SOME NEGATIVE THEOREMS OF APPROXIMATION THEORY

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Recently there has been some interest in so-called "negative theorems" of approximation theory, or more precisely, in lower bounds for the degree of approximation possible to functions of a given class by means of prescribed approximating functions. The main general tools developed thus far for obtaining lower bounds are the computation of "widths" (see for example Lorentz [3], [4], Tihomirov [12]) and "entropies" (Kolmogorov and Tihomirov [2], Lorentz [4], Vituškin [14]). Negative theorems of more special character are also known for special methods of approximation, for example, Fejér sums (Nikolski [9]), or for projection maps (theorem of Lozinski and Haršiladze, see [1, p. 242]).

A particularly simple type of negative theorem is the corollary to Theorem A, below, due essentially to S. Bernstein. In Section 1 we prove a generalization of this corollary, using the Baire category theorem. In Section 2 an analogous theorem is proved for generalized rational functions. In Section 3 a lower bound is established concerning the degree of approximation by generalized rational functions, and by the analogous nonlinear class that uses a product instead of a quotient of linear forms. The technique here is similar to the estimate of widths in [3], but a new idea is required, namely the use of the combinatory Theorem 5, which seems to be new and perhaps has independent interest. The tendency of the results of Sections 2 and 3 is to suggest that, if one wishes to approximate to functions of a rather "thick" class such as Lip α, then the nonlinear methods considered do not enable one to improve the order of magnitude of the approximation beyond what is possible by a linear method (or even by ordinary polynomial approximation) having the same number of parameters. This is also in accord with the deep investigations of Vituškin [14]. Since it is known on the other hand that rational functions furnish spectacular improvement over polynomials, in certain questions of approximation (Newman [5]), one might surmise that the main strength of rational approximation lies in the approximation of functions with special analytic properties. Finally, we mention that an essentially different type of nonlinear approximation is studied in [8].

1. A GENERAL NEGATIVE THEOREM CONCERNING LINEAR APPROXIMATION

The following theorem (see Timan [12, p. 50]) is in its essential ideas due to S. N. Bernstein. Here $E(x, B)$ denotes $\inf_{y \in B} \|x - y\|$.

THEOREM A. *Let* X *be a Banach space, and* $\{x_n\}$ *a linearly independent, total sequence of elements of* X. *Let* $\{d_n\}$ *be a nonincreasing sequence of positive numbers tending to zero. Then there exists an* $x \in X$ *such that* $E(x, X_n) = d_n$ $(n = 1, 2, \cdots)$. *Here* X_n *denotes the linear manifold spanned by* $x_1, \cdots, x_n$.

The following is an important consequence of this theorem.

COROLLARY. *Under the given hypotheses, there exists an* $x \in X$ *such that the relation* $E(x, X_n) = O(d_n)$ *does not hold. In other words, there exist elements to which the approximation by the* x_i *is arbitrarily poor.*

Received March 17, 1964.

The purpose of this section is to establish the corollary under weaker hypotheses under which the X_n need not be finite-dimensional. Our method of proof, which seems very natural in such problems, is to apply the Baire Category Theorem.

THEOREM 1. *Let* X *be a Banach space, and* $\{X_n\}$ *a sequence of proper closed subspaces. Let* $\{d_n\}$ *be a sequence of positive numbers that tends to zero. Then there exists an* $x \in X$ *such that the relation* $E(x, X_n) = O(d_n)$ *does not hold.*

Proof. With m denoting a positive integer, let

$$Y_m = \{x: E(x, X_n) \leq md_n \text{ for all } n\},$$

$$Y = \{x: E(x, X_n) = O(d_n)\}.$$

Clearly $Y = \bigcup_{m=1}^{\infty} Y_m$; to prove the theorem it therefore suffices, in view of the Baire category theorem, to show that, for each m, Y_m is nowhere dense. Suppose the contrary, and let Y_{m_0} be dense in the ball with center at x_0 and radius $r > 0$. Then Y_{m_0}, being closed, contains this ball, that is, the set of all elements of the form $x_0 + ry$ with $\|y\| \leq 1$. Thus, for every y of norm 1, the elements $x_0 + ry$, $x_0 - ry$, and $-x_0 + ry$ lie in Y_{m_0} (the latter because Y_{m_0} is symmetric about 0). Since Y_{m_0} is convex, it contains $\frac{1}{2}[(x_0 + ry) + (-x_0 + ry)] = ry$, that is, the ball with the center at 0 and radius r. Now choose n_0 so large that $m_0 d_{n_0} < r$. We then deduce that every $x \in X$ of norm r is at a distance at most $m_0 d_{n_0} < r$ from the proper closed subspace X_{n_0}. This contradicts a well-known theorem of F. Riesz ([1, p. 61]), and the theorem is proved.

2. A NEGATIVE THEOREM ON GENERALIZED RATIONAL APPROXIMATION

By a similar argument we may deduce an analogous theorem concerning approximation by generalized rational functions (see [6] for further background on this subject).

Let T denote a compact Hausdorff space, and $C = C(T)$ the Banach space of real-valued continuous functions on T with $\|f\| = \max_{t \in T} |f(t)|$. For $f \in C$ we denote by G(f) the set $\{t: f(t) \neq 0\}$. If two functions $f, g \in C$ have the properties that (i) G(g) is dense in T, and (ii) $f(t)/g(t)$ is uniformly continuous on the set G(g), we denote by $h = f/g$ the unique element of C(T) that coincides with $f(t)/g(t)$ for $t \in G(g)$. By this convention, division of functions in C gives a function in C in certain cases where the denominator vanishes at some points.

THEOREM 2. *Let* $\{X_n\}$, $\{Y_n\}$ *be two sequences of finite-dimensional linear subspaces of* C(T), *and let* R_n *denote the set of* $h \in C(T)$ *of the form* f/g *with* $f \in X_n$, $g \in Y_n$. *Suppose moreover that* T *contains infinitely many points, and let* $\{d_n\}$ *be a sequence of positive numbers tending to zero. Then there exists a function* $k \in C(T)$ *such that the relation* $E(k, R_n) = O(d_n)$ *does not hold.*

Proof. With m denoting a positive integer, let

$$S_m = \{k: E(k, R_n) \le md_n \text{ for all } n\},$$

$$S = \{k: E(k, R_n) = O(d_n)\}.$$

As in the proof above, it suffices to show that for each m, S_m contains no ball. Suppose then, on the contrary, that S_{m_0} contains the ball with center at h_0 and radius $r > 0$.

We remark next that there is no loss of generality in assuming $X_n = Y_n$, for if the theorem were known to be true for $X_n = Y_n$, the (apparently) more general case would follow if we replace both X_n and Y_n by $X_n \oplus Y_n$. We therefore assume $X_n = Y_n$. Since the set of functions in C of the form f/g with f and g in X_n contains $h + a$ for each constant a, if it contains h, it follows that R_n (and hence also each S_m) is invariant under the transformation $h(t) \to h(t) + a$.

Since T is compact and not finite, it contains a point t_0 that is not isolated. Hence, S_{m_0} contains a ball of radius r about p, where $p(t) = h_0(t) - h_0(t_0)$. Choose n_0 so large that $m_0 d_{n_0} < r/2$, and let s denote the dimension of X_{n_0}. Let N denote a neighborhood of t_0 such that $|p(t)| < r/2$ for $t \in N$. Every $h \in R_{n_0}$ has the form $f(t)/g(t)$, where f and g are in X_{n_0}. Let $f_i(t)$ $(i = 1, \cdots, s)$ be a basis for X_{n_0}, and let Z denote the space spanned by the s^2 functions $\{f_i(t) f_j(t)\}$ $(i, j = 1, \cdots, s)$. Then, if

$$h(t) = \frac{f(t)}{g(t)} = \frac{f(t)\,g(t)}{g(t)^2},$$

the relation $\operatorname{sgn} h(t) = \operatorname{sgn}[f(t)g(t)]$ holds for $t \in G(g)$. Choose any $q = s^2 + 1$ distinct points $t_1, t_2, \cdots, t_q$ in $N \cap G(g)$. Since $f(t)g(t) \in Z$ and Z has dimension s^2, there exist real numbers a_i, not all zero, such that $\Sigma_{i=1}^{q} a_i f(t_i) g(t_i) = 0$ for all f and g in X_{n_0}. Writing

$$b_i = \begin{cases} \operatorname{sgn} a_i & (a_i \neq 0), \\ 1 & (a_i = 0), \end{cases}$$

we see that there exists no function $h \in R_{n_0}$ such that $\operatorname{sgn} h(t_i) = b_i$ for $i = 1, \cdots, q$. (This type of argument is well-known; see for example [3, p. 26]). Now, there exists a function $u(t) \in C(T)$ such that $\|u\| = r$ and $u(t_i) = b_i r$ $(i = 1, \cdots, q)$. By assumption $p(t) + u(t)$ lies in S_{m_0}, and at t_i it has the sign b_i and absolute value at least $r/2$. Therefore some $h \in R_{n_0}$ differs from $p + u$ by less than $r/2$ at every point. For this h, $\operatorname{sgn} h(t_i) = b_i$, and this contradiction proves the theorem.

Remarks. One can modify the above argument so that it applies to the case where the functions are complex-valued. Also, one can prove analogous results for other nonlinear combinations of functions such as fg, fgh, $\cdots$ or, indeed, any quotient of polynomials in $f, g, h, \cdots$, instead of f/g.

3. DEGREE OF APPROXIMATION BY GENERALIZED RATIONAL FUNCTIONS

In this section we establish an analogue of a theorem of Lorentz and Tihomirov (see [3, p. 28]); after some preliminary definitions we shall state the latter theorem, for purposes of reference, as Theorem B. Let T be a compact metric space with distance function $d(t_1, t_2)$, and let $\omega(u)$ be a function defined for $u \geq 0$, bounded, nondecreasing, continuous, subadditive, and vanishing at 0 (a so-called "modulus-of-continuity function"). Let Λ^{ω} denote the set of $f \in C(T)$ such that for all $u > 0$ the condition $d(t_1, t_2) \leq u$ implies $|f(t_1) - f(t_2)| \leq \omega(u)$. For any sets A and B we write $E(A, B) = \sup_{x \in A} E(x, B)$.

THEOREM B. *Suppose* T *contains* $n+1$ *points at mutual distance at least* 2a. *Let* X *denote any* n-*dimensional subspace of* C(T). *Then* $E(\Lambda^{\omega}, X) \geq \frac{1}{2}\omega(a)$.

The proof of our next theorem is based on certain combinatorial-geometric properties of Euclidean space. Since the material has independent interest, we state these properties in formal theorems.

THEOREM 3. *Let* X *and* Y *denote subspaces of* C(T), *of dimension* m *and* n *respectively, and let* R *be the set of* $h \in C(T)$ *of the form* f/g *with* $f \in X$, $g \in Y$. *Suppose that* T *contains* N *points at mutual distance at least* 2a, *where* N *satisfies the inequality*

$$(1) \qquad 4\left[\binom{N}{m-1} + \binom{N}{m-3} + \cdots\right]\left[\binom{N}{n-1} + \binom{N}{n-3} + \cdots\right] < 2^N$$

(*here the sums extend to* $\binom{N}{1}$ *or to* $\binom{N}{0}$ *according to the parity of* m, n). *Then* $E(\Lambda^{\omega}, R) \geq \frac{1}{2}\omega(a)$. *In particular,* (1) *holds if* $N = 8(m+n)$.

THEOREM 4. *Let the maximum number of connected components into which Euclidean* k-*space may be partitioned by means of* n *hyperplanes be denoted by* G(k, n). *Then for* $n \geq 1$,

$$(2) \qquad G(k, n) = 2\left[\binom{n}{k-1} + \binom{n}{k-3} + \cdots\right].$$

(By "hyperplane" we mean as usual a (k - 1)-dimensional linear manifold; in particular, it contains the origin. As will be seen from the proof below, any n hyperplanes in "general position" partition k-space into precisely G(k, n) components; note also that the sum in (2) terminates with $\binom{n}{0}$ or $\binom{n}{1}$, according as k is odd or even.) In principle, Theorem 4 is certainly known; the problem of the partitioning of 2-and 3-dimensional space by lines and planes, respectively, that do not necessarily contain the origin, is discussed in [15, p. 176] and [10, p. 43]. The discussion in these books is based on the memoir [11] of Jakob Steiner, from which the method of proof used below is also adapted. The author is indebted to G. Pólya and I. J. Schoenberg for this reference.

Proof of Theorem 4. The result follows by induction from the recurrence relation

(3) $$G(k, n) = G(k, n - 1) + G(k - 1, n - 1) \qquad (k \geq 2,\ n \geq 1)$$

together with the evident formulas

(4) $$G(1, 0) = 1, \qquad G(1, n) = 2 \quad \text{for } n \geq 1,$$

(5) $$G(k, 0) = 1 \quad \text{for } k \geq 1.$$

To prove (3), consider a partition of k-space into $G(k, n)$ components by n hyperplanes "in general position," by which we mean that for each $r \leq \min(k, n)$, the intersection of r of these hyperplanes is a linear manifold of dimension $k - r$. Let π denote one of these hyperplanes. The remaining $n - 1$ hyperplanes each intersect π in a $(k - 2)$-dimensional subspace, and these subspaces partition π (considered as a $(k - 1)$-dimensional space in its own right) into $G(k - 1, n - 1)$ components. Imagine now the configuration that arises from the suppression of π. The remaining $n - 1$ hyperplanes divide the k-space into $G(k, n - 1)$ components K_i. Since π is partitioned into $G(k - 1, n - 1)$ components by these hyperplanes, it follows that π intersects precisely $G(k - 1, n - 1)$ of the K_i, each of which it subdivides into two components. This proves (3), and by an elementary induction we deduce (2).

THEOREM 5. *Let* E_n *denote Euclidean* n*-space (conceived as* n*-tuples* $x = (x_1, \cdots, x_n)$ *in the usual way), and let* K *be a* k*-dimensional subspace. The maximal number of open orthants that* K *can intersect is the number* $G(k, n)$ *defined by* (2). (An *orthant* is one of the 2^n components of E_n determined by the sequence of signs of the coordinates.)

Proof. Let $A = \|a_{ij}\|$ $(i = 1, \cdots, n;\ j = 1, \cdots, k)$ denote any $n \times k$ matrix of rank k, and consider the map $x = Ay$ from E_k into E_n defined by

$$x_i = \sum_{j=1}^{k} a_{ij} y_j \qquad (i = 1, \cdots, n).$$

The range of this mapping is a k-dimensional subspace K of E_n, and any k-dimensional subspace of E_n may be so obtained. The n hyperplanes $\Sigma_{j=1}^{k} a_{ij} y_j = 0$ divide E_k into $G(k, n)$ components; the component to which any point $y \in E_k$ belongs is uniquely determined by the sequence of signs of the numbers $\Sigma_{j=1}^{k} a_{ij} y_j$. Thus, there are precisely $G(k, n)$ sequences of signs possible for the coordinates $(x_1, \cdots, x_n)$ of a point in K, and this proves the theorem.

We can now easily prove Theorem 3. By the argument used to prove Theorem B, it is enough to show that if N satisfies (1) and $t_1, \cdots, t_N$ are distinct points of T, then there exists a sequence $b_1, \cdots, b_N$, with $b_i = \pm 1$, such that for every $h \in R$ the relations $\operatorname{sgn} h(t_i) = b_i$ $(i = 1, 2, \cdots, N)$ are impossible. Now, the sequences $[f(t_1), \cdots, f(t_N)]$ with $f \in X$ form a linear manifold of N-tuples of dimension at most m, and hence by Theorem 5 the maximum possible number of sign sequences is $G(m, N)$. Similarly, the maximum possible number of sign sequences for $[g(t_1), \cdots, g(t_N)]$ is $G(n, N)$. Thus, the maximum possible number of sign sequences for $[h(t_1), \cdots, h(t_N)]$, where $h = f/g$, is at most $G(m, N)G(n, N)$. If this is less than 2^N, some sign sequence cannot be attained. This proves the theorem, except for the last statement.

Suppose now that $N = (m + n)A$, where $m \leq n$. Then, since $G(k, r) \leq r^k/(k - 1)!$,

$$G(m, N)\,G(n, N) \le \frac{mn\,N^{m+n}}{m!\,n!} \le \frac{mn\,e^{m+n}\,(m+n)^{m+n}\,A^{m+n}}{m^m\,n^n}.$$

With the notation $R = n/m$, we can write

$$\frac{(m+n)^{m+n}}{m^m\,n^n} = \left(1+\frac{1}{R}\right)^n (1+R)^{n/R} \le 2^n e^n,$$

and therefore

$$G(m, N)\,G(n, N) \le mn(2Ae^2)^{m+n} \le (4Ae^2)^{m+n}.$$

The last member is less than $2^N = 2^{(m+n)A}$, provided $4Ae^2 < 2^A$. The latter inequality holds for $A \ge 8$, and therefore the proof of Theorem 3 is complete.

Remarks. The theorem clearly remains true if instead of quotients we consider functions of the form $f(t)g(t)$ with $f \in X$, $g \in Y$. We remark also that the main point of our theorem is that we do not assume $g(t)$ to be of constant sign: if $g(t) > 0$, the present argument is unnecessary, since $\operatorname{sgn}[f(t)/g(t)] = \operatorname{sgn} f(t)$, and one obtains a stronger result by the argument in [3]. Note also that the analogous theorem with (1) replaced by $N > mn$ is also true, by a quite trivial argument, but that this theorem is much weaker than Theorem 3.

To illustrate possible applications of Theorem 3, we state the following result; it can also be obtained from the theorems of Vituškin [14] (which, however, are much less elementary).

THEOREM 6. *Let* $T = [0, 1]$, *let* X, Y *be subspaces of* $C(T)$ *of dimension* n, *and let* R *denote the set of* $h \in C(T)$ *of the form* f/g *with* $f \in X$, $g \in Y$. *Then* $E(\Lambda^\omega, R) > A\omega(1/n)$, *where* A *is a positive absolute constant. Moreover, the same results hold if* R *denotes the set of* $h \in C(T)$ *of the form* $f(t)g(t)$.

Remark. This theorem shows that insofar as approximation of the whole class Λ^ω is concerned, the nonlinearity made possible by the passage to generalized rational functions yields no essential improvement over linear approximation, or even (in view of Jackson's theorem) over approximation by ordinary polynomials.

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