ON CERTAIN ALGEBRAS OF ANALYTIC FUNCTIONS

K. V. Rajeswara Rao

1. INTRODUCTION

Given a Riemann surface W, let $\mathrm{Log}^+(W)$ be the set of all analytic functions of bounded characteristic on W (see Heins [1]), together with the constants. Mizumoto and Ozawa [3] have shown that if W_1 and W_2 are hyperbolic plane regions and the algebras $\mathrm{Log}^+(W_1)$ and $\mathrm{Log}^+(W_2)$ are isomorphic, then W_1 and W_2 are conformally equivalent. The present note extends this result to the case where W_1 and W_2 are hyperbolic Riemann surfaces of arbitrary finite genus. Here the method of Mizumoto and Ozawa, which depends on the existence of a univalent function, is not applicable; our approach stems essentially from Royden [4], [5].

To contrast our result with known characterizations (see Royden [4]) of conformal structure in terms of bounded analytic functions, we recall (see Heins [1]) that a hyperbolic Riemann surface of finite genus may not admit nonconstant bounded analytic functions.

In Section 2 we determine the complex-valued homomorphisms of a general class of algebras. In Section 3 we deal with homomorphisms of certain algebras of "rational" functions. In Section 4 we show that the conformal structure of a hyperbolic surface W of finite genus is determined by the set of analytic functions of bounded characteristic on W. We conclude with a remark on the general class of algebras considered in Section 2.

2. SPECTRA OF CERTAIN ALGEBRAS

We consider, on a Riemann surface W, algebras $\mathscr{A}$ (over the complex numbers) of analytic functions. As usual, the operations are point-wise, and $\mathscr{A}$ is always supposed to contain the constants. Further, by a *homomorphism* we always mean a homomorphism that preserves the complex constants. Our interest centers around algebras $\mathscr{A}$ satisfying

(2.1) if $f, g \in \mathscr{A}$ and f/g is analytic on W, then $f/g \in \mathscr{A}$

and

(2.2) $\mathscr{A}$ contains a function, say h, of finite valence.

PROPOSITION 1. *If the algebra* $\mathscr{A}$ *satisfies* (2.1) *and* (2.2), *then to every homomorphism* η *of* $\mathscr{A}$ *into the complex numbers there corresponds a (not necessarily unique)* $p \in W$ *such that*

(2.3) $$\eta(f) = f(p) \qquad (\forall f \in \mathscr{A}).$$

Received April 6, 1964.

This work was supported (in part) by the Air Force Office of Scientific Research.

Proof. Setting $h_1 = h - \eta(h)$, we see that $\eta(h_1) = 0$. This implies that h_1 must vanish somewhere on W; for otherwise $1/h_1 \in \mathscr{A}$ by (2.1), and

$$1 = \eta(1) = \eta\left(h_1 \cdot \frac{1}{h_1}\right) = \eta(h_1)\cdot\eta(1/h_1) = 0.$$

Let then $p_1, \cdots, p_n$ be the zeros of h_1 on W, and μ_i the multiplicity of h_1 at p_i $(1 \leq i \leq n)$. Now, if f is an arbitrary member of $\mathscr{A}$, then, by (2.1),

$$F(z) \equiv \frac{\prod_i \{f(z) - f(p_i)\}^{\mu_i}}{h_1(z)}$$

belongs to $\mathscr{A}$; hence,

$$\prod_i \{\eta(f) - f(p_i)\}^{\mu_i} = \eta(h_1 \cdot F) = \eta(h_1)\cdot\eta(F) = 0.$$

Thus, to each $f \in \mathscr{A}$ there corresponds at least one i $(1 \leq i \leq n)$ such that

$$\eta(f) = f(p_i). \tag{2.4}$$

For each $f \in \mathscr{A}$, let

$$E(f) = \{p_i \mid f(p_i) = \eta(f)\}.$$

The proposition will be established if we show that

$$\bigcap_{f\in\mathscr{A}} E(f) \neq \square.$$

To this end, we observe that, in view of a standard compactness argument, it is enough to show that

$$E(f_0) \cap E(f_1) \cap \cdots \cap E(f_m) \neq \square \tag{2.5}$$

for any finite subset $\{f_0, \cdots, f_m\}$ of $\mathscr{A}$. Consider then the polynomials

$$P(t) \equiv \sum_{j=0}^{m} \eta(f_j)t^j = \eta\left(\sum_{j=0}^{m} f_j t^j\right)$$

and

$$P_i(t) \equiv \sum_{j=0}^{m} f_j(p_i)t^j \qquad (i = 1, 2, \cdots, n).$$

By (2.4), there exists for each t an $i = i(t)$ such that $P(t) = P_i(t)$; together with the Dirichlet box-principle, this implies that, for some i, $P(t) \equiv P_i(t)$, that is, $\eta(f_j) = f_j(p_i)$ $(j = 0, 1, 2, \cdots, m)$, so that (2.5) is valid. ∎

Remark. Note that we do not need the full force of (2.1) to get the conclusion of Proposition 1.

3. ALGEBRAS OF "RATIONAL" FUNCTIONS

In this section, S (with or without subscripts) denotes a closed Riemann surface, and W (with or without subscripts) a proper open subsurface of S. Let $\mathscr{A} \equiv \mathscr{A}(W, S)$ be the set of those meromorphic functions on S that are analytic on W. Clearly, $\mathscr{A}$, regarded as an algebra on W, satisfies (2.1). It also satisfies (2.2), since h can be taken as any meromorphic function (on S) with a single pole outside W.

Also, $\mathscr{A}(W, S)$ separates points of W. For let $z_1, z_2 \in W$, $z_1 \neq z_2$. Pick $g \not\equiv 0$ in $\mathscr{A}(W, S)$ with a zero, of order m, say, at z_1. Let g_2 be a meromorphic function on S with its sole pole, of order n, say, at z_1 and such that $g_2(z_2) = 0$. Then $g = g_1^n \cdot g_2^m$ is in $\mathscr{A}(W, S)$, $g(z_1) \neq 0$, $g(z_2) = 0$. Observe that essentially the same argument shows that $\mathscr{A}(W, S)$ also separates points of S.

We further claim that the algebra $\mathscr{A}(W, S)$ satisfies the following condition:

(δ) if $\{x_n\}$ is a sequence of points of W that does not converge in W, then there exists an f in the algebra such that $\{f(x_n)\}$ does not converge (in the finite plane).

To see this, we first note that since S is compact and satisfies the first countability axiom, $\{x_n\}$ must have limit points in S. If at least one of these limit points, say z, lies outside W, then f can be taken to be any meromorphic function on S with its sole pole at z. On the other hand, if $\{x_n\}$ has two distinct limit points in W, say z_1 and z_2, then f can be taken to be any member of the algebra that separates z_1 and z_2.

We can now readily establish the following result.

PROPOSITION 2. (a) *Let* $\mathscr{B}$ *be the algebra of all analytic functions on the Riemann surface* R, *and let*

$$T: \mathscr{A}(W, S) \to \mathscr{B}$$

be a homomorphism. Then there exists a complex analytic map $\phi: R \to W$ *such that*

$$T(f) = f \circ \phi \qquad (\forall f \in \mathscr{A}(W, S)). \tag{3.1}$$

(b) *If* $W_i \subset S_i$ $(i = 1, 2)$ *and* $T: \mathscr{A}(W_1, S_1) \to \mathscr{A}(W_2, S_2)$ *is an isomorphism (onto), then there exists a conformal homeomorphism* ϕ *of* W_2 *onto* W_1 *such that*

$$T(f) = f \circ \phi \qquad (f \in \mathscr{A}(W_1, S_1)). \tag{3.2}$$

(c) *The* ϕ *in* (3.2) *can be extended to a conformal homeomorphism of* S_2 *onto* S_1.

Proof. Under the hypothesis of (2), the existence of a continuous ϕ satisfying (3.1) follows from Proposition 8 of Royden's paper [5], our Proposition 1, and the observations of this section. The analyticity of ϕ follows from equation (3.1) itself and Riemann's theorem on isolated singularities. Thus (a) is valid. Part (b) now follows on applying part (a) to T and T^{-1}.

To establish part (c), we recall that $\mathscr{A}(W_i, S_i)$, and hence the quotient field Q_i of $\mathscr{A}(W_i, S_i)$, separates points of S_i $(i = 1, 2)$. Further, Q_i contains the constants. Hence, by an observation of Heins [2, p. 270], Q_i is $M(S_i)$, the field of all meromorphic functions on S_i. Thus, the isomorphism T of part (b) induces an isomorphism (again to be denoted by T) of $M(S_1)$ onto $M(S_2)$. Hence, by a classical result (see, for instance, Heins [2], or Royden [4]), there exists a conformal homeomorphism ψ of S_2 onto S_1 such that

$$(Tf)(z) = (f \circ \psi)(z) \qquad (f \in M(S_1),\ z \in S_2).$$

Together with (3.2) and the fact that $\mathscr{A}(W_1, S_1)$ separates points of S_1, this shows that $\psi(z) = \phi(z)$ $(z \in W_2)$, thus completing the proof of Proposition 2.

Remarks. Parts (a) and (b) of Proposition 2 could have been deduced from the results of Heins [2] or Royden [4] and the observation of Heins mentioned in the proof of (c) above. However, the present argument is needed in the next section.

4. FUNCTIONS OF BOUNDED CHARACTERISTIC

It is known (see Heins [1]) that if W is an arbitrary Riemann surface, then the algebra $\mathscr{A} = \mathrm{Log}^+(W)$ satisfies (2.1). If W is of finite genus, hyperbolic, and (as is possible) imbedded in the closed surface S, then (see Heins [1]) $\mathscr{A}(W, S) \subset \mathrm{Log}^+(W)$; thus $\mathrm{Log}^+(W)$ satisfies (2.2) and condition (δ), and it separates points of W. Thus, the reasoning that established parts (a) and (b) of Proposition 2 also shows that (a) and (b) of Proposition 2 remain valid if $\mathscr{A}(W, S)$ is replaced by $\mathrm{Log}^+(W)$ and $\mathscr{A}(W_i, S_i)$ by $\mathrm{Log}^+(W_i)$ $(i = 1, 2)$; here, W and W_i $(i = 1, 2)$ are hyperbolic Riemann surfaces of finite genus. In particular, this establishes the extension, mentioned in the Introduction, of the result of Mizumoto and Ozawa.

5. CONCLUSION

Conjecture. Let $\mathscr{A}_i$ $(i = 1, 2)$ be a separating algebra of analytic functions on the Riemann surface W_i satisfying (2.1) and (2.2). If $T: \mathscr{A}_1 \to \mathscr{A}_2$ is an isomorphism (onto), then there exists a conformal homeomorphism ϕ of W_2 onto W_1 such that

$$Tf = f \circ \phi \qquad (f \in \mathscr{A}_1). \tag{5.1}$$

The existence of a one-to-one mapping ϕ of W_2 onto W_1 satisfying (5.1) follows readily from Proposition 1 and a standard argument. We have not been able to establish the continuity (or equivalently, the analyticity) of ϕ. In this connection, it may be observed that the proof of Theorem C of Heins's paper [2] implies that the quotient field of $\mathscr{A}_i$ $(i = 1, 2)$ provides local coordinates at *every* point of W_i. Also, if the conjecture is true, one can describe all possible isomorphisms between two (not necessarily separating) algebras satisfying (2.1) and (2.2).

REFERENCES

1. M. Heins, *Lindelöfian maps*, Ann. of Math. (2) 62 (1955), 418-446.

2. ———, *Algebraic structure and conformal mapping,* Trans. Amer. Math. Soc. 89 (1958), 267-276.

3. H. Mizumoto and H. Ozawa, *On rings of analytic functions*, Japan. J. Math. 29 (1959), 114-117.

4. H. L. Royden, *Rings of meromorphic functions*, Proc, Amer. Math. Soc. 9 (1958), 959-965.

5. ———, *Function algebras,* Bull. Amer. Math. Soc. 69 (1963), 281-298.

Harvard University