows directly. Relations (4.1.6) and (4.1.7) now follow from (4.1.11) and (4.1.16) respectively. This completes the proof of the Lemma.

LEMMA 4.2. *Let* $x_i(t)$ $(i = 1, \cdots, q)$ *be a* Q*-minimizing sheaf, and let* $\tilde{\xi} \in C_0$, $\xi_i \in C_1$ $(i = 1, \cdots, q)$, *and the function* $\sigma(t)$ *from* T *to* S *be as in relations* (4.0.1). *Let* $h_{i,j}(t)$ *be as in Lemma* 4.1. *There exist a positive number* Δ *and a compact subset* D_1 *of* V (*both independent of* Q) *with the property that if the* Δ*-neighborhood of* Q *contains* P, *then there exist uniquely defined functions* $x(t, p)$ *from* $T \times P$ *to* D_1 *and* $h_j(t, p)$ $(j = 1, \cdots, n)$ *from* $T \times P$ *to* E_n, *absolutely continuous on* T *for every* p *in* P, *and satisfying the conditions*

$$\text{(4.2.1)} \quad \begin{cases} \dot{x}(t, p) = f(x(t, p), t, p, \sigma(t)) \quad \text{a.e. in } T \quad (p \in P) \\ x(t_0, p) = c_0(\tilde{\xi}), \end{cases}$$

$$\text{(4.2.2)} \quad \begin{cases} \dot{h}_j(t, p) = -f_x^T(x(t, p), t, p, \sigma(t))\, h_j(t, p) \quad \text{a.e. in } T \quad (p \in P;\ j = 1, \cdots, n), \\ h_j(t_0, p) = \delta_j \quad (j = 1, \cdots, n), \end{cases}$$

$$\text{(4.2.3)} \quad x(t, p_i) = x_i(t), \quad h_j(t, p_i) = h_{i,j}(t) \qquad (t \in T;\ p_i \in Q;\ j = 1, \cdots, n).$$

Proof. Let D be the compact set referred to in Assumption 2.3. Since $D \subset V$ and V is open, there exists a positive number α such that the compact set D_1 of all the points within a distance α of D is contained in V. Let c be the constant referred to in (2.3.2).

By Assumption (2.3.1), there exists a positive Δ such that

$$|g(x, t, p, \rho) - g(x, t, p', \rho)| \leq \frac{\alpha c}{\exp c(t_1 - t_0) - 1} \qquad \text{for } (x, t, \rho) \in V \times T' \times R,$$

provided p and p' are within a Δ-neighborhood of each other in P.

Let now $p \in P$, and let p_i in Q be in a Δ-neighborhood of p. The system

$$\dot{x}(t) = f(x(t), t, p, \sigma(t)) \quad \text{a.e. in } T,$$

$$x(t_0) = c_0(\tilde{\xi})$$

has a unique solution $x(t)$ (as a consequence of (2.3.1), (2.3.2), and (4.1.8)), and this solution can be extended to the boundary of $V \times T$. Assume, by way of contradiction, that it cannot be extended beyond $\bar{t}$ ($\bar{t} < t_1$) and that therefore $x(\bar{t})$ is on the boundary of V. Then (4.0.1) implies that

$$\dot{x}_i(t) = f(x_i(t), t, p_i, \sigma(t)) \qquad \text{a.e. in } T;$$

hence, by Assumption (2.3.2),

$$(4.2.4)\quad \left\{\begin{aligned} |\dot{x}(t) - \dot{x}_i(t)| &\leq |f(x(t), t, p, \sigma(t)) - f(x(t), t, p_i, \sigma(t)| \\ &\quad + |f(x(t), t, p_i, \sigma(t)) - f(x_i(t), t, p_i, \sigma(t)| \\ &\leq \int_R |g(x(t), t, p, \rho) - g(x(t), t, p_i, \rho)|\, d\sigma(t) \\ &\quad + \int_R |g(x(t), t, p_i, \rho) - g(x_i(t), t, p_i, \rho)|\, d\sigma(t) \\ &\leq \frac{\alpha c}{\exp c(t_1 - t_0) - 1} + c|x(t) - x_i(t)| \qquad \text{a.e. in } [t_0, \bar{t}]. \end{aligned}\right.$$

This differential inequality, combined with the initial condition

$$|x(t_0) - x_i(t_0)| = 0,$$

implies that $|x(t) - x_i(t)| \leq \alpha$ ($t_0 \leq t \leq \bar{t}$), hence $x(\bar{t}) \in D_1$, contrary to the assumption that $x(\bar{t})$ is on the boundary of V. Thus $\bar{t} = t_1$, and the system (4.2.1) has a unique solution contained in D_1 for every p in P.

It follows from Assumption (2.3.2) that $|f_x(x(t), t, p, \sigma(t))| \leq c$ on $T' \times P$, and from (4.1.8) that this matrix is measurable. Thus system (4.2.2), being linear in h_j, has a unique solution for every p in P. Relations (4.2.3) can now be derived by comparing system (4.2.1) with (4.0.1) and comparing system (4.2.2) with (4.1.1) and (4.1.2).

5. PASSING TO THE LIMIT

Let Δ be the positive number referred to in Lemma 4.2, and let

$$\Delta > \varepsilon_1 > \varepsilon_2 > \cdots, \qquad \varepsilon_s \to 0 \text{ as } s \to \infty.$$

Since the set P is compact, we can find, for every positive integer s, a finite subset P_s of P such that the ε_s-neighborhood of P_s contains P. We may also assume that $P_s \subset P_{s+1}$ ($s = 1, 2, \cdots$).

For $(t, p) \in T \times P$, $s = 1, 2, \cdots$, and $j = 1, 2, \cdots, n$, let

$$x(t, p, s), \qquad h_j(t, p, s), \qquad \sigma(t, s), \qquad \tilde{\xi}(s)$$

represent the functions $x(t, p)$, $h_j(t, p)$, and $\sigma(t)$ and the point $\tilde{\xi}$ defined in Lemma 4.2 for $Q = P_s$. By "subsequence" we shall henceforth mean "infinite subsequence."

LEMMA 5.1. *The functions* $x(t, p, s)$ *and* $h_j(t, p, s)$ $(j = 1, 2, \cdots, n;\ s = 1, 2, \cdots)$ *are uniformly bounded and uniformly continuous on* $T \times P$. *Over some subsequence* Σ *of the positive integers, they converge to functions* $x^*(t, p)$ *and* $h_j^*(t, p)$ $(j = 1, \cdots, n)$ *that are uniformly continuous on* $T \times P$.

Proof. Since, by Assumption 2.3, $|g(x, t, p, \rho)| \leq c$ on $V \times T' \times P \times R$, hence $|f(x, t, p, \sigma)| \leq c$ on $V \times T' \times P \times S$, and since $x(t_0, p, s) = c_0(\tilde{\xi}(s)) \in B_0$, it easily follows that the $x(t, p, s)$ are uniformly continuous in t on $T \times P \times \{1, 2, \cdots\}$ and that they are uniformly bounded. We can also show, using an inequality analogous to (4.2.4), that the $x(t, p, s)$ are uniformly continuous in p on $T \times P \times \{1, 2, \cdots\}$. Thus the $x(t, p, s)$ $(s = 1, 2, \cdots)$ are uniformly bounded and uniformly continuous on $T \times P$.

We also observe that, as a consequence of (2.3.1), (2.3.2), and (4.1.8), the functions $f_x(x(t, p, s), t, p, \sigma(t, s))$ $(s = 1, 2, \cdots)$ are measurable in t for every p in P, that they are uniformly bounded, and that they are uniformly continuous in p on $T \times P$. Thus the solutions

$$h_j(t, p, s) \qquad (j = 1, \cdots, n;\ s = 1, 2, \cdots;\ p \in P)$$

of the linear system (4.2.2) are uniformly bounded and uniformly continuous in t. The uniform continuity of $h_j(t, p, s)$ $(j = 1, \cdots, n;\ s = 1, 2, \cdots)$ in p now follows by the argument that we applied to $x(t, p, s)$.

The convergence of $x(t, p, s)$ and $h_j(t, p, s)$ $(j = 1, \cdots, n)$ over some subsequence Σ of the integers s now follows from Arzela's theorem.

LEMMA 5.2. *There exist uniformly continuous functions* $\bar{\xi}(p, s)$ $(s = 1, 2, \cdots)$ *from* P *to* C_1 *such that*

$$c_1(\bar{\xi}(p, s)) = x(t_1, p, s) \qquad \textit{for } p \in P_s \quad (s = 1, 2, \cdots).$$

They converge uniformly to a function $\bar{\xi}^*(p)$ *from* P *to* C_1, *over some subsequence* Σ_1 *of the sequence* Σ *defined in Lemma* 5.1.

Proof. Since B_1 is homeomorphic to a compact and convex euclidean set C_1 of dimension n (Assumptions (2.3.5) and (2.3.6)), there exists a homeomorphic mapping $\phi(b) = (\phi^1(b), \cdots, \phi^n(b))$ from B_1 onto the closed unit cube in E_n. By Lemma 5.1, by (4.2.3), and by (4.0.1),

$$x(t_1, p_i, s) = c_1(\xi_i(s)) \in B_1 \qquad \text{for } p_i \in P_s,$$

and the $x(t_1, p, s)$ $(s = 1, 2, \cdots)$ are uniformly continuous on P, hence on P_s. Thus the mappings $\phi(x(t_1, p, s))$ $(s = 1, 2, \cdots)$ are uniformly continuous mappings of the sets P_s into the unit cube, and there exists a modulus of continuity $\delta(\varepsilon)$ such that $\lim_{\varepsilon \to 0+} \delta(\varepsilon) = 0$ and

$$|\phi^j(x(t_1, p, s)) - \phi^j(x(t_1, p', s))| \leq \delta(|p - p'|)$$

$$(j = 1, 2, \cdots, n;\ p \in P_s;\ p' \in P_s;\ s = 1, 2, \cdots),$$

where $|p - p'|$ represents the distance in P. Since $0 \leq \phi^j \leq 1$ $(j = 1, \cdots, n)$, it follows from [3, Theorem 20, p. 117] that we may assume $\delta(\varepsilon)$ to be concave and nondecreasing. Then, by a theorem of McShane [3, Theorem 21, p. 117] (the proof of which remains valid for functions over metric spaces), we can, for each s and j, extend the function $\phi^j(x(t_1, p, s))$ from P_s to the unit interval to a continuous function $\chi^j(p, s)$ from P to the unit interval in such a manner that $\delta(\varepsilon)$ remains the modulus of continuity.

Now let c_1^{-1} and ϕ^{-1} denote the inverses of the mappings c_1 and ϕ, respectively, and let $\chi = (\chi^1, \cdots, \chi^n)$. Then the functions

$$\bar{\xi}(p, s) = c_1^{-1}(\phi^{-1}(\chi(p, s))) \qquad (s = 1, 2, \cdots)$$

have the desired properties.

LEMMA 5.3. *Let* $x(t, p, s)$, $x^*(t, p)$, $h_j(t, p, s)$, $h_j^*(t, p)$ *be defined as in Lemma* 5.1, *and let* Σ_1 *be defined as in Lemma* 5.2. *There exist a subsequence* Σ_2 *of* Σ_1, *a measurable subset* T^* *of* T' *of measure* $t_1 - t_0$, *and a function* $\sigma^*(t)$ *from* T *to* S *such that*

(5.3.1) $f(x, t, p, \sigma^*(\tau))$ *and* $f_x(x, t, p, \sigma^*(\tau))$ *are measurable functions of* τ *on* T *for all* $(x, t, p) \in V \times T^* \times P$,

$$(5.3.2) \qquad \begin{cases} \dot{x}^*(t, p) = f(x^*(t, p), t, p, \sigma^*(t)) & \textit{on } T^* \times P, \\ \dot{h}_j^*(t, p) = -f_x^T(x^*(t, p), t, p, \sigma^*(t))\, h_j^*(t, p) & \textit{on } T^* \times P \quad (j = 1, \cdots, n). \end{cases}$$

Proof. Consider first the statement

(5.3.2)' for every p in P there exists a measurable subset T(p) of T', of measure $t_1 - t_0$, such that the differential equations of (5.3.2) hold on T(p).

The proof of the Lemma with statement (5.3.2) replaced by (5.3.2)' is almost identical with that of [9, Theorem 4.1, p. 124], except that the Banach space $\mathscr{B}$ of continuous functions over R now includes the functions

$$g^i(x, t, p, \rho) \quad \text{and} \quad \frac{\partial g^i(x, t, p, \rho)}{\partial x^j} \qquad (i, j = 1, \cdots, n;\ (x, t, p) \in V \times T' \times P)$$

and $k_N(\phi; \tau)$ is redefined as $\int_{t_0}^{\tau} d\theta \int_R \phi(\rho)\, d\sigma(\theta, N)$. (We observe that $\int_R \phi(\rho)\, d\sigma(t, N)$ is measurable and integrable over T, by [9, Theorem 4.1, p. 124].)

We now proceed to prove statement (5.3.2). Let $P^\#$ be a denumerable dense subset of P, and let $T^\# = \bigcap_{p \in P^\#} T(p) \cap T^*$. Then $T^\#$ is of measure $t_1 - t_0$.

Consider an arbitrary t in $T^\#$ and an arbitrary p in P. By Lemma 5.1 and Assumption 2.3, there exists for every $\varepsilon > 0$ a point p' in $P^\#$ such that

$$(5.3.3) \qquad |g(x^*(\tau, p), \tau, p, \rho) - g(x^*(\tau, p'), \tau, p', \rho) \leq \varepsilon/3 \text{ on } T \times R.$$

Let now $h_1 > 0$ be such that

$$(5.3.4) \qquad \left|\dot{x}^*(t, p') - \frac{1}{h}\int_t^{t+h} f(x^*(\tau, p'), \tau, p', \sigma^*(\tau))\, d\tau\right| \leq \varepsilon/3$$

$$\text{for } |h| \leq h_1,\ t + h \in T.$$

Such an h_1 exists, since by (5.3.2)' the left-hand side of (5.3.4) converges to 0 with h.

By (5.3.2)', (5.3.3), and (5.3.4),

$$\left| \frac{1}{h} (x^*(t+h, p) - x^*(t, p)) - f(x^*(t, p), t, p, \sigma^*(t)) \right|$$

$$\leq \left| \frac{1}{h} \int_t^{t+h} d\tau \int_R \{g(x^*(\tau, p), \tau, p, \rho) - g(x^*(\tau, p'), \tau, p', \rho)\} d\sigma^*(\tau) \right|$$

$$+ \left| \frac{1}{h} \int_t^{t+h} f(x^*(\tau, p'), \tau, p', \sigma^*(\tau)) d\tau - f(x^*(t, p'), t, p', \sigma^*(t)) \right|$$

$$+ \left| \int_R \{g(x^*(t, p'), t, p', \rho) - g(x^*(t, p), t, p, \rho)\} d\sigma^*(t) \right| \leq \varepsilon \qquad \text{for } |h| \leq h_1 .$$

Since ε is arbitrary, this proves that $\dot{x}^*(t, p)$ exists and

$$\dot{x}^*(t, p) = f(x^*(t, p), t, p, \sigma^*(t)) \qquad \text{on } T^\# \times P .$$

The proof that the second equation of (5.3.2) holds on $T^\# \times P$ is entirely analogous. Since $T^\# \subset T^*$, the Lemma now holds with $T^\#$ replacing T^*.

6. CONVERGENCE OF MEASURES

Let $\zeta_i^0(s)$ and $\zeta_i(s)$ $(i = 1, \cdots, q;\ s = 1, 2, \cdots)$ be defined as ζ_i^0 and ζ_i, respectively, in Lemma 4.1, for $Q = P_s$, and let

$$d(s) = \sum_{i=1}^{q} \sum_{j=0}^{n} |\zeta_i^j(s)| .$$

For every s and for $j = 0, \cdots, n$, we then define the signed measure $\mu^j(s)$ on Borel subsets of P as concentrated at the points of P_s with the "mass" (not necessarily positive) $\zeta_i^j(s)/d(s)$ at the point p_i. This definition is permissible, since $d(s) \neq 0$ by Lemma 4.1.

LEMMA 6.1. *Let* Σ_2 *be the sequence defined in Lemma* 5.3. *Then there exists a subsequence* Σ_3 *of* Σ_2 *such that the measures* $\mu^j(s)$ $(j = 0, \cdots, n)$ *converge weakly to regular bounded measures* μ^{*j} $(j = 0, \cdots, n)$ *over* Σ_3. *(*γ_k *converges weakly to* γ *if, for every continuous* $\phi(p)$, $\int_P \phi(p)\, d\gamma_k$ *converges to* $\int_P \phi(p)\, d\gamma$ *as* $k \to \infty$.*) Furthermore, at least one of the measures* μ^{*j} $(j = 0, \cdots, n)$ *is not identically* 0.

Proof. By their definition, the variations of the measures $\mu^j(s)$ $(j = 0, \cdots, n;\ s = 1, 2, \cdots)$ are bounded by 1. The existence of a subsequence Σ_3 of Σ_2 over which they converge weakly to limit measures follows thus from Helly's selection theorem.

Let now $\mu^j(s, A)$ ($j = 0, \cdots, n$; s in Σ_3) represent the $\mu^j(s)$-measure of the Borel subset A of P, and let $|\mu^j|(s, A)$ be the absolute $\mu^j(s)$-measure of A; that is, let

$$|\mu^j|(s, A) = \sup_{A' \subset A} (|\mu^j(s, A')| + |\mu^j(s, A - A')|).$$

By the definition of $\mu^j(s)$, we see that

$$\sum_{j=0}^{n} |\mu^j|(s, P) = 1 \qquad (s = 1, 2, \cdots).$$

It follows easily that there exist a subsequence Σ_4 of Σ_3 and some index a $(0 \leq a \leq n)$ such that $|\mu^a|(s, P) \geq 1/(n+1)$ over Σ_4.

Let $K(s, A) = \{k |\ p_k \in P_s \cap A \text{ and } \zeta_k^a(s) \geq 0\}$ (s in Σ_4) for every Borel subset A of P, and let measures $\pi_0^j(s)$ and $\pi_1^j(s)$ be defined by

$$\text{(6.1.1)} \quad \begin{cases} \pi_0^j(s, A) = \sum_k \zeta_k^j(s)/d(s) & (k \in K(s, A);\ j = 0, \cdots, n;\ s \text{ in } \Sigma_4), \\ \pi_1^j(s, A) = \mu^j(s, A) - \pi_0^j(s, A) & (j = 0, \cdots, n;\ s \text{ in } \Sigma_4). \end{cases}$$

Since the variation of each of the measures $\pi_0^j(s)$ and $\pi_1^j(s)$ is bounded by 1, the sequences $\pi_0^j(s)$ and $\pi_1^j(s)$ ($s = 1, 2, \cdots$) converge weakly to limit measures π_0^{*j} and π_1^{*j}, over some subsequence Σ_5 of Σ_4. We see that $\pi_0^{*j} + \pi_1^{*j} = \mu^{*j}$ ($j = 0, \cdots, n$) and

$$\text{(6.1.2)} \qquad \pi_0^a(s, P) - \pi_1^a(s, P) = |\mu^a|(s, P) \geq 1/(n+1) \qquad (s \text{ in } \Sigma_5).$$

We shall now assume, by way of contradiction, that the measures μ^{*j} ($j = 0, \cdots, n$) are all identically 0. By definition, $\pi_0^a(s)$ and $-\pi_1^a(s)$ are nonnegative. Since, by assumption, $\mu^a(s, P)$ converges to 0 over Σ_5, (6.1.1) and (6.1.2) yield

$$\text{(6.1.3)} \qquad \pi_0^{*a}(P) - \pi_1^{*a}(P) = -2\pi_1^{*a}(P) = 2\pi_0^{*a}(P) \geq 1/(n+1).$$

By (4.1.3), $\zeta_k^0(s) \geq 0$ for all k and s, hence $\pi_1^0(s)$ is identically 0. Thus π_1^{*0} and $\pi_0^{*0} = \mu^{*0} - \pi_1^{*0}$ are identically 0. It follows now from (6.1.3) that $a \neq 0$.

Let $\bar{\xi}(p, s)$ ($s = 1, 2, \cdots$; $p \in P$) be defined as in Lemma 5.2. Then relation (4.1.5) implies that

$$\text{(6.1.4)} \quad \left(\zeta_i^0(s)\,\delta_1 - \sum_{j=1}^{n} \zeta_i^j(s) h_j(t_1, p_i, s)\right) \cdot c_{1,\xi}(\bar{\xi}(p_i, s))(\bar{\xi}(p_i, s) - \xi_1) \leq 0,$$

for every s in Σ_5, every $\xi_1 \in C_1$, every $p_i \in P_s$. Let now $\xi(p)$ be an arbitrary continuous function from P to C_1. Then (6.1.4) implies (if summed first over all i in $K(s, P)$ and then over all the remaining indices i) that

$$(6.1.5)\quad \int_P \left(d\pi_k^0(s)\,\delta_1 - \sum_{j=1}^{n} d\pi_k^j(s)\,h_j(t_1, p, s) \right)\cdot c_{1,\xi}(\bar\xi(p, s))(\bar\xi(p, s) - \xi(p)) \le 0$$

$$(k = 0, 1;\ s \text{ in } \Sigma_5).$$

We apply Lemmas 5.1 and 5.2, use the relation $\pi^{*0} = 0$, let $s \to \infty$ over Σ_5, and deduce that

$$(6.1.6)\qquad - \int_P \sum_{j=1}^{n} d\pi_k^{*j} h_j^*(t_1, p)\cdot c_{1,\xi}(\bar\xi^*(p))\cdot(\bar\xi^*(p) - \xi(p)) \le 0 \quad (k = 0, 1)$$

for every continuous function $\xi(p)$ from P to C_1.

Now $\pi_0^{*j} + \pi_1^{*j} = \mu^{*j}$ $(j = 0, 1, \cdots, n)$, and by assumption the μ^{*j} are all identically 0. Thus (6.1.6) yields

$$(6.1.7)\qquad \int_P \sum_{j=1}^{n} d\pi_0^{*j} h_j(t_1, p)\cdot c_{1,\xi}(\bar\xi^*(p))(\bar\xi^*(p) - \xi(p)) = 0$$

for every continuous $\xi(p)$ from P to C_1.

Let $\omega(A) = \sum_{j=0}^{n} |\mu^{*j}|(A)$. Since $\sum_{j=0}^{n} |\mu^j|(s, P) = 1$ for all s, we conclude that $\omega(P) = 1$. Since

$$|\pi_0^j(A, s)| \le \sum_{i=0}^{n} |\mu^i|(s, A) \qquad (j = 1, \cdots, n)$$

for all s in Σ_5 and all Borel subsets A of P, it follows that the π_0^{*j} $(j = 1, \cdots, n)$ are absolutely continuous with respect to ω. Thus, by the Radon-Nikodym theorem, there exist finite measurable scalar functions $\phi^j(p)$ $(j = 1, \cdots, n)$ such that $\pi_0^{*j}(A) = \int_A \phi^j(p)\,d\omega$. Since $\pi_0^{*a}(P) \ge 1/(2n + 2)$ (see (6.1.3)), $\phi^a(p) \ne 0$ over some set of positive ω-measure. The vectors $h_j^*(t_1, p)$ $(j = 1, \cdots, n)$ are linearly independent for each $p \in P$, since, for each p, the $h_j^*(t, p)$ $(j = 1, \cdots, n)$ constitute a fundamental set of solutions of linear differential equations (the second system of (5.3.2)). By Assumption (2.3.6), the matrix $c_{1,\xi}(\bar\xi^*(p))$ is nonsingular for every p in P. Thus the vector function

$$\chi(p) = c_{1,\xi}^T(\bar\xi^*(p)) \sum_{j=1}^{n} \phi^j(p)\,h_j^*(t_1, p)$$

is nonvanishing over some set Ω of positive ω-measure. Furthermore, relation (6.1.7) yields

$$(6.1.8)\qquad \int_P \chi(p)\cdot(\bar\xi^*(p) - \xi(p))\,d\omega = 0$$

for every continuous $\xi(p)$ from P to C_1.

We now observe that if $\xi(p)$ is continuous from P to C_1 and u(p) is a continuous scalar function over P such that $0 \leq u(p) \leq 1$, then $u(p)\xi(p) + (1 - u(p))\xi^*(p)$ is continuous from P to the convex set C_1; hence, by (6.1.8),

$$\int_P \chi(p) \cdot (\bar{\xi}^*(p) - \xi(p)) u(p) \, d\omega = 0. \tag{6.1.9}$$

But every continuous scalar function on P is a linear combination of continuous functions with range in [0, 1]; hence (6.1.9) holds for all continuous u(p), whence

$$\chi(p) \cdot (\bar{\xi}^*(p) - \xi(p)) = 0 \quad \text{a.e. with respect to } \omega, \tag{6.1.10}$$

for every continuous $\xi(p)$ from P to C_1.

Since $|\chi(p)| \neq 0$ on some set Ω of positive ω-measure, it easily follows from (6.1.10) that there exists a subset Ω' of Ω of equal ω-measure such that

$$\chi(p) \cdot (\bar{\xi}^*(p) - \xi) = 0 \qquad \text{on } \Omega'$$

for all ξ in some dense denumerable subset of C_1, hence for all ξ in C_1. This implies, however, that C_1 is contained in a supporting hyperplane normal to nonvanishing $\chi(p)$ for every p in the nonempty set Ω', contrary to Assumption (2.3.6), which states that C_1 is n-dimensional.

Thus the assumption that the measures μ^{*j} $(j = 0, \cdots, n)$ are all identically 0 leads to a contradiction.

7. THE WEIERSTRASS E-CONDITION

LEMMA 7.1. *Let* $\sigma^*(t)$, μ^{*j}, $x^*(t, p)$, *and* $h_j^*(t, p)$ *be as in Section* 6, *and* T^* *as in Lemma* 5.3. *Then*

$$\sum_{j=1}^{n} \int_P h_j^*(t, p) \cdot f(x^*(t, p), t, p, \sigma^*(t)) \, d\mu^{*j}$$

$$= \operatorname*{Min}_{\sigma \in S} \sum_{j=1}^{n} \int_P h_j^*(t, p) \cdot f(x^*(t, p), t, p, \sigma) \, d\mu^{*j} \quad \textit{on } T^*.$$

Proof. Let Σ_3, x(t, p, s), $h_j(t, p, s)$, $\mu^j(s)$, and $\sigma(t, s)$ be defined as in Section 6. We have, by (4.1.6), for every fixed σ in S,

$$\begin{aligned} &\sum_{j=1}^{n} \int_P h_j(t, p, s) \cdot f(x(t, p, s), t, p, \sigma(t, s)) \, d\mu^j(s) \\ &\leq \sum_{j=1}^{n} \int_P h_j(t, p, s) \cdot f(x(t, p, s), t, p, \sigma) \, d\mu^j(s) \end{aligned} \tag{7.1.1}$$

a.e. in T for every s in Σ_3.

Since $\dot{x}(t, p, s) = f(x(t, p, s), t, p, \sigma(t, s))$ a.e. in T and, by Lemmas 5.1 and 6.1 and Assumption 2.3, $h_j(t, p, s)$, $f(x, t, p, \sigma)$, and $\mu^j(s)$ $(j = 1, \cdots, n;\ s$ in $\Sigma_3)$ are uniformly bounded and continuous on $T' \times P$, (7.1.1) yields

$$\begin{aligned}(7.1.2)\qquad &\sum_{j=1}^{n} \int_P d\mu^j(s) \int_t^{t+h} h_j(\tau, p, s)\cdot \dot{x}(\tau, p, s)\, d\tau \\ &\leq \sum_{j=1}^{n} \int_P d\mu^j(s) \int_t^{t+h} h_j(\tau, p, s)\cdot f(x(\tau, p, s), \tau, p, \sigma)\, d\tau\end{aligned}$$

for $t \in T$, s in Σ_3, and $t + h \in T$.

Let $\delta(h)$ $(h > 0)$ be a common modulus of continuity with respect to t of $h_j(t, p, s)$ (it exists, by Lemma 5.1). Then

$$\int_t^{t+h} h_j(\tau, p, s)\cdot \dot{x}(\tau, p, s)\, d\tau \geq h_j(t, p, s)\cdot (x(t+h, p, s) - x(t, p, s)) - ch\,\delta(h)$$

$$(j = 1, \cdots, n;\ (t, p) \in T \times P;\ s \text{ in } \Sigma_3;\ t + h \in T;\ h > 0),$$

the measures $\mu^j(s)$ converge weakly to μ^{*j}, and $\sum_{j=1}^{n} |\mu^j(s)|(P) \leq 1$; hence the limit inferior over Σ_3 of the left-hand side of (7.1.2) is at least

$$\sum_{j=1}^{n} \int_P h_j^*(t, p)\cdot (x^*(t+h, p) - x^*(t, p))\, d\mu^{*j} - ch\,\delta(h).$$

Since $g(x, t, p, \rho)$ is Lipschitz continuous in x with constant c, $f(x, t, p, \sigma)$ has the same property, and the right-hand side of (7.1.2) converges over Σ_3 to

$$\sum_{j=1}^{n} \int_P d\mu^{*j} \int_t^{t+h} h_j^*(\tau, p)\cdot f(x^*(\tau, p), \tau, p, \sigma)\, d\tau;$$

hence

$$\begin{aligned}(7.1.3)\qquad &\sum_{j=1}^{n} \int_P d\mu^{*j} \int_t^{t+h} \frac{1}{h} h_j^*(\tau, p)\cdot f(x^*(\tau, p), \tau, p, \sigma)\, d\tau \\ &\geq \sum_{j=1}^{n} \int_P \frac{1}{h} h_j^*(t, p)\cdot (x^*(t+h, p) - x^*(t, p))\, d\mu^{*j} - c\,\delta(h)\end{aligned}$$

$$((t, p) \in T \times P,\ h > 0,\ t + h \in T).$$

Since, by Assumption (2.3.1), $g(x, t, p, \rho)$ is continuous on $V \times T' \times P$, uniformly in ρ, it follows that

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} h_j^*(\tau, p)\cdot f(x^*(\tau, p), \tau, p, \sigma)\, d\tau = h_j^*(t)\cdot f(x^*(t, p), t, p, \sigma)$$

on $T' \times P$, hence also on $T^* \times P$. By Lemma 5.3,

$$\lim_{h \to 0} \frac{1}{h} (x^*(t+h, p) - x^*(t, p)) = \dot{x}^*(t, p) = f(x^*(t, p), t, p, \sigma^*(t))$$

on $T^* \times P$. Furthermore, the integrands in (7.1.3) are uniformly bounded. Letting $h \to 0$ in (7.1.3), we now derive the conclusion of the Lemma.

8. PROOF OF THEOREM 3.1

We shall continue to use the notation as it appears in the statements of Lemmas 4.2 to 6.1. Let now Σ^* be a subsequence of Σ_3 over which $\tilde{\xi}(s)$ converges to some $\tilde{\xi}^*$ in the compact set C_0. We shall prove that Theorem 3.1 holds, with

$$x(t, p) = x^*(t, p), \qquad h_j(t, p) = h_j^*(t, p) \quad (j = 1, \cdots, n),$$

$$\sigma(t) = \sigma^*(t), \qquad \tilde{\xi} = \tilde{\xi}^*, \qquad \bar{\xi}(p) = \bar{\xi}^*(p), \qquad \omega = \sum_{j=0}^{n} |\mu^{*j}|$$

(where $|\mu^{*j}|$ is the absolute measure of μ^{*j}), and

$$\psi^j(p) = \frac{d\mu^{*j}}{d\omega} \text{ at } p \qquad (j = 0, \cdots, n).$$

Proof of (3.1.1). By Lemma 5.3, $\dot{x}^*(t, p) = f(x^*(t, p), t, p, \sigma^*(t))$ on $T^* \times P$. Since $\tilde{\xi}(s) \to \tilde{\xi}^*$ and c_0 is continuous, we see (taking limits over Σ^*) that

$$x^*(t_0, p) = \lim x(t_0, p, s) = \lim c_0(\tilde{\xi}(s)) = c_0(\tilde{\xi}^*) \in B_0 \qquad (p \in P).$$

Now, by (4.2.3), $x(t, p, s)$ $(p \in P_s)$ is a P_s-minimizing sheaf. Thus $x(t_1, p, s) \in B_1$ for $p \in P_s$ and s in Σ^*. Since B_1 is closed and P_s converges over Σ^* to a dense subset of P, and since $x(t_1, p, s)$ (s in Σ^*) are uniformly continuous by Lemma 5.1, we conclude that $x^*(t_1, p) \in B_1$ for $p \in P$. Furthermore, since $x^*(t_1, p)$ is continuous, $\operatorname{Max}_{p \in P} x^{*1}(t_1, p)$ exists. Thus $\sigma^*(t)$ is an admissible relaxed control.

Let now $\sigma^{\#}(t)$ be any admissible relaxed control, and let $\sigma^{\#}(t)$, $x^{\#}(t, p)$, and $b_0^{\#}$ satisfy Definition 2.2. Then $x^{\#}(t, p)$ $(p \in P_s)$ is a P_s-admissible sheaf, since it satisfies relations (4.0.1). Thus

$$\operatorname*{Max}_{p \in P_s} x^1(t_1, p, s) \le \operatorname*{Max}_{p \in P_s} x^{\#1}(t_1, p) \le \operatorname*{Max}_{p \in P} x^{\#1}(t_1, p),$$

and, letting $s \to \infty$ over Σ^*, we see that

$$\operatorname*{Max}_{p \in P} x^{*1}(t_1, p) \le \operatorname*{Max}_{p \in P} x^{\#1}(t_1, p).$$

Thus $\sigma^*(t)$ is a minimizing control.

The last part of statement (3.1.1) follows now from statement (5.3.1) and Assumption 2.3.

Proof of statement (3.1.2). Let s be a fixed element of Σ^*, and let $Q = P_s = \{p_1, \cdots, p_q\}$. By its definition, a P_s-minimizing sheaf $x_i(t) = x(t, p_i, s)$ $(i = 1, \cdots, q)$ satisfies relations (4.0.1). Therefore it follows from [9, Theorem 2.2, p. 113] that the $x(t, p_i, s)$ $(i = 1, \cdots, q)$ are uniform limits of curves $y_N(t, p_i, s)$ $(N = 1, 2, \cdots)$ from T to V which, by [9, (2.2.11), p. 116], satisfy the relations

$$\dot{y}_N(t, p_i, s) = g(y_N(t, p_i, s), t, p_i, \rho(t, s, N)) \text{ a.e. in } T \qquad (i = 1, \cdots, q),$$

$$y_N(t_0, p_i, s) = x(t_0, p_i, s) = c_0(\tilde{\xi}(s)) \qquad (i = 1, \cdots, q)$$

for functions $\rho(t, s, N)$ $(N = 1, 2, \cdots)$ from T to R. Furthermore, by its construction, $\rho(t, s, N)$ is for each N a step function over each set T_r $(r = 1, 2, \cdots)$ referred to in Assumption 2.3; hence $g(x, t, p, \rho(t, s, N))$ is measurable on T for each $(x, p) \in V \times P$ $(N = 1, 2, \cdots)$.

Let $N(s)$ be such that $|y_N(t, p_i, s) - x(t, p_i, s)| \leq 1/s$ for all $N \geq N(s)$, $t \in T$, and $p_i \in P_s$. Let $\rho_s(t) = \rho(t, s, N(s))$. For each p in P, we now consider the system

$$\dot{x}_s(t, p) = g(x_s(t, p), t, p, \rho_s(t)) \text{ a.e. in } T,$$

$$x_s(t_0, p) = c_0(\tilde{\xi}(s)).$$

By the argument used in Lemma 4.2, we can prove that this system has a unique solution $x_s(t, p)$ contained in a compact set $D_2 \subset V$, provided s is large enough, say, s is in a subsequence Σ^{**} of Σ^*. Furthermore, the argument in Lemma 5.1 shows that the $x_s(t, p)$ (s in Σ^{**}) are uniformly continuous on $T \times P$. Since the $x(t, p, s)$ converge uniformly to $x^*(t, p)$ over Σ^{**}, it easily follows that the $x_s(t, p)$ have the same property.

Proof of (3.1.3.1). By Lemma 6.1, at least one of the measures μ^{*j} $(j = 0, 1, \cdots, n)$ is not identically 0, hence $\omega(P) > 0$. Since

$$|\mu^{*j}|(A) = \int_A |\psi^j(p)|\, d\omega \qquad (j = 0, \cdots, n),$$

we deduce that $\sum_{j=0}^{n} |\psi^j(p)| = 1$ a.e. with respect to ω.

Let $\mu^j(s, A)$ ($j = 0, \cdots, n$; s in Σ^*; A a Borel subset of P) be defined as in Lemma 6.1. Then, by (4.1.3), $\mu^0(s, A) \geq 0$ for every s and every A. It follows that μ^{*0}, the weak limit of $\mu^0(s)$, is nonnegative.

Assume now that $\omega(Z) > 0$, and let $x^{*0} = \operatorname{Max}_{p \in P} x^{*1}(t_1, p)$,

$$Z_k = \{p \in P \mid 1/(k+1) < x^{*0} - x^{*1}(t_1, p) \leq 1/k\} \qquad (k = 0, 1, 2, \cdots)$$

and $L = \{\ell \mid Z_\ell \text{ is nonempty}\}$. Let $\ell \in L$ be arbitrary. By Lemma 5.1, the function $x^{*1}(t_1, p)$ is continuous, hence both the sets Z_ℓ and

$$Z'_\ell = \{p \in P \mid x^{*0} - x^{*1}(t_1, p) \leq 1/(2\ell + 2)\}$$

are closed. Neither set is empty, since Z'_ℓ contains the point p that maximizes $x^{*1}(t_1, p)$. The sets are also disjoint. It follows that there exists a continuous function $\phi(p)$ on P such that $0 \leq \phi(p) \leq 1$, $\phi(p) = 1$ on Z_ℓ and $\phi(p) = 0$ on Z'_ℓ.

Let now Σ^{**} be a subsequence of Σ^* such that

$$|x^{*1}(t_1, p) - x^1(t_1, p, s)| < 1/(6\ell + 6)$$

on P and

$$\operatorname*{Max}_{p' \in P} x^1(t_1, p', s) - \operatorname*{Max}_{p' \in P_s} x^1(t_1, p', s) < 1/(6\ell + 6)$$

for s in Σ^{**}. Then the equality $x^1(t_1, p, s) = \operatorname*{Max}_{p' \in P_s} x^1(t_1, p', s)$ implies

$$x^{*0} - x^{*1}(t_1, p) < \operatorname*{Max}_{p' \in P} x^{*1}(t_1, p') - x^1(t_1, p, s) + 1/(6\ell + 6)$$

$$< \operatorname*{Max}_{p' \in P} x^1(t_1, p', s) - \operatorname*{Max}_{p' \in P_s} x^1(t_1, p', s) + 1/(3\ell + 3) < 1/(2\ell + 2);$$

hence $A_s = \{p \in P \mid x^1(t_1, p, s) = \operatorname*{Max}_{p' \in P_s} x^1(t_1, p', s)\} \subset Z'_\ell$ (s in Σ^{**}).

By (4.1.3) and the definition of $\mu^j(s, A)$, we see that $\mu^0(s, P - A_s) = 0$ (s in Σ^{**}) and $\mu^0(s, A) \geq 0$ for all Borel sets A, hence

$$\int_P \phi(p)\, d\mu^0(s) = \int_{A_s} \phi(p)\, d\mu^0(s) \leq \int_{Z'_\ell} \phi(p)\, d\mu^0(s) = 0.$$

Letting $s \to \infty$ over Σ^{**}, we get the relations $0 = \int_P \phi(p)\, d\mu^{*0} \geq \int_{Z_\ell} \phi(p)\, d\mu^{*0}$; hence

$$\int_{Z_\ell} \phi(p)\, d\mu^{*0} = \mu^{*0}(Z_\ell) = \int_{Z_\ell} \psi^0(p)\, d\omega = 0.$$

Now $Z = \bigcup_{\ell \in L} Z_\ell$, hence $\int_Z \psi^0(p)\, d\omega = 0$, and this completes the proof that $\psi^0(p) = 0$ a.e. in Z with respect to ω.

The proof that $\omega(U) = 0$ will follow the proof of statement (3.1.3.4).

Proof of (3.1.3.2). The differential equations have been derived in Lemma 5.3. Clearly, $x^*(t_0, p) = \lim x(t_0, p, s) = \lim c_0(\tilde{\xi}(s)) = c_0(\tilde{\xi}^*)$ and

$$h_j^*(t_0, p) = \lim h_j(t_0, p, s) = \delta_j$$

on P, the limits being over Σ^*. By Lemma 5.2,

$$x(t_1, p, s) = c_1(\bar{\xi}(p, s)) \quad (s \text{ in } \Sigma^*;\ p \in P) \qquad \text{and} \qquad \lim \bar{\xi}(p, s) = \bar{\xi}^*(p),$$

hence $x^*(t_1, p) = c_1(\bar{\xi}^*(p)) \in B_1$.

Proof of (3.1.3.3). This follows directly from Lemma 7.1.

Proof of (3.1.3.4). Rewriting (4.1.4) for $Q = P_s$ (s in Σ^*) and letting $\mu(s) = (\mu^1(s), \cdots, \mu^n(s))$, we get the relation

$$\mu(s, P) \cdot c_{0,\xi}(\tilde{\xi}(s))\tilde{\xi}(s) = \underset{\xi \in C_0}{\text{Min}} \ \mu(s, P) \cdot c_{0,\xi}(\tilde{\xi}(s))\xi .$$

By Lemma 6.1, the $\mu(s, P)$ converge over Σ^* to

$$\mu^*(P) = (\mu^{*1}(P), \cdots, \mu^{*n}(P)) = \int_P \psi(p)\, d\omega .$$

Also, $\lim c_{0,\xi}(\xi(s)) = c_{0,\xi}(\xi^*)$ (over Σ^*), by the continuity of $c_{0,\xi}$ (Assumption (2.3.5)). This proves the first support condition.

Let now $\xi(p)$ be a continuous function from P to C_1. Then, by (4.1.5),

$$\int_P \left(d\mu^0(s)\,\delta_1 - \sum_{j=1}^{n} d\mu^j(s) h_j(t_1, p, s) \right) \cdot c_{1,\xi}(\bar{\xi}(p, s))(\bar{\xi}(p, s) - \xi(p)) \leq 0$$

for every s in Σ^*. Because of bounded convergence, we may write

$$\lim_s \int_P = \int_P \lim_s \quad \text{and} \quad \psi^j(p)\, d\omega = d\mu^{*j} \quad (j = 0, \cdots, n);$$

if we let

$$a(p) = c^T_{1,\xi}(\bar{\xi}^*(p)) \left(\psi^0(p)\,\delta_1 - \sum_{j=1}^{n} \psi^j(p) h^*_j(t_1, p) \right),$$

this yields the inequality

$$\int_P a(p) \cdot (\bar{\xi}^*(p) - \xi(p))\, d\omega \leq 0 .$$

Since $\xi(p)$ is an arbitrary continuous function from P to C_1 and C_1 is convex, we may replace $\xi(p)$ by $\bar{\xi}^*(p) + \alpha(p)(\xi(p) - \bar{\xi}^*(p))$, where $\alpha(p)$ is an arbitrary continuous function, except for the restriction $0 \leq \alpha(p) \leq 1$. Thus

$$\int_P a(p) \cdot (\bar{\xi}^*(p) - \xi(p))\, \alpha(p)\, d\omega \leq 0,$$

hence $a(p) \cdot (\bar{\xi}^*(p) - \xi(p)) \leq 0$ a.e. with respect to ω for every continuous $\xi(p)$ from P to C_1. The second support condition now follows directly.

Completion of the proof of (3.1.3.1). It now remains to prove that $\omega(U) = 0$, where U is defined as in (3.1.3.1).

By (3.1.3.4) and Assumption (2.3.6), we can state that, a.e. in P with respect to ω, either $\bar{\xi}(p)$ belongs to the boundary of C_1 or $\psi^0(p)\,\delta_1 - \sum_{j=1}^n \psi^j(p) h_j(t_1, p)$ vanishes. Since $x(t_1, p) \in$ Interior of B_1 for $p \in U$, and since c_1 is a homeomorphism, it follows that

$$\bar{\xi}(p) = c_1^{-1}(x(t_1, p)) \in \text{Interior of } C_1 \qquad \text{for } p \in U.$$

We have previously shown that $\psi^0(p) = 0$ a.e. in Z with respect to ω. Since $U \subset Z$, it follows that $\sum_{j=1}^{n} \psi^j(p) h_j(t_1, p)$ vanishes a.e. in U with respect to ω. Now the $h_j(t_1, p)$ $(j = 1, \cdots, n)$ are linearly independent for every $p \in P$, since $h_j(t, p)$ $(j = 1, \cdots, n)$ is a fundamental set of solutions of linear differential equations (in (3.1.3.2)). It follows that $\psi^j(p) = 0$ $(j = 0, 1, \cdots, n)$ a.e. in U with respect to ω. Since $\sum_{j=0}^{n} |\psi^j(p)| = 1$ a.e. with respect to ω, we conclude that $\omega(U) = 0$.

This completes the proof of Theorem 3.1.

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