ISOMETRIC IMMERSIONS OF CONSTANT CURVATURE MANIFOLDS

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INTRODUCTION

Let M^d and $\overline{M}^{d+k}$ be complete, differentiable (C^∞) Riemannian manifolds of constant sectional curvature C and $\overline{C}$ respectively. It is known that for $C < \overline{C}$ and $k < d - 1$, there exist no isometric immersions of M^d in $\overline{M}^{d+k}$. On the other hand, if $C \geq \overline{C}$ there are many such immersions, even for $k = 1$. We shall investigate the character of these immersions in the critical case $C = \overline{C}$, using a refinement of the method applied to the flat case in [2]. The general idea is that if M^d is not totally geodesic in $\overline{M}^{d+k}$, it must be bent along rather special submanifolds. We prove a precise formulation of this in the next section, and draw some consequences from it in Section 3.

THE MAIN THEOREM

Let M^d *and* $\overline{M}^{d+k}$ *be manifolds with the same constant curvature* C, M^d *being assumed complete. We assume further that* ψ: $M^d \to \overline{M}^{d+k}$ *is an isometric immersion, with* $k < d$. Our notation will be essentially that in [2]. In particular, we express the second fundamental form information of ψ in terms of a tensor T related to the classical operators S_z by the identity $\langle T_x(y), z\rangle = \langle S_z(x), y\rangle$, where $x, y \in M_m$ and $z \in (M_m)^\perp$. (Here $(M_m)^\perp$ denotes the orthogonal complement of $d\psi(M_m)$ in $\overline{M}_{\psi(m)}$.)

If $m \in M$, let $\mathscr{N}(m)$ be the space of null-vectors at m, that is, the subspace of M_m consisting of all vectors x such that $T_x = 0$. There is a useful result (Theorem 2 of [1]) which, though stated for the flat case, applies also in the case at hand. It asserts that *for each point* $m \in M$, *there exists a vector* $y \in \mathscr{N}(m)^\perp$ *such that* T_y *is one-one on* $\mathscr{N}(m)^\perp$. (Here the orthogonal complement is only in M_m.)

Let n be the minimum value of the dimension of $\mathscr{N}(m)$ on M, and let G be the (open) set of M on which this minimum occurs. Then $\mathscr{N}$ is a differentiable field of n-planes on G. Using this notation, we can state our main result.

THEOREM 1. *The field* $\mathscr{N}$ *is integrable on* G; *its leaves are complete, totally geodesic,* n-*dimensional submanifolds of* M^d, *with* $n \geq d - k$. *Furthermore, each leaf is totally geodesic in* $\overline{M}^{d+k}$ *relative to* ψ.

Proof. The last assertion follows immediately from the definition of $\mathscr{N}$. The lower bound for n is a consequence of the theorem of Chern and Kuiper stated above. The proof that $\mathscr{N}$ is integrable on G and that its leaves are totally geodesic is the same as in the flat case. This is true because the proof involves only the relative position of $\psi(M)$ in $\overline{M}$, that is, involves only the second fundamental form and Codazzi equation of ψ. However, the essential feature of the theorem is the *completeness* of the leaves. This depends not merely on relative information, but also

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on the intrinsic geometry of M. The completeness proof we now give will, when reduced to the case $C = \overline{C} = 0$, yield a simplified version of the proof in [2].

We use the following criterion for completeness of a Riemannian manifold L: for some fixed number $\varepsilon > 0$, every geodesic segment $\gamma: [0, b) \to L$ of length less than ε has a geodesic extension $\tilde{\gamma}: [0, c) \to L$ with $c > b$. In applying this criterion to prove the completeness of a leaf L of $\mathcal{N}$, let $\varepsilon = \varepsilon(M)$ be $\pi/2\sqrt{C}$ if the curvature C of M is positive; otherwise let $\varepsilon(M) = 1$.

We need a frame field of a rather special type.

LEMMA 1. *Let* $\gamma: [0, b) \to L$ *be a unit speed geodesic in a leaf* L *of* $\mathcal{N}$ *in* G, *with* $b < \varepsilon(M)$. *Then there exists a frame field* $E = (E_1, \cdots, E_d)$ *on a neighborhood of* γ *in* G *such that*

1. *The geodesic* γ *is an integral curve of* E_1;
2. *Each integral curve of* E_1 *is a geodesic of* M;
3. *The vector fields* $E_1, \cdots, E_n$ *are contained in* $\mathcal{N}$;
4. *The frame field* E *is parallel on* γ.

Proof. Let $x_1, \cdots, x_d$ be a Frobenius coordinate system for $\mathcal{N}$ defined on a neighborhood U of the point $m = \gamma(0)$. We can arrange for $\partial/\partial x_1(m)$ to be the initial velocity $\gamma'(0)$ of γ. Then let Σ be the slice, x_1 constant, of U through m. Denote by $\Sigma \cdot L$ the slice of U through m which has $x_1, x_{n+1}, \cdots, x_d$ constant. Now the (totally geodesic) leaf L has constant curvature C, and the length of γ is less than $\pi/2\sqrt{C}$ if C is positive. Thus, reducing the size of U if necessary, we can find a differentiable vector field E_1 on $\Sigma \cdot L$, tangent to L, such that the geodesics of L with initial velocity E_1 fill a neighborhood of γ in L. Extend E_1 to a vector field (also denoted by E_1) on Σ, with E_1 contained in $\mathcal{N}$. We want to show that those geodesics of M with initial velocities given by E_1 will fill a neighborhood of γ in M. First we state rigorously what this means. Consider the differentiable function $F: \Sigma \times [0, b) \to M$ such that $F(s, t) = \sigma_{E_1(s)}(t)$. (Here σ_x denotes the geodesic of M with initial velocity x.) Note that the curve $t \to F(m, t)$ is the given geodesic γ. We must prove that F *is regular at each point of* $\{m\} \times [0, b)$. By our original choice of the geodesics in L, the restriction of F to $(\Sigma \cdot L) \times [0, b)$ is regular on $\{m\} \times [0, b)$. Now let K be the slice of U through m, complementary to L, and let f be the restriction of F to $K \times [0, b)$. To prove the required regularity of F, it suffices to show that for each $t \in [0, b)$ *the function* $f_t: K \to M$ *such that* $f_t(k) = F(k, t)$ *is regular at* m, *and that the image of* K_m *under the differential map of* f_t *has intersection zero with the tangent space of* L *at* $f_t(m) = \gamma(t)$.

But, using standard properties of foliations, we can deduce this from the following facts:

(1) The above assertion is true for $t = 0$, since f_0 is the inclusion map of K in M.

(2) For each point $k \in K$, the curve $t \to F(k, t)$ lies in the leaf L(k) of $\mathcal{N}$ through k—at least for as long as this curve remains in G.

The preceding construction takes place in M. But since the geodesic γ is in G and G is open in M, we can reduce to G by replacing the geodesics used to define the function F by their maximum initial segments in G. It is now easy to obtain the required frame field. First, define $E = (E_1, \cdots, E_d)$ differentiably on Σ, with E_1 as above and $E_2, \cdots, E_n$ also in $\mathcal{N}$. Then parallel translate out the geodesic segments just mentioned.

It remains to prove the following lemma.

LEMMA 2. *The leaves of* $\mathcal{N}$ *in* G *are complete.*

Proof. We use the criterion given earlier; hence we suppose that $\gamma: [0, b) \to L$ is a unit speed geodesic in a leaf L, with $b < \varepsilon(M)$. We want a geodesic extension of γ in L. Since γ is also a geodesic of M (complete), there exists a geodesic extension $\tilde{\gamma}$ in M. It suffices to show that the point $\tilde{\gamma}(b)$ is in G. We assume the contrary and derive a contradiction. Let $E = (E_1, \cdots, E_d)$ be a frame field as given by Lemma 1. Let $\phi = (\phi_{ij})$ be the connection form of M relative to this frame field, where $1 \leq i, j \leq d$. We exclude the totally geodesic case $n = d$, and adopt the index conventions $1 \leq a, b \leq n$, $n + 1 \leq r, s \leq d$. If $t \in [0, b)$, denote by P(t) the $(d - n) \times (d - n)$ matrix which is the value of $(\phi_{r1}(E_s))$ at $\gamma(t)$. Thus P is a differentiable matrix-valued function on $[0, b)$. Using the Codazzi equation exactly as in [2], we find

$$(*) \qquad \int_0^t \text{trace } P \to +\infty \qquad \text{as } t \to b.$$

We interrupt the proof of this lemma to establish the portion of its proof which is intrinsic to M.

LEMMA 3. *The matrix function* P *defined above satisfies the differential equation* $P' = -P^2 - CI$ *on* $[0, b)$.

Proof. We use this part of the second structural equation:

$$d\phi_{r1} = -\sum \phi_{ri} \wedge \phi_{i1} + C\omega_r \wedge \omega_1 .$$

From properties 1 and 4 in Lemma 1, we find that $\phi(E_1) = 0$ on γ. From property 2, we conclude that $\phi_{r1}(E_1) = 0$ on the whole domain of E. Thus, applying the above structural equation to E_1, E_s yields, on γ, the equation

$$E_1(\phi_{r1}(E_s)) = \phi_{r1}([E_1, E_s]) - C\delta_{rs} .$$

In general,

$$[E_1, E_s] = \sum (\phi_{is}(E_1) - \phi_{i1}(E_s)) E_i .$$

But $\phi(E_1) = 0$ on γ; and, since the leaves of $\mathcal{N}$ are totally geodesic, from property 3 we see that

$$\phi_{r1}(E_a) = \langle \nabla_{E_a}(E_1), E_r \rangle = 0 .$$

Thus the bracket term above reduces to $-\Sigma \phi_{r1}(E_q) \phi_{q1}(E_s)$ on γ, so we have obtained the required differential equation.

Resuming the proof of Lemma 2, we shall show that the differential equation for P is incompatible with the limit condition (*). Consider, for each $t \in [0, b)$, the (complex) eigenvalues and eigenvectors of the (real) matrix P(t). The equation $P' = -P^2 - CI$ shows that P is actually an analytic function, and one can deduce that the eigenvectors of P(t) do not depend on t. It follows that for each eigenvector λ_0 of P(0), we obtain a complex-valued function λ on the interval $[0, b)$ such that

(1) $\lambda(0) = \lambda_0$;

(2) The function λ satisfies the same differential equation as P itself, that is, $\lambda' = -\lambda^2 - C$;

(3) If $t \in [0, b)$, then for all distinct choices of eigenvalues λ_0 of P(0), the numbers $\lambda(t)$ give all distinct eigenvalues of P(t).

To contradict (*), it suffices to show that trace P is bounded above on the interval $[0, b)$. Note that if λ is real-valued, the differential equation in (2) shows that λ is bounded above on the interval. We shall prove that *either* λ is real-valued *or* the magnitude $|\lambda|$ of λ is bounded on the interval.

There are four cases to be considered.

Case 1. C arbitrary, $\lambda_0 = 0$.

Here λ is real-valued on the interval.

Case 2. $C > 0$, $\lambda_0 \neq 0$.

Solving the differential equation for λ yields the result

$$\lambda(t) = \frac{\lambda_0\sqrt{C} - \sqrt{C}\tan(t\sqrt{C})}{\lambda_0\tan(t\sqrt{C}) + \sqrt{C}} \qquad (0 \leq t < b).$$

Since $b < \pi/2\sqrt{C}$, the magnitude $|\lambda|$ of λ is bounded in this case, unless the denominator of this expression is zero for $t = b$. But in that event, $\lambda_0 = -1/\tan(b\sqrt{C})$, so λ is real-valued.

Case 3. $C = 0$, $\lambda_0 \neq 0$.

Then $\lambda(t) = \lambda_0(1 + t\lambda_0)^{-1}$. Thus either $\lambda_0 = -1/b$, and λ is real-valued, or $|\lambda|$ is bounded.

Case 4. $C < 0$, $\lambda_0 \neq 0$.

This case is similar to Case 2, with $\tan(t\sqrt{C})$ replaced by $\tanh(t\sqrt{-C})$.

CONSEQUENCES

We prove three corollaries of Theorem 1. In each case, M(C) denotes a complete manifold of constant curvature C.

COROLLARY 1. *If* $C > 0$ *and* $2k \leq d$, *then every isometric immersion of* $M^d(C)$ *in* $\overline{M}^{d+k}(C)$ *is totally geodesic.*

Proof. If such an immersion exists which is not totally geodesic, we deduce from Theorem 1 that $M^d(C)$ must contain two disjoint, complete, totally geodesic submanifolds, the sum of whose dimensions is at least d. But this is impossible.

This shows, for example, that the sphere $S^d(C)$ is rigid in the sphere $S^{d+k}(C)$ if $2k \leq d$. The next corollaries are analogous to Corollary 2 of [2].

COROLLARY 2. *If* $C < 0$, *there exist no bounded isometric immersions of* $M^d(C)$ *in the hyperbolic space* $H^{2d-1}(C)$.

COROLLARY 3. *If* $C > 0$ *and diameter* $(M^d(C)) < \pi/\sqrt{C}$, *there exist no isometric immersions of* $M^d(C)$ *in the sphere* $S^{2d-1}(C)$.

These two results follow from this consequence of Theorem 1: an immersion must carry some entire geodesic of $M^d(C)$ onto a geodesic of $H^{2d-1}(C)$ or $S^{2d-1}(C)$ respectively.

REFERENCES

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