ON THE COMPLETENESS OF BIORTHOGONAL SYSTEMS

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1. One method of proving the completeness of the eigenfunctions of a Sturm-Liouville boundary value problem uses the fact that an orthonormal sequence which is sufficiently close to a complete orthonormal sequence must be complete. This fact, related to a theorem of Paley and Wiener [4, p. 208] on biorthogonal sequences, is fairly easy to prove directly [1, p. 307]. Since there exist asymptotic estimates for the eigenfunctions of a Sturm-Liouville boundary value problem which show that these eigenfunctions are close to a complete sequence of trigonometric functions [3, p. 66], the completeness of the eigenfunctions follows immediately. The drawback of this approach is that although the necessary Hilbert space theorem is quite easy, the asymptotic estimates require a rather complicated analysis. Much of the effort is in a sense wasted, since the asymptotic estimates are also valid for non-self-adjoint boundary value problems, while the applicability of the Hilbert space theorem is restricted to the self-adjoint case. It is therefore natural to ask whether some appropriate Hilbert space theorem dealing with biorthogonal sequences can be combined with the asymptotic estimates to yield a proof of completeness of the eigenfunctions of a non-self-adjoint boundary value problem. Although the Paley-Wiener theorem mentioned above is not suitable, there exists a similar theorem that serves the purpose. This theorem will be proved, and the result will then be applied to the study of non-self-adjoint boundary value problems.

2. Let H be a separable Hilbert space. A pair of sequences $\{x_n\}$, $\{y_n\}$ of elements of H is said to form a *normalized biorthogonal system* if

$$(x_n, y_m) = \delta_{mn} = \begin{cases} 1 & (n = m), \\ 0 & (n \neq m). \end{cases}$$

This biorthogonal system is said to be *complete* if every $f \in H$ can be written in the form

$$f = \sum_{n=1}^{\infty} (f, y_n)x_n = \sum_{n=1}^{\infty} (f, x_n)y_n .$$

A sequence $\{\phi_n\}$ of elements of H is said to be *orthonormal* if

$$(\phi_n, \phi_m) = \delta_{mn} .$$

An orthonormal sequence $\{\phi_n\}$ is said to be *complete* if every $f \in H$ can be written in the form

$$f = \sum_{n=1}^{\infty} (f, \phi_n)\phi_n .$$

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THEOREM 1. *Let* $\{\phi_n\}$ *be a complete orthonormal sequence, and let* $\{x_n\}$, $\{y_n\}$ *be a normalized biorthogonal system such that*

$$(1) \qquad \sum_{n=1}^{\infty} \|\phi_n - x_n\|^2 < \infty, \qquad \sum_{n=1}^{\infty} \|\phi_n - y_n\|^2 < \infty.$$

Then the system $\{x_n\}$, $\{y_n\}$ *is complete.*

Proof. We begin by proving the theorem with (1) replaced by the stronger condition

$$(2) \qquad \sum_{n=1}^{\infty} \|\phi_n - x_n\|^2 < 1, \qquad \sum_{n=1}^{\infty} \|\phi_n - y_n\|^2 < 1.$$

We define a linear transformation K_1 of H into itself by

$$K_1 f = \sum_{n=1}^{\infty} (f, \phi_n)(\phi_n - x_n)$$

for every $f \in H$. This transformation is well-defined, since the series converges in H. In fact,

$$\|K_1 f\|^2 = \left\| \sum_{n=1}^{\infty} (f, \phi_n)(\phi_n - x_n) \right\|^2 \le \left[\sum_{n=1}^{\infty} |(f, \phi_n)| \cdot \|\phi_n - x_n\| \right]^2$$

$$\le \sum_{n=1}^{\infty} |(f, \phi_n)|^2 \sum_{n=1}^{\infty} \|\phi_n - x_n\|^2 = \|f\|^2 \sum_{n=1}^{\infty} \|\phi_n - x_n\|^2,$$

by the triangle inequality, the Schwarz inequality, and the Parseval equality. In view of (2), this shows that K_1 is a bounded operator, with $\|K_1\| < 1$. We now define $T_1 = I - K_1$, where I is the identity transformation. Since $\|K_1\| < 1$, T_1^{-1} exists. Also, an easy calculation gives

$$T_1 \phi_k = \phi_k - K_1 \phi_k = \phi_k - \sum_{n=1}^{\infty} (\phi_k, \phi_n)(\phi_n - x_n)$$

$$= \phi_k - (\phi_k - x_k) = x_k \qquad [k = 1, 2, \cdots].$$

In exactly the same say, we define another linear transformation K_2 by

$$K_2 f = \sum_{n=1}^{\infty} (f, \phi_n)(\phi_n - y_n).$$

This transformation is bounded, $\|K_2\| < 1$, and if $T_2 = I - K_2$, then T_2^{-1} exists. Also, $T_2 \phi_k = y_k$ $[k = 1, 2, \cdots]$.

Since $T_1 \phi_k = x_k$ and $T_2 \phi_k = y_k$,

$$\delta_{mn} = (x_n, y_m) = (T_1 \phi_n, T_2 \phi_m) = (\phi_n, T_1^* T_2 \phi_m) = (T_2^* T_1 \phi_n, \phi_m) = (\phi_n, \phi_m),$$

where T_1^* and T_2^* are the adjoints of T_1 and T_2, respectively. This implies that $\phi_m - T_1^* T_2 \phi_m$ is orthogonal to every ϕ_n, whence $T_1^* T_2 \phi_m = \phi_m$. Since $\{\phi_m\}$ is a complete set, $T_1^* T_2 f = f$ for every $f \in H$, and T_1^* is a left inverse of T_2. Since we know that T_2 has an inverse, $T_1^* = T_2^{-1}$. In particular, $T_2 T_1^* f = f$ for every $f \in H$. Now

$$f = T_2 T_1^* f = T_2 \sum_{n=1}^{\infty} (T_1^* f, \phi_n)\phi_n$$

$$= T_2 \sum_{n=1}^{\infty} (f, T_1 \phi_n)\phi_n = T_2 \sum_{n=1}^{\infty} (f, x_n)\phi_n = \sum_{n=1}^{\infty} (f, x_n)y_n.$$

An analogous argument shows that $T_2^* T_1 f = T_1 T_2^* f = f$ for every $f \in H$, and this leads to the expansion $f = \Sigma_{n=1}^{\infty} (f, y_n)x_n$. This completes the proof of the theorem with the hypothesis (2) instead of (1).

To complete the proof, we assume (1) and choose an integer N large enough so that

$$(3) \qquad \sum_{n=N+1}^{\infty} \|\phi_n - x_n\|^2 < 1, \qquad \sum_{n=N+1}^{\infty} \|\phi_n - y_n\|^2 < 1.$$

Let S denote the intersection of the following two subspaces of H: S_1, spanned by $x_{N+1}, x_{N+2}, \cdots$, and S_2, spanned by $y_{N+1}, y_{N+2}, \cdots$. Being the intersection of subspaces, S is also a subspace. Let $S^\perp$ denote the orthogonal complement of S. Since $x_1, \cdots, x_N$ are orthogonal to S_2 and $y_1, \cdots, y_N$ are orthogonal to S_1, this orthogonal complement $S^\perp$ contains $x_1, \cdots, x_N$ and $y_1, \cdots, y_N$. In view of (3), the part of the theorem already proved shows that S contains the subspace spanned by $\phi_{N+1}, \phi_{N+2}, \cdots$. This implies that $S^\perp$ is contained in the subspace spanned by $\phi_1, \cdots, \phi_N$, which has dimension N. It is easy to verify, by means of the biorthogonality, that $x_1, \cdots, x_N$ are linearly dependent, and therefore $x_1, \cdots, x_N$ form a basis of $S^\perp$. Thus every $f \in S^\perp$ can be written in the form $f = \Sigma_{n=1}^{N} \alpha_n x_n$, and

$$(f, x_k) = \sum_{n=1}^{N} \alpha_n (x_n, y_k) = \alpha_k.$$

Since H is the direct sum of S and $S^\perp$, every $f \in H$ can be written in the form $f = \Sigma_{n=1}^{\infty} \alpha_n x_n$, with $\alpha_n = (f, y_n)$. An analogous argument shows $f = \Sigma_{n=1}^{\infty} (f, x_n)y_n$, and this completes the proof of the theorem in its full generality.

The theorem extends a result of Birkhoff and Rota [1, p. 307] to the effect that if $\{\phi_n\}$ and $\{x_n\}$ are orthonormal sequences in a Hilbert space H, if $\{\phi_n\}$ is complete, and if

$$(4) \qquad \sum_{n=1}^{\infty} \|\phi_n - x_n\|^2 < \infty,$$

then $\{x_n\}$ is complete. Obviously, this follows from Theorem 1 if we take $y_n = x_n$ for each n.

Theorem 1 is clearly related to a theorem of Paley and Wiener [4, p. 208] which says that if $\{\phi_n\}$ is a complete orthonormal sequence and $\{x_n\}$ is a sequence such that

$$\sum_{n=1}^{\infty} \|\phi_n - x_n\|^2 < 1, \tag{5}$$

then there exists a sequence $\{y_n\}$ such that $\{x_n\}$, $\{y_n\}$ is a complete normalized biorthogonal system. This theorem does not contain the Birkhoff-Rota theorem. It is easy to see that if the sequence $\{x_n\}$ is orthonormal, then the sequence $\{y_n\}$ constructed in the Paley-Wiener theorem is the same as $\{x_n\}$. However, it is impossible to relax the condition (5) to (4) in the Paley-Wiener theorem, and the applications to eigenfunction expansions in [1] and in this paper require the weaker hypotheses (1) or (4).

3. Let L be a linear ordinary differential operator

$$Lx = x^{(n)} + p_1(t)\,x^{(n-1)} + \cdots + p_n(t)\,x,$$

where p_k is a complex-valued function of class C^{n-k} on $a \le t \le b$. Let $Ux = 0$ denote a set of n boundary conditions

$$U_j x = \sum_{k=1}^{n} [M_{jk}\, x^{(k-1)}(a) + N_{jk}\, x^{(k-1)}(b)] = 0.$$

To the eigenvalue problem

$$Lx = \lambda x, \qquad Ux = 0 \tag{6}$$

there corresponds an adjoint problem of the same type,

$$L^+ y = \lambda y, \qquad U^+ y = 0, \tag{7}$$

where L^+ is the adjoint differential operator

$$L^+ y = (-1)^n x^{(n)} + (-1)^{n-1}(\bar{p}_1(t)\,x)^{(n-1)} + \cdots + \bar{p}_n(t)\,x,$$

and where $U^+ y = 0$ denotes another set of boundary conditions determined by $Ux = 0$. It is well-known (see for example [2, p. 310], that the eigenvalues of (7) are the complex conjugates of the eigenvalues of (6) and that the eigenfunctions $x_k(t)$ of (6) and $y_k(t)$ of (7) can be chosen so as to form a normalized biorthogonal system in $L^2(a, b)$.

It is also known [3, p. 66] that for a large class of boundary conditions, called regular boundary conditions, the boundary value problem (6) has an infinite sequence of eigenvalues and two sequences of eigenfunctions $x_n^{(1)}(t)$ and $x_n^{(2)}(t)$ such that

$$x_n^{(1)}(t) = a_n\left[\cos\frac{2\pi nt}{b-a} + O\left(\frac{1}{n}\right)\right], \qquad (n = 0, 1, 2, \cdots)$$
$$x_n^{(2)}(t) = b_n\left[\sin\frac{2\pi nt}{b-a} + O\left(\frac{1}{n}\right)\right].$$

The functions $\cos\frac{2\pi nt}{b-a}$, $\sin\frac{2\pi nt}{b-a}$ are of course complete in $L^2(a, b)$, and

$$\text{(8)} \qquad \left\|x_n^{(1)}(t) - a_n\cos\frac{2\pi nt}{b-a}\right\|^2 = O\left(\frac{1}{n^2}\right), \qquad (n = 0, 1, 2, \cdots)$$
$$\left\|x_n^{(2)}(t) - b_n\sin\frac{2\pi nt}{b-a}\right\|^2 = O\left(\frac{1}{n^2}\right).$$

Since the adjoint problem (7) is of the same type as (6), it also has two sequences of eigenfunctions $y_n^{(1)}(t)$ and $y_n^{(2)}(t)$ such that

$$\text{(9)} \qquad \left\|y_n^{(1)}(t) - a\ \cos\frac{2\pi nt}{b-a}\right\|^2 = O\left(\frac{1}{n^2}\right), \qquad (n = 0, 1, 2, \cdots)$$
$$\left\|y_n^{(2)}(t) - b_n\sin\frac{2\pi nt}{b-a}\right\|^2 = O\left(\frac{1}{n^2}\right).$$

The estimates (8) and (9) together with the convergence of $\Sigma\, 1/n^2$ show that Theorem 1 can be applied. This yields the following result:

THEOREM 2. *The eigenfunctions of a boundary value problem* (6) *with regular boundary conditions, together with the eigenfunctions of the adjoint problem, can be chosen so as to form a normalized biorthogonal system that is complete in* $L^2(a, b)$.

It should be remarked that Theorem 2 is not a best possible result. It can be shown that, in fact, the eigenfunction expansion of a function f is equiconvergent with the Fourier series expansion of f. Here we have shown only that the eigenfunction expansion of a function in $L^2(a, b)$ converges in $L^2(a, b)$. The more precise results require sharper asymptotic estimates of Green's functions and contour integration in the complex plane.

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