THE INSUFFICIENCY OF BARYCENTRIC SUBDIVISION

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A theorem one would like to prove for a reasonable category of spaces is that complexes realized by the same space have isomorphic subdivisions. A tactical question in the formulation of such a theorem concerns the kind of subdivision to be admitted. J. W. Alexander, in his elegant paper on combinatorial topology [1], showed that for homogeneous, finite simplicial complexes, piecewise linear subdivision and stellar subdivision are equally satisfactory. We now present a theorem showing that for locally-finite simplicial complexes, barycentric subdivision is insufficient. It requires only a few pictures to show that under the restriction to barycentric subdivision, the theorem we first mentioned cannot be proved, even for complexes that are realized by a disk. Our main theorem shows that if only barycentric subdivision is allowed, one cannot prove the equivalence theorem for complexes realized by any space save a point.

THEOREM 1. *Let* K *and* L *be connected, locally-finite, simplicial complexes, and let* BK *and* BL *be the complexes of their first barycentric subdivisions. If* BK *and* BL *are isomorphic, then* K *and* L *are isomorphic. If the underlying space* $|K|$ *is not a* 1*-manifold and* K *is neither a simplex nor the boundary of a simplex, then the isomorphism of* BK *and* BL *induces an isomorphism of* K *onto* L.

This theorem implies that if the nth barycentric subdivision $B^n K$ of K is isomorphic to the mth barycentric subdivision $B^m L$, then one of the two complexes K and L is already isomorphic to a barycentric subdivision of the other. If $m > n$, then K is isomorphic to $B^{m-n}L$. Barycentric subdivision is a rigid mechanism.

To illustrate the rigidity of barycentric subdivision, let us partition the vertices of BK by defining $V^i(K)$ to be the set of vertices of BK that appear as barycenters of i-simplices of K. We think of $V^0(K)$ as identical with the collection of vertices of K. In general, it may not be easy to tell whether a given vertex of BK belongs to $V^0(K)$. For example, let $|K|$ be a 2-manifold, and suppose that v is a vertex of K whose star in K contains exactly three 2-simplices. In BK, we cannot tell the star of v from the star of a vertex of $V^2(K)$. Surprisingly, we can still distinguish the set $V^0(K)$ from any other $V^i(K)$, provided $|K|$ is not a 1-manifold and K is neither a simplex nor the boundary of a simplex. Under these conditions, $f(V^0(K)) = V^0(L)$. In fact, the vertex map $f \mid V^0(K)$ defines an isomorphism $\hat{f}$ of K onto L in the familiar way. In addition, f and $\hat{f}$ induce the same homeomorphism of $|K|$ onto $|L|$.

From this last statement it follows that one cannot use barycentric subdivision to construct new simplicial homeomorphisms of a triangulated space onto itself, unless $|K|$ is a 1-manifold or K is either a simplex or the boundary of a simplex. For example, let $|K|$ be a torus, and let h be a homeomorphism of $|K|$ onto itself. If h is simplicial with respect to $B^m K$ for some integer $m \geq 0$, then h is also simplicial with respect to K.

The proof of Theorem 1 depends upon two other combinatorial theorems. One of these, Theorem 3, characterizes the boundary of an $(n + 1)$-simplex $(n > 1)$ as the only connected simplicial complex in which each $(n - h)$-face lies in exactly $(h + 1)$ n-simplices.

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Theorem 1 does not hold for regular cell complexes. One cannot expect a cell complex and its dual to be isomorphic. A 3-simplex is self-dual, but a three-dimensional cube is not. Of course, the first subdivision of a regular complex is simplicial, so that cell complexes aren't completely beyond the reach of the theorem.

1. DEFINITIONS

In what follows, complexes will always be locally-finite and simplicial. If E is a complex, BE will denote the complex determined by the first barycentric subdivision of E. We make no distinction between a vertex in E and the vertex it determines in the complex BE; also, if s is a simplex of E, then bs will denote both the barycenter of s and the vertex determined by it in BE. If E has dimension n, then the *order of* s *in* E, denoted by o(s, E), is the number of n-simplices of E that contain s (of which s is a face). Thus o(bs, BE) denotes the number of n-simplices of BE that contain the vertex bs of BE. If two complexes E and E' are isomorphic under the correspondence g, then o(s, E) = o(g(s), E').

Simplices are to be regarded as closed. The star of a simplex s in E, denoted by st(s, E), is the subcomplex of E consisting of all faces of all simplices of E that *contain* s. The link of s in E, denoted by lk(s, E), is the subcomplex of E determined by the simplices of st(s, E) that do not meet s.

A complex E is said to be *homogeneous* if it has finite dimension n and each simplex of E is a face of some n-simplex of E.

2. THE PROOF IN OUTLINE

Let f be an isomorphism of BK onto BL. Rather than prove the theorem in one stage, we prove it first for homogeneous K and L, and then proceed with relative ease to more general complexes, using their natural decomposition into homogeneous parts. In this section we outline the proof for homogeneous complexes, and we state two combinatorial theorems on which the proof depends. The extension to general complexes is accomplished in Section 7.

Suppose that K and L are homogeneous and that they have dimension $n > 0$ (the proof for $n = 0$ may safely be left as an exercise). We distinguish between two cases, according to whether K is a single n-simplex or contains more than one simplex of dimension n.

In case K contains more than one n-simplex, the principal steps consist in proving that 1) $f(V^0(K) \cup V^n(K)) = V^0(L) \cup V^n(L)$ and 2) if $f(V^0(K)) \cap V^0(L)$ is not empty and f carries some vertex of K to a vertex of L, then $f(V^0(K)) = V^0(L)$ and $f(V^n(K)) = V^n(L)$. It then follows quickly that if f carries a vertex of K to a vertex of L, then f induces an isomorphism of K onto L that agrees with f on vertices of K.

It also follows that if f carries no vertex of K to a vertex of L, then $f(V^0(K)) = V^n(L)$ and $f(V^n(K)) = V^0(L)$. This means that each vertex of K lies in exactly $n + 1$ n-simplices of K, and that each vertex of L lies in exactly $n + 1$ n-simplices of L. If $n = 1$, then $|K|$ and $|L|$ are determined by the fact that they are homeomorphic to one of the two possible 1-manifolds, and it is easy to construct an isomorphism of K onto L. If $n > 1$, then K and L are each isomorphic to the boundary of an $(n + 1)$-simplex, as a consequence of Theorems 2 and 3 (proved in Sections 4 and 5).

THEOREM 2. *If* E *is an* n-*dimensional, homogeneous complex whose mod*-2 *boundary is empty, then* $o(s^{n-h}, E) \geq h + 1$ *for each* (n - h)-*simplex of* E. *If* $o(v, E) = n + 1$ *for each vertex of* E, *then* $o(s^{n-h}, E) = h + 1$ *for each* (n - h)-*simplex of* E.

An (n - 1)-simplex of a homogeneous complex E of dimension n is said to lie in the *mod*-2 *boundary* $M_2(E)$ of E provided it lies in an odd number of n-simplices of E. As a subcomplex of E, $M_2(E)$ is the collection of all faces of such (n - 1)-simplices. A simplex of E that does not lie in $M_2(E)$ is said to be *internal*. Note that M_2 "commutes" with B and with f, and that the barycenter of an n-simplex of E is an internal vertex of BE.

THEOREM 3. *Let* E *be a connected complex of dimension greater than* 1. *If each* (n - h)-*simplex lies in exactly* h + 1 n-*simplices of* E, *then* E *is isomorphic to the boundary of an* (n + 1)-*simplex.*

One can show by induction on n that if K and L are isomorphic to the boundary of an (n + 1)-simplex, there exist $2^n(n + 2)!$ isomorphisms of BK onto BL, in contrast to the (n + 2)! isomorphisms of K onto L.

To return to the proof of Theorem 1 for homogeneous complexes, suppose now that K consists of the faces of a single n-simplex s. Unless f(bs) lies in $V^n(L)$, a vertex of lk(f(bs), BL) lies in $V^n(L)$. But no vertex of f(lk(bs, BK)) = lk(f(bs), BL) is internal. Hence f(bs) is the barycenter of an n-simplex t of L. Hence

$$Bt = f(st(bs, BK)) = f(BK) = BL,$$

and L = t. There are (n + 1)! isomorphisms of K onto L. Note that once again there exists no one-to-one correspondence between isomorphisms of K and isomorphisms of BK, because there are $2^{n-1}(n + 1)!$ isomorphisms of the latter onto BL: one for each of the $2^{n-1}(n + 1)!$ isomorphisms of B(bdry K) onto B(bdry L).

3. PRELIMINARY LEMMAS

LEMMA 1. *If* s^h *is an* h-*face of the* n-*simplex* s^n, *each* h-*simplex of* Bs^h *lies in* (n - h)! n-*simplices of* Bs^n.

The lemma may be proved by mathematical induction as follows: if h < n, then s^h lies in (n - h) (n - 1)-simplices of s^n, and in the barycentric subdivision of each (n - 1)-simplex, each h-simplex lies in (n - 1 - h)! (n - 1)-simplices. Also, the lemma is true for n = 1.

From the fact that the barycenter of s^h lies in (h + 1)! h-simplices in Bs^h comes the following corollary.

COROLLARY 1. *If* s^h *is an* h-*face of the* n-*simplex* s^n, *then*

$$o(bs^h, Bs^n) = (n - h)!\,(h + 1)!\ .$$

From the corollary we draw the following conclusions for later use. If v is the barycenter of an (n - 1)-face of s^n, then $o(v, Bs^n) = n!$. If v is the barycenter of an h-face of s^n with $0 < h < (n - 1)$, then $o(v, Bs^n) < n!$. If s^h lies in i n-simplices of an n-dimensional complex E, then $o(bs^h, BE) = i(n - h)!\,(h + 1)!$. If v is the barycenter of s^n, then $o(v, Bs^n) = (n + 1)!$.

Let e be an edge (1-simplex) of Bs^h that joins bs^h to a vertex v of s^h. From Corollary 1 we know that $o(v, Bs^h) = h!$. Moreover, $o(e, Bs^h) = o(v, Bs^h)$ because $st(v, Bs^h) = st(e, Bs^h)$. Hence $o(e, Bs^h) = h!$. Since each h-simplex of Bs^h lies in $(n - h)!$ n-simplices of Bs^n, we have a second corollary to Lemma 1.

COROLLARY 2. *Let* s^h *be an* h-*face of an* n-*simplex* s^n, *with* $1 \leq h \leq n$, *and let* e *be an edge of* Bs^h *that joins* bs^h *to a vertex of* s^h. *Then*

$$o(e, Bs^n) = (n - h)!\, h!\ .$$

If s^h *lies in* i n-*simplices of an* n-*dimensional complex* E, *then*

$$o(e, BE) = i(n - h)!\, h!\ .$$

LEMMA 2 (the Vertex Lemma). *Let* s *and* t *be simplices of the same dimension, and suppose that* f *is an isomorphism of* Bs *onto* Bt. *If* f(v) *is a vertex of* t *for some vertex* v *of* s, *then* f *carries each vertex of* s *to a vertex of* t.

Thus f induces a one-to-one correspondence between vertices of s and vertices of t.

The argument is clear for dim s = 1. Suppose the lemma is proved for $1 \leq \dim s < n$. Let w be a vertex of s different from v, and let s^{n-1} be an (n - 1)-face of s that contains both v and w. Such a face exists, because $n > 1$. By Corollary 1, $o(bs^{n-1}, Bs) = n!$. Hence $o(f(bs^{n-1}), Bt) = n!$. The only vertices of Bt in st(f(v), Bt), excluding f(v) itself, that are of order n! are barycenters of (n - 1)-faces of Bt (again by Corollary 1). Hence $f(bs^{n-1})$ is the barycenter of an (n - 1)-face t^{n-1} of t. By the induction hypothesis, f carries vertices of s^{n-1} to vertices of t^{n-1}. In particular, f(w) is a vertex of t, and the lemma is proved.

LEMMA 3 (the Face Lemma). *Let* s *and* t *be* n-*simplices, and suppose that* f *is an isomorphism of* Bs *onto* Bt. *If, for* $h < n$, f *carries the subdivision* Bs^h *of an* h-*face* s^h *of* s *onto the subdivision* Bt^h *of an* h-*face* t^h *of* t, *then* f *carries each vertex of* s *to a vertex of* t.

If n = 1, the Face Lemma reduces to the Vertex Lemma. Suppose henceforth that $n > 1$.

If $h < n - 1$, we know from Corollary 1 that a vertex of Bs^h has order n! in Bs if and only if it is a vertex of s and hence of s^h. The analogous statement holds for vertices of Bt^h. Hence, if v is a vertex of s^h, then f(v) must be a vertex of Bt^h with order n! . That is, f(v) must be a vertex of t^h. Apply the Vertex Lemma.

If $h = n - 1$, we conclude from Corollary 1 that Bs^{n-1} has $n + 1$ vertices of order n! in Bs^n. Of these, only one, bs^{n-1}, has all the other vertices in its star in Bs^{n-1} (other vertices occur, because $n > 1$). Since the analogous statement holds for t^{n-1}, it follows that $f(bs^{n-1}) = bt^{n-1}$ and f carries each vertex of s^{n-1} to a vertex of t^{n-1}. Apply the Vertex Lemma.

4. THE PROOF OF THEOREM 2

In [1], we find the following lemma (to which we have added corollaries). Alexander's argument treats finite homogeneous complexes, but it is easily extended to prove the lemma for locally-finite homogeneous complexes as well.

LEMMA 4. *A simplex* s *is internal in a homogeneous complex* F *if and only if* $M_2(\mathrm{lk}(s, F))$ *is empty.*

Note that while $M_2(E)$ is assumed to be empty, in Theorem 2, no such assumption is made here about $M_2(F)$.

COROLLARY 3. *If* s *is both an internal simplex of* F *and a proper face of a simplex* t *of* F, *then* $o(s, F) > o(t, F)$.

COROLLARY 4. *If* v *is an internal vertex of* F, *then* $o(v, F) \geq n + 1$.

We prove Corollary 3 by showing that if $\dim t = 1 + \dim s$, then $o(s, F) > o(t, F)$. Because s is internal, t is internal, so that both $M_2(\mathrm{lk}(s, F))$ and $M_2(\mathrm{lk}(t, F))$ are empty. Let v be the vertex of t not included in s. Certainly $o(s, F) \geq o(t, F)$, so suppose that $o(s, F) = o(t, F)$. Then, with Alexander's notation,

$$\mathrm{st}(t, F) = \mathrm{st}(s, F),$$

$$t \cdot \mathrm{lk}(t, F) = s \cdot \mathrm{lk}(s, F),$$

$$v \cdot s \cdot \mathrm{lk}(t, F) = s \cdot \mathrm{lk}(s, F),$$

$$v \cdot \mathrm{lk}(t, F) = \mathrm{lk}(s, F).$$

By comparing the mod-2 boundary of the complex on each side of the last equation, one sees that $\mathrm{lk}(t, F)$ is empty, which is possible only if $\dim t = \dim F$. But then $\dim s = \dim F - 1$, so that $o(s, F) \geq 2$ while $o(t, F) = 1$.

To prove Corollary 4, let s^n be an n-simplex containing v, and let

$$v = s^0 < s^1 < \cdots < s^i < \cdots < s^n$$

be an ascending sequence of faces of s^n with $\dim s^i = i$. Each s^i is internal in F, because each s^i contains v. Hence $o(s^i, F) > o(s^{i+1}, F)$, so that $o(v, F) \geq n + 1$ because $o(s^n, F) = 1$. Corollary 4 will be used in the proof of Theorem 1.

To prove Theorem 2, we need only Corollary 3. Let E be an n-dimensional, homogeneous complex whose mod-2 boundary is empty, and let s^{n-h} be an $(n - h)$-simplex of E. Let s^n be an n-simplex of E that contains s^{n-h}, and let

$$s^n > s^{n-1} > \cdots > s^{n-h} > \cdots > s^0$$

be a sequence of faces of s^n with $\dim s^i = i$. Each s^i is internal, because $M_2(E)$ is empty. To see that $o(s^{n-h}, E) \geq h + 1$, we apply Corollary 3 to the sequence from s^n to s^{n-h}. To see that $o(s^{n-h}, E) = h + 1$ if $o(s^0, E) = n + 1$, we apply Corollary 3 to the sequence from s^{n-h} to s^0 and deduce that $o(s^0, E) > n + 1$ if

$$o(s^{n-h}, E) > h + 1.$$

5. THE PROOF OF THEOREM 3

We use induction on the dimension of E. Let v be a vertex of E. If $\dim E = 2$, $\mathrm{st}(v, E)$ contains three 2-simplices. Since each edge emanating from v lies in two of these 2-simplices, the three 2-simplices fit together to form a disk, the boundary $\mathrm{lk}(v, E)$ of which consists of three edges e, e_1, and e_2. Let x_i denote the vertex

$e \cap e_i$ of e. One of the two 2-simplices containing e is $v \cdot e$. Let s be the other. Since s does not lie in st(v, E), we can now account for the three 2-simplices of each of the stars $st(x_i, E)$, namely $v \cdot e_i$, $v \cdot e$, and s. Hence s must contain each e_i as well as e. The subcomplex $E' = (st(v, E) \cup s)$ is isomorphic to the boundary of a 3-simplex. That E' is all of E follows from the connectedness of E: If Cl(E - E') is not empty, its intersection with E' contains a vertex whose star contains more than it should.

Suppose that the theorem has been proved for dimension n - 1, and suppose that dim E = n. Then st(v, E) is the union of all faces of all n-simplices containing v, since each simplex of E containing v lies in at least one n-simplex containing v. Hence lk(v, E) is the union of all faces of all (n - 1)-simplices of st(v, E) that do not contain v: $v \cdot lk(v, E) = st(v, E)$. Let s^{n-1-h} be a simplex of lk(v, E). Then $v \cdot s^{n-1-h}$ is an (n - h)-simplex t^{n-h} of st(v, E). By hypothesis, $o(t^{n-h}, E) = h + 1$. Each n-simplex containing t^{n-h} lies in st(v, E), because it contains v. Each n-simplex containing t^{n-h} has an (n - 1)-face in lk(v, E), and these (n - 1)-faces are distinct for distinct n-simplices. Each of these (n - 1)-faces contains s^{n-1-h}. Therefore s^{n-1-h} lies in h + 1 (n - 1)-simplices in lk(v, E). Therefore lk(v, E) is isomorphic to the boundary of an n-simplex, by the induction hypothesis.

Let s_1 be one of the n + 1 (n - 1)-simplices of lk(v, E). By hypotheses, exactly two n-simplices of E contain s_1. One of them is $v \cdot s_1$. Call the other t^n. Note that t^n does not belong to st(v, E) (if it did, it would contain the n + 1 vertices of the n-simplex $v \cdot s_1$, from which we have supposed it to be distinct). Let s_2 be any other (n - 1)-simplex of lk(v, E). Then $s_2 \cap s_1$ is an (n - 2)-simplex, s^{n-2}: lk(v, E) is built that way, because it is isomorphic to the boundary of an n-simplex. By hypothesis, s_2 belongs to exactly two n-simplices of E. One of them is $s_2 \cdot v$. Call the other r^n. By hypothesis, s^{n-2} belongs to exactly three n-simplices of E. But we have now named four n-simplices containing s^{n-2}. Of these $s_2 \cdot v$, $s_1 \cdot v$, and t^n are known to be distinct. Hence r^n must be one of these three. But r^n does not lie in st(v, E) (if it did, it would contain all n + 1 vertices of the n-simplex $s_2 \cdot v$, from which it is supposed to be distinct). Therefore $r^n = t^n$, and s_2 is a face of t^n. Thus each (n - 1)-simplex of lk(v, E) is a face of t^n, so that lk(v, E) is the boundary of t^n.

We next show that $E' = st(v, E) \cup t^n$ is identical with E. Each vertex x of E' lies in n + 1 n-simplices of E'. If x = v, then x lies in the n + 1 n-simplices of st(v, E). If x lies in lk(v, E), then x lies in n n-simplices of st(v, E) and in t^n as well. That E = E' follows from the connectivity of E. We can now construct an isomorphism between E and the boundary S of an (n + 1)-simplex s.

COROLLARY 5. *If* E *is an* n*-dimensional connected complex, if* $|E|$ *is locally-euclidean, and if* $o(v, E) = n + 1$ *for each vertex* v *of* E, *then* E *is isomorphic to the boundary of an* (n + 1)*-simplex.*

6. THE PROOF OF THEOREM 1 FOR HOMOGENEOUS COMPLEXES

Let K and L be homogeneous connected complexes of dimension n, and let f be an isomorphism of BK onto BL. We first prove that if K is not a simplex, then $f(V^0(K) \cup V^n(K)) = (V^0(L) \cup V^n(L))$.

Let v be an internal vertex of K. Since f(v) lies in $V^h(L)$ for some h, with $0 \leq h \leq n$, we can show that f(v) lies in $V^0(L) \cup V^n(L)$ by showing that if h > 0, then $h \geq n$. Let r = o(v, K). Then $o(v, BK) = r \cdot n!$, while $o(f(v), BL) = i \cdot (h + 1)!\,(n - h)!$, where i is the number of n-simplices containing the h-face s^h of L of which f(v) is the barycenter (see Corollary 1). Equating the two orders, we see that

$$i = \frac{r \cdot n!}{(h+1)!\,(n-h)!} .$$

Let e be an edge of BL joining bs^h to a vertex of s^h. Such an edge exists, since $h > 0$. According to Corollary 2, $o(e, BL) = i \cdot h!\,(n-h)!$, which becomes

$$o(e, BL) = \frac{r}{h+1} \cdot n!$$

when we substitute for i. We now find an upper bound on o(e, BL) by investigating $o(f^{-1}(e), BK)$. The edge $f^{-1}(e)$ connects v to the barycenter w of some m-simplex s^m of K, so that by Corollary 2 $o(f^{-1}(e), BK) = j \cdot m!\,(n-m)!$, where $j = o(s^m, K)$. Hence

$$\frac{r}{h+1} \cdot n! = j \cdot m!\,(n-m)! .$$

If $m = n$, then $j = 1$, and $r = h + 1$. But $r \geq n + 1$, since v is internal (Corollary 4), so that $h \geq n$. Even if $m < n$, still m is positive (the edge $f^{-1}(e)$ exists, after all) so that

$$m!\,(n-m)! \leq (n-1)! .$$

Because v is internal, $j < r$ (Corollary 3), and

$$j \cdot m!\,(n-m)! \leq (r-1)(n-1)! .$$

Therefore

$$\frac{r}{h+1} \cdot n \leq r - 1 ,$$

which is possible for $r > 0$ only if $h \geq n$.

A slight modification of the argument just presented for internal vertices of K shows that if v is the barycenter of an n-simplex s^n of K, then f(v) lies in $V^0(L) \cup V^n(L)$. As before, f(v) lies in $V^h(L)$ for some h with $0 \leq h \leq n$, and we show that if $h > 0$, then $h \geq n$. This time, however, $o(v, BK) = (n+1)!$, so that

$$o(e, BL) = \frac{n+1}{h+1} \cdot n! .$$

On the other hand, $f^{-1}(e)$ joins v to a vertex w of B(bdry s^n). Since $o(w, Bs^n) \leq n!$ (by Corollary 1) and $o(f^{-1}(e), BK) = o(f^{-1}(e), Bs^n) = o(w, Bs^n)$, it follows that

$$\frac{n+1}{h+1} \cdot n! \leq n! ,$$

which is possible only if $h \geq n$.

We can now show that if v is a noninternal vertex of K, then f(v) lies in $V^0(L) \cup V^n(L)$. Let s be an n-simplex of st(v, K). Then f(bs) lies in $V^n(L)$, because if it lay in $V^0(L)$, then f(v), and hence v, would be internal. Hence

$$f(Bs) = f(st(bs, BK)) = st(f(bs), BL) = Bt$$

for some n-simplex t of L. We consider two cases.

Case 1: st(v, K) contains more than one n-simplex. Let s_1 and s_2 be distinct n-simplices containing v, and let t_i be the n-simplex of L of which $f(bs_i)$ is the barycenter. Since v lies in $s_1 \cap s_2$, $s_1 \cap s_2$ is a (nonempty) face common to s_1 and s_2 that is carried to the common face $t_1 \cap t_2$ of t_1 and t_2. Since s_1 and s_2 are distinct, $\dim(s_1 \cap s_2) < n$, so that f(v) is a vertex of L by the Face Lemma.

Case 2: st(v, K) has only one n-simplex, s. Since K is homogeneous and connected, and since we have assumed that $K \neq s$, the simplex s contains a vertex w that meets another n-simplex of K. Even if w is not internal, $o(w, K) \geq 2$, so that f(w) is a vertex, by a previous argument. Since f(Bs) = Bt for some n-simplex t of L, f(v) is a vertex by the Vertex Lemma.

Thus $f(V^0(K) \cup V^n(K)) \subset (V^0(L) \cup V^n(L))$. By analogy,

$$f^{-1}(V^0(L) \cup V^n(L)) \subset (V^0(K) \cup V^n(K)),$$

so that $f(V^0(K) \cup V^n(K)) = (V^0(L) \cup V^n(L))$.

It is appropriate to say a word about the restriction that K itself not be a simplex. In Section 2 we showed that if K is a simplex, then L is a simplex. The equation we have just derived may not hold for these simplices, because f may well carry a vertex of K to the barycenter of a bounding (n - 1)-simplex of L.

The next step in proving Theorem 1 for homogeneous complexes is to show that if $f(V^0(K)) \cap V^0(L)$ is not empty, then $f(V^0(K)) = V^0(L)$. Let v be a vertex of K with f(v) in $V^0(L)$. If s is an n-simplex containing v, then f(bs) lies in $V^n(L)$, because no point of $V^0(L)$, besides f(v), lies in st(f(v), BL). Hence f carries each vertex of s to $V^0(L)$, because no point of $V^n(L)$, save f(bs), lies in st(f(bs), BL). In particular, if e is a 1-simplex of K containing v, then f carries both vertices of e to $V^0(L)$, because e lies in some n-simplex of K. That $f(V^0(K)) \subset V^0(L)$ now follows from the connectedness of K: each vertex w of K lies on a path of edges of K emanating from v, and f carries each vertex of the path to $V^0(L)$. By analogy, $f^{-1}(V^0(L)) \subset V^0(K)$, so that $f(V^0(K)) = V^0(L)$.

Because $f(V^0(K) \cup V^n(K)) = (V^0(L) \cup V^n(L))$, we can also conclude that $f(V^n(K)) = V^n(L)$ if $f(V^0(K)) = V^0(L)$. It is also easy to verify that if $f(V^0(K)) \cap V^0(L)$ is empty, then $f(V^n(K)) = V^0(L)$ and $f(V^0(K)) = V^n(L)$. Accordingly, we divide the remainder of the proof into two parts, depending on whether f carries a vertex of K to a vertex of L.

Part 1. Let K and L be connected, homogeneous complexes containing more than one simplex of top dimension, and suppose that f is an isomorphism of BK onto BL that carries a vertex of K to a vertex of L. There exists an isomorphism f' of K onto L that agrees with f on $V^0(K)$.

We proceed by induction on dim K. The assertion is obviously true for dim K = 1. Suppose that it is true for dim K < n, and let dim K = n. Because $f(V^i(K)) = V^i(L)$ for i = 0 and i = n, f carries the n-simplices of K to the n-simplices of L, so that $f \mid BK^{n-1}$ (K^{n-1} is the (n - 1)-skeleton of K) is an isomorphism of BK^{n-1} onto BL^{n-1} carrying vertices of K^{n-1} to vertices of L^{n-1}. Let f' be the hypothetical isomorphism of K^{n-1} onto L^{n-1} agreeing with $f \mid BK^{n-1}$

on $V^0(K^{n-1}) = V^0(K)$. Extend f' to include the n-simplices of K, by defining $f'(s^n)$ to be that n-simplex of L to which f carries bs^n. We observe that f' is a well-defined one-to-one correspondence between K and L. Let s and t be distinct simplices of K. It remains to show that $f'(s \cap t) = f'(s) \cap f'(t)$. If neither simplex has dimension n, then s and t are simplices of K^{n-1}, on which f' is already supposed to be an isomorphism. Suppose that t has dimension n, and that s does not. Then

$$f'(s \cap t) = f'(s \cap \text{bdry } t) = f'(s) \cap f'(\text{bdry } t),$$

because s and bdry t lie in K^{n-1}. But $f'(\text{bdry } t) = \text{bdry } f'(t)$, because f carries the vertices of t to the vertices of f'(t), and f' agrees with f on $V^0(K)$. Hence $f'(s \cap t) = f'(s) \cap \text{bdry } f'(t)$. Since f'(s) lies in L^{n-1},

$$f'(s) \cap f'(t) \subset (L^{n-1} \cap f'(t)) = \text{bdry } f'(t).$$

Hence $f'(s) \cap \text{bdry } f'(t) = f'(s) \cap f'(t)$.

If s and t are both n-simplices, then

$$f'(s \cap t) = f'(\text{bdry } s \cap \text{bdry } t) = f'(\text{bdry } s) \cap f'(\text{bdry } t) = \text{bdry } f'(s) \cap \text{bdry } f'(t).$$

Since f'(s) and f'(t) are distinct n-simplices of L,

$$f'(s) \cap f'(t) = \text{bdry } f'(s) \cap \text{bdry } f'(t).$$

Therefore $f'(s \cap t) = f'(s) \cap f'(t)$.

Part 2. Let K and L be connected, n-dimensional, homogeneous complexes containing more than one n-simplex, and suppose that the isomorphism f of BK onto BL carries no vertex of K to a vertex of L. Then $f(V^0(K)) = V^n(L)$, and each vertex of K is internal, so that $M_2(K)$ is empty. Similarly, $M_2(L)$ is empty. (This can be deduced either from the fact that M_2 commutes with B, or from the fact that $f(V^n(K)) = V^0(L)$.)

If v is a vertex of K, then Corollary 1 implies that

$$o(f(v), BL) = (n+1)! = o(v, BK) = n!\, o(v, K),$$

so that $o(v, K) = n + 1$. By Theorem 2, each (n - h)-simplex of K lies in the minimum number h + 1 of n-simplices of K. By Theorem 3, if $n > 1$, then K is isomorphic to the boundary of an (n + 1)-simplex. If n = 1, then $|K|$ is one of the two possible 1-manifolds. Since analogous statements hold for L here, K and L are isomorphic. It is clear that no isomorphism of K and L can be induced directly by f, because f doesn't carry vertices to vertices.

Since we have already proved Theorem 1 for the case where K is an n-simplex, the completion of Part 2 establishes the theorem in general for connected, homogeneous complexes.

7. THE PROOF OF THEOREM 1 FOR NONHOMOGENEOUS COMPLEXES

Let K and L be connected complexes that are not homogeneous. If there exists an isomorphism f of BK onto BL, then $f(V^0(K)) = V^0(L)$, and there exists an isomorphism f' of K onto L that agrees with f on $V^0(K)$.

Let K_h be the subcomplex of K determined by the h-simplices of K that do not belong to (h + 1)-simplices of K. Define L_h analogously, and observe that if K_h is not empty, then $f(BK_h) = BL_h$. Since K is not homogeneous, K_h is nonempty for at least two distinct values of $h \geq 1$. Let i be one of these values, and let CK_i be a component of K_i. For some j different from i, $CK_i \cap K_j$ is not empty, because K is connected. Let CK_j be a component of K_j for which $CK_i \cap CK_j$ is not empty. Let CL_i and CL_j be the components of L_i and of L_j for which $f(BCK_i) = BCL_i$ and $f(BCK_j) = BCL_j$. Because

$$[V^0(CL_i) \cup V^i(CL_i)] \cap [V^0(CL_j) \cup V^j(CL_j)] \subset [V^0(CL_i) \cap V^0(CL_j)],$$

f carries a vertex of $CK_i \cap CK_j$ to $V^0(CL_i) \cap V^0(CL_j)$. In particular, f carries a vertex of CK_i to a vertex of CL_i. Since CK_i is homogeneous as well as connected, the arguments for homogeneous complexes show that f carries the vertices of CK_i to the vertices of CL_i and that there exists an isomorphism f_i of CK_i onto CL_i that agrees with f on vertices. There is an analogous isomorphism f_j of CK_j onto CL_j, and f_j and f_i agree on $CK_i \cap CK_j$, because they agree on vertices of this intersection. The construction of an isomorphism of K onto L agreeing with f on $V^0(K)$ is now routine.

REFERENCE

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