BOUNDED CONTINUOUS VECTOR-VALUED FUNCTIONS ON A LOCALLY COMPACT SPACE

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1. INTRODUCTION

In this paper we prove a representation theorem for the strict dual of the space of bounded continuous functions from a locally compact space X into a locally convex linear topological space E. As one would expect, the representation is accomplished by means of an identification between the strict dual and a certain space of measures on X whose values lie in the dual of E. Guided by the now familiar technique of de Branges [3], we apply this representation theorem to obtain a generalized Stone-Weierstrass theorem for the strict topology. The study was inspired by the main results in a paper of R. C. Buck [4], and it may be regarded as an extension of Buck's investigation.

Before stating our main result, we introduce some notations. Let X be a locally compact Hausdorff space, and E a complex linear space with a locally convex topology described by a family of seminorms N. By C(X: E) we denote the space of bounded, complex, continuous functions from X into E, and by $C_0(X: E)$ the subspace of C(X: E) consisting of all functions that vanish at infinity. In case E is the complex field, we denote the spaces by C(X) and $C_0(X)$. The uniform topology σ on C(X: E) is defined by the seminorms

$$\|f\|_p = \sup_{x \in X} p(f(x)),$$

where p ranges over N. A weaker locally convex topology on C(X: E) is the strict topology β defined by the seminorms

$$\|f\|_{\phi,p} = \|\phi f\|_p = \sup_{x \in X} p(\phi(x) f(x)),$$

where ϕ ranges over $C_0(X)$ and p ranges over N. See [4] for general properties, and [4], [7], [9] for applications involving the strict topology.

M(X) denotes the set of all finite, complex, regular Borel measures on X, and M(X: E*) denotes the set of all measures μ whose values lie in the dual space E* of E and which satisfy the following conditions:

(1) $\mu(\cdot)s \in M(X)$ for every $s \in E$.

(2) There exist a seminorm $p \in N$ and a constant K such that $\sup |\Sigma \mu(A_i) s_i| \leq K$, where the supremum is taken over all partitions of X into a finite number of disjoint Borel sets $\{A_i\}$ and all finite collections of elements $\{s_i\}$ in E such that $p(s_i) \leq 1$.

THEOREM 1. *If* L *is a strictly continuous linear functional on* C(X: E), *then there exists a* $\mu \in$ M(X: E*) *such that*

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(3) $$L(f) = \int d\mu f$$

for each f ∈ C(X: E). *Conversely, for every* $\mu \in$ M(X: E*) *the formula* (3) *defines a strictly continuous linear functional on* C(X: E).

The integral in (3) is of the Riemann-Stieltjes type obtained by taking the limit under successive refinements of sums of the form $\Sigma\mu(A_i)f(x_i)$, where $\{A_i\}$ is a finite partition of X into disjoint Borel sets and $\{x_i\}$ is a sequence in X such that $x_i \in A_i$. All integrals in the sequel extend over X.

In case E is the complex field, Theorem 1 reduces to a theorem of Buck [4, Theorem 2, p. 99] which states that the strict dual of C(X) is the space M(X), the pairing between functional and measure being that induced by integration.

2. PRELIMINARY RESULTS

We begin with a lemma that expresses the essential character of the strict topology for C(X).

LEMMA 1. *If* $\alpha \in$ M(X), *then there exist a* $\gamma \in$ M(X) *and a* $\phi \in C_0(X)$ *such that* $\alpha = \phi\gamma$, *that is,*

$$\alpha(A) = \int_A \phi d\gamma$$

for every Borel subset A *of* X.

For a proof, see [4, p. 100].

We shall need an analogous result for M(X: E*). For every $\mu \in$ M(X: E*) and every s ∈ E, μs denotes the measure in M(X) whose value at the Borel set A is $\mu(A)s$; $|\mu s|$ denotes the usual variation measure with total variation $\|\mu s\|$. If μ satisfies condition (2) for the seminorm p ∈ N, we define the p-variation $|\mu|_p$ of μ by

(4) $$|\mu|_p(A) = \sup |\Sigma\mu(A_i)s_i|,$$

where the supremum is exactly as in (2) except that only partitions of A are considered. The reader may easily verify that $|\mu|_p$ belongs to M(X). We set

$$\|\mu\|_p = |\mu|_p(X).$$

LEMMA 2. *If* $\mu \in$ M(X: E*), *there exist a* $\nu \in$ M(X: E*) *and a* $\phi \in C_0(X)$ *such that* $\mu = \phi\nu$, *that is,*

(5) $$\mu(A)s = \int_A \phi d\nu s$$

for each s ∈ E *and each Borel set* A *in* X.

Proof. Let p ∈ N be a seminorm for which (2) holds with respect to μ. Since $|\mu|_p$ belongs to M(X) and therefore determines a strictly continuous functional on

$C(X)$, there exists a real $\phi \in C_0(X)$ such that $\left|\int f\,d|\mu|_p\right| < 1$ whenever f is in

$$V = \{f \in C(X): \|\phi f\| < 1\}.$$

Now suppose α is any measure in $M(X)$ with the property $\left|\int f\,d\alpha\right| < 1$ for every $f \in V$ (see [7]). Since $f/[\|\phi f\| + \varepsilon]$ $(\varepsilon > 0)$ is in V for every $f \in C(X)$, it follows that $\left|\int f\,d\alpha\right| \leq \|\phi f\|$. Therefore the mapping T defined by

$$T(\phi f) = \int f\,d\alpha \tag{6}$$

is a σ-continuous functional on the subspace $\phi C(X)$ of $C_0(X)$ with norm $\|T\| \leq 1$. The functional T has (by the Hahn-Banach theorem) a norm-preserving extension T' to $C_0(X)$, where (by the Riesz representation theorem) there exists a $\gamma \in M(X)$ with $\|\gamma\| \leq 1$ such that

$$T'(f) = \int f\,d\gamma \qquad (f \in C_0(X)).$$

Combining this with (6), we see that

$$\int f\phi\,d\gamma = \int f\,d\alpha \qquad (f \in C(X));$$

hence $\alpha = \phi\gamma$. We ensure uniqueness by requiring that γ vanish on the set $S(\phi) = \{x \in X: \phi(x) = 0\}$. Since $|\mu|_p$ satisfies the condition imposed on α, we may write $|\mu|_p = \phi\nu_p$ for some positive measure ν_p in $M(X)$ with $\|\nu_p\| \leq 1$. In fact, the same argument shows that, for each s in E, there is a unique measure νs in $M(X)$ that vanishes on $S(\phi)$ and satisfies the conditions $\|\nu s\| \leq p(s)$ and $\mu s = \phi\nu s$. This assertion follows directly from the fact (see (2) and (4)) that $|\mu s|$ is majorized by $p(s)\,|\mu|_p$ and from the resulting inequality

$$\left|\int f\,d\mu s\right| \leq \int |f|\,d|\mu s| \leq p(s)\int |f|\,d|\mu|_p = p(s)\int |f\phi|\,d\nu_p$$

$$\leq p(s)\,\|\phi f\|\,\|\nu_p\| < p(s) \qquad (f \in V).$$

We are now in a position to describe ν. For each Borel set $A \subset X$, let $\nu(A)$ be the functional on E whose value at the point s is $\nu(A)s$ (the value of the measure νs at A). The identities $\mu(s+t) = \mu s + \mu t$ and $\mu s = \phi\nu s$ show that

$$\nu(A)(s+t) = \nu(A)s + \nu(A)t,$$

and the relation $|\nu(A)s| \leq p(s)$ follows from the inequality $\|\nu s\| \leq p(s)$. Hence $\nu(A) \in E^*$, and ν satisfies condition (1).

Finally, the inequality

$$\left|\sum \nu(A_i)s_i\right| = \left|\sum \int_{A_i} \phi^{-1}\, d\mu s_i\right| \leq \sum \int_{A_i} \phi^{-1}\, d|\mu s_i|$$

$$\leq \sum \int_{A_i} \phi^{-1}\, d|\mu|_p = \int \phi^{-1}\, d|\mu|_p = \int d\nu_p \leq 1$$

($p(s_i) \leq 1$, and $\{A_i\}$ a finite partition of X into disjoint Borel sets) shows that ν satisfies condition (2). Hence $\nu \in M(X\colon E^*)$, and the proof is complete.

LEMMA 3. *If* $\nu \in M(X\colon E^*)$, $\phi \in C(X)$, $\mu = \phi\nu$, *and* ν *satisfies* (2) *for the seminorm* $p \in N$, *then* $|\mu|_p = |\phi|\,|\nu|_p$.

The proof follows standard lines and will be omitted.

The final lemma of this section establishes the form of the σ-continuous linear functionals on $C_0(X\colon E_p)$. (We adopt this notation to mean that N contains only the seminorm p.) This problem was treated by Gowurin [8] and Bochner and Taylor [2] in the special case where $X = [0, 1]$ and E is a Banach space. Indeed, the literature abounds with elegant treatments of variations of this problem [5, p. 543]. However, no reference seems to be readily available to the variation we have in mind, and for this reason we offer a proof.

LEMMA 4. *The general form of the* σ*-continuous linear functional* L *on* $C_0(X\colon E_p)$ *is*

$$L(f) = \int d\mu f, \tag{7}$$

where $\mu \in M(X\colon E_p^*)$ *and* $\|L\| = \|\mu\|_p$.

Proof. For each $s \in E$ and $g \in C_0(X)$, the mapping $gs\colon x \to g(x)s$ represents a member of $C_0(X\colon E_p)$. If L is a σ-continuous linear functional on $C_0(X\colon E_p)$, define the functional L_s ($s \in E_p$) on $C_0(X)$ by $L_s(g) = L(gs)$. The functional L_s is linear, and the inequality

$$|L_s(g)| = |L(gs)| \leq \|L\|\,\|gs\|_p = \|L\|\,p(s)\,\|g\|$$

shows that $\|L_s\| \leq \|L\|\,p(s)$. By the Riesz representation theorem there exists a unique measure $\mu s \in M(X)$ such that $\|\mu s\| = \|L_s\|$ and $L_s(g) = \int g\,d\mu s$. The relation $\mu(as + bt) = a\mu s + b\mu t$ (a, b complex and s, t in E) comes from the fact that L is a bilinear map on $C_0(X) \times E_p$ and, together with the inequality $\|\mu s\| \leq \|L\|\,p(s)$, it shows that the map $s \to \mu(A)s$ defines a continuous linear functional on E_p for each fixed Borel set A in X. It remains to show that the vector-valued measure μ thus defined satisfies condition (2).

Let $\{A_i\}_{i=1}^n$ be a partition of X into disjoint Borel sets, and let $\{s_i\}_{i=1}^n$ be a sequence in E_p. Let ε be positive. Because all measures involved are regular, there exist compact sets $F_i \subset A_i$ such that $|\mu s_i|(A_i \setminus F_i) < \varepsilon/2n$, and open sets $V_i \supset F_i$ such that

$$|\mu s_i|(V_i \setminus F_i) < \varepsilon/2n \qquad (i = 1, 2, \cdots, n)$$

with $V_i \cap V_j = \emptyset$ $(i \neq j)$. Next choose continuous functions g_i with values in $[0, 1]$ such that $g_i(F_i) = 1$ and $g_i(V_i') = 0$. Clearly, the function

$$h = \sum_{i=1}^{n} g_i s_i \tag{8}$$

is in $C_0(X: E_p)$, $\|h\|_p \leq \max\{p(s_1), p(s_2), \cdots, p(s_n)\}$, and

$$L(h) = \sum_{i=1}^{n} L(g_i s_i) = \sum_{i=1}^{n} \int g_i \, d\mu \, s_i .$$

This information, combined with the inequalities $\left|\mu(A_i)s_i - \int g_i \, d\mu \, s_i\right| < \varepsilon/n$, yields the relation

$$\left|\sum_{i=1}^{n} \mu(A_i)s_i\right| < \varepsilon + |L(h)| \leq \varepsilon + \|L\| \max\{p(s_1), p(s_2), \cdots, p(s_n)\} .$$

Letting $\varepsilon \to 0$, we obtain condition (2) for μ.

Simple estimates on approximating sums show that $\int d\mu(gs) = \int g \, d\mu \, s$ for $g \in C_0(X)$ and $s \in E_p$. Thus

$$L(h) = \int d\mu \, h .$$

Since functions of the form (8) with $g_i \in C_0(X)$ $(s_i \in E_p)$ are σ-dense in $C_0(X: E_p)$, it follows in the usual way that $L(f)$ has the form (7).

The argument for the last assertion of the lemma is routine, and we omit it.

3. PROOF OF THEOREM 1

The strict topology has a base of neighborhoods about the origin O in $C(X: E)$ of the form

$$U = \{f \in C(X: E): \|\phi_1 f\|_{p_1} < 1, \|\phi_2 f\|_{p_2} < 1, \cdots, \|\phi_n f\|_{p_n} < 1\},$$

where $\phi_i \in C_0(X)$ and $p_i \in N$ $(i = 1, 2, \cdots, n)$. Since

$$p = \max\{p_1, p_2, \cdots, p_n\} \tag{9}$$

is a seminorm on E, it is clear that U contains the neighborhood

$$V = \{f \in C(X: E): \|\phi f\|_p < 1\}, \tag{10}$$

where $\phi = \max\{|\phi_1|, |\phi_2|, \cdots, |\phi_n|\}$. The new system of seminorms obtained by adjoining to N all possible seminorms p and kp $(k > 0)$ of the form (9) defines the same topology on E. Henceforth we assume that the family N has been so extended, and that accordingly every strict neighborhood of $O \in C(X: E)$ contains a strict neighborhood V defined by a single inequality (10).

Let L be a β-continuous functional on C(X: E). There exist a seminorm $p \in N$ and a $\phi \in C_0(X)$ such that $|L(f)| < 1$ whenever f is in the neighborhood V defined by (10). For $f \in C(X: E)$ and $\varepsilon > 0$, the function $f/[\|\phi f\|_p + \varepsilon]$ is in V; therefore $|L(f)| < \|\phi f\|_p + \varepsilon$, and letting $\varepsilon \to 0$, we see that $|L(f)| \leq \|\phi f\|_p$. This last inequality implies that if we view $M = \phi C(X: E)$ as a subspace of $C_0(X: E_p)$ (see Lemma 4), then the functional T defined by $T(\phi f) = L(f)$ represents a σ-continuous functional on M whose norm does not exceed 1. By the Hahn-Banach theorem, T has an extension T' to $C_0(X: E_p)$ such that

$$|T'(f)| \leq \|f\|_p \qquad (f \in C_0(X: E_p)).$$

According to Lemma 4 there exists a measure $\nu \in M(X: E_p^*) \subset M(X: E^*)$ such that

$$T'(f) = \int d\nu f \qquad (f \in C_0(X: E_p)).$$

Consequently,

$$L(f) = \int d\nu(\phi f) \qquad (f \in C(X: E)). \tag{11}$$

If we set $\mu = \phi\nu$, the integral on the right can be rewritten in the form $L(f) = \int d\mu f$, which is the required representation (3).

It remains to show that every $\mu \in M(X: E^*)$ defines a β-continuous functional on C(X: E) by means of (3). By virtue of Lemma 2, μ has the decomposition $\mu = \phi\nu$ for some $\phi \in C_0(X)$, $\nu \in M(X: E^*)$. Thus (3) may be written in the form (11), which clearly defines a β-continuous functional on C(X: E).

4. A STONE-WEIERSTRASS THEOREM FOR C(X: E)

The purpose of this section is to show how Theorem 1 provides a natural tool in the study of Stone-Weierstrass theorems for C(X: E). Our procedure is that outlined in [7].

THEOREM 2. *Let* X *be a locally compact space, and* E *a locally convex topological space whose topology is given by a family of seminorms* N. *Let* A *be a* β-*closed subspace of* C(X: E) *that satisfies the following conditions:*

(i) $A(x) = \{f(x): f \in A\} = E$ *for every* $x \in X$.

(ii) *If* $\phi \in C(X)$ *has values in* [0, 1] *and* $f \in A$, *then* $\phi f \in A$.

Then A = C(X: E).

Proof. Suppose the conclusion is false. Then there exists a nonzero measure μ in $M(X: E^*)$ such that $\mu \perp A$, in other words, such that $\int d\mu f = 0$ for every $f \in A$. Since μ determines a strictly continuous linear functional on C(X: E), there exists a ϕ in $C_0(X)$ and a p in N such that

$$\left|\int d\mu f\right| < 1 \quad \text{whenever} \quad \|\phi f\|_p < 1. \tag{12}$$

The set V^0 of all measures μ for which (12) holds and $\mu \perp A$ is a convex and weak* compact subset of $M(X: E^*)$, which by the Krein-Milman theorem has an extreme point μ. According to Lemma 2, there exists a ν in $M(X: E^*)$ such that $\mu = \phi\nu$, and since μ is extreme, ν has p-variation equal to 1 on X, that is,

$$|\nu|_p(X) = \|\nu\|_p = 1.$$

Choose an h in C(X) such that $0 < h < 1$. Now observe that the measures $h\mu/\|h\nu\|_p$ and $(1 - h)\mu/\|(1 - h)\nu\|_p$ are in V^0 and

$$\mu = \|h\nu\|_p \frac{h\mu}{\|h\nu\|_p} + \|(1 - h)\nu\|_p \frac{(1 - h)\mu}{\|(1 - h)\nu\|_p},$$

where, by Lemma 3,

$$\|h\nu\|_p + \|(1 - h)\nu\|_p = \int (1 - h)\, d|\nu|_p + \int h\, d|\nu|_p = \int d|\nu|_p = 1.$$

Thus $\mu = h\mu/\|h\nu\|_p$, since μ is extreme. Set $c^{-1} = \|h\nu\|_p$. Then

$$\int g\, d\mu s = c \int gh\, d\mu s$$

for each s in E and g in C(X). Thus h is constant on the support of μs, and since h can be chosen so that it separates points, this implies that the support set is a single point x_0. The identity $\mu(s + t) = \mu s + \mu t$, which holds for each pair $s, t \in E$, shows that each of the measures μs $(s \in E)$ is a point mass at x_0. Since $\mu \perp A$, we conclude that

$$0 = \int d\mu f = \mu(\{x_0\}) f(x_0) \qquad (f \in A). \tag{13}$$

Because μ is nonzero, $\mu(\{x_0\})$ is a nonzero continuous linear functional on E^* whose kernel, according to (13), contains $A(x_0)$, contrary to (i). This contradiction establishes the theorem.

An earlier result of this type [4, Theorem 5, p. 102] required that X be metrizable and E be finite-dimensional. More general results can be established. For example, the result in [7] has a complete analogue in C(X: E).

As an illustration of Theorem 2, consider the family A of all finite linear combinations of functions of the form

$$x \to \phi(x) f(\cdot + x) \qquad (\phi \in C(G),\ f \in L^1(G))$$

from a locally compact abelian group G into its group algebra $L^1(G)$. Each element of A is bounded and continuous with respect to the L^1-norm, and A evidently satisfies conditions (i) and (ii). Hence A is a strictly dense subset of $C(G: L^1(G))$.

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