ON MAHLER'S FUNCTION θ_1

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1. INTRODUCTION

Let $P(x) = \Sigma_{i=0}^{n} a_i x^i$ be a polynomial with rational integral coefficients whose leading coefficient a_n is not 0. We call n the *degree*, and $h = \max_{i \leq n} |a_i|$ the *height* of the polynomial P(x). To every algebraic number α there corresponds a polynomial P(x) of lowest degree with $P(\alpha) = 0$ and such that its coefficients are rational integers without a common divisor. The degree and the height of this polynomial are called the *degree* and the *height* of α, respectively. We denote the set of all polynomials with rational integers as coefficients, and whose degrees and heights are $n \geq 1$ and $h \geq 1$, respectively, by $\mathfrak{P}(n, h)$. In order to characterize transcendental numbers and to decompose the set of all transcendental numbers into different classes, K. Mahler [5] introduced the following functions:

$$w_n(h, \gamma) = \min_{\substack{P \in \mathfrak{P}(n,h) \\ P(\gamma) \neq 0}} |P(\gamma)|,$$

$$w_n(\gamma) = \overline{\lim_{h \to \infty}} - \frac{\log w_n(h, \gamma)}{\log h}$$

$$w(\gamma) = \overline{\lim_{n \to \infty}} \frac{w_n(\gamma)}{n}.$$

It is immediately clear that $w(\gamma) \geq 0$ for all complex numbers γ. It can be shown that $w(\gamma) = 0$ if and only if γ is an algebraic number. Or in other words: γ is transcendental if and only if $w(\gamma) > 0$. (The proof of this and the following unproved statements can be found in T. Schneider's *Einführung in die transzendenten Zahlen* [7], which gives a careful introduction into the subject.) Although it is not immediately apparent from the definition, no real numbers exist for which $0 < w(\gamma) < 1$. In fact, it is not difficult to show that for all real transcendental numbers $\theta_n(\gamma) \geq 1$, and hence also $\theta(\gamma) \geq 1$, where the functions $\theta_n(\gamma)$ and $\theta(\gamma)$ are defined by

$$\theta_n(\gamma) = \frac{w_n(\gamma)}{n},$$

$$\theta(\gamma) = \sup_{n=1}^{\infty} \theta_n(\gamma).$$

The following questions arise: If an arbitrary number $c \geq 1$ is given, do there exist transcendental numbers γ such that $\theta(\gamma) = c$, and is it possible to find such numbers? And secondly, if they exist, what is the value of the fractional dimension

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of the set of all real numbers γ for which $\theta(\gamma) = c$? (For the definition of the fractional dimension see Section 3, Definition 3.)

In this paper we will answer the corresponding questions for the function $\theta_1(\gamma)$ as a first step towards the solution of the problems above. For any real number $c \geq 1$ and any positive integer n, let the set of numbers γ for which $\theta_n(\gamma) = c$ be denoted by $R_n(c)$, and let the set of numbers γ for which $\theta_n(\gamma) \geq c$ be denoted by $S_n(c)$. For the function θ, let the corresponding sets be denoted by $R(c)$ and $S(c)$, respectively. We show that for each $c \geq 1$ the set $R_1(c)$ is not empty. We construct elements of $R_1(c)$ explicitly by means of continued fractions. We find further—and this is our main theorem—that for any $c \geq 1$, the fractional dimension of $R_1(c)$ is given by

$$\dim R_1(c) = \frac{2}{c+1}. \tag{1}$$

A direct consequence of the results about the function $\theta_1(\gamma)$ is the fact that $S_n(c)$ and $S(c)$ are not empty. Since $S(c) \supseteq R_1(c)$, we find from (1), by property (2) of fractional dimension (see Section 3), that

$$\dim S(c) \geq \dim R_1(c) = \frac{2}{c+1}.$$

In addition, it is possible to give a nontrivial lower bound for the fractional dimension of $S_n(c)$ (see the corollary to Theorem 4). The fact that, for any $c \geq 1$, the set $S(c)$ is not empty shows that there exist values of c for which $R(c)$ is not empty. For any given $c > 1$, however, the questions of whether the set $R(c)$ is empty or not, and what its fractional dimension is, remain unanswered.

One reason that it is difficult to generalize the argument used to prove the result about the dimension of $R_1(c)$ is that nonrational algebraic numbers are not as regularly distributed on the real line as rational numbers (see Section 4). Therefore it seems to be necessary to find results about the minimal and maximal distance between algebraic numbers of given height and given degree in the unit interval. Lower estimates for the differences between zeros of different polynomials and between different zeros of the same polynomial are derived in a previous paper of the author [3]. So far, however, no results about the maximal difference of such algebraic numbers are known. This might be a subject of further investigation. Also, it might be possible, by use of the results of [3], at least to generalize Theorem 4 and to obtain the fractional dimension of the set of all real numbers for which $\theta_n(\gamma) \geq c$, where $c > 1$ is otherwise arbitrary.

2. THE CONSTRUCTION OF NUMBERS γ FOR WHICH $\theta_1(\gamma) = c$

The definition of $\theta_n(\gamma)$ given in Section 1 is equivalent to the following alternate one:

Definition 1. Let γ be a real number. Then $\theta_n(\gamma) = c$ if and only if

(a) for each $\varepsilon > 0$ there exists an h_0 such that for all polynomials $P(x)$ of degree n and height $h > h_0$

$$|P(\gamma)| > h^{-cn-\varepsilon}$$

and

(b) for each $\varepsilon > 0$ there exist infinitely many polynomials $P(x) \in \mathfrak{P}(n, h)$ for which

$$|P(\gamma)| < h^{-cn+\varepsilon}.$$

Condition (a) is equivalent to the inequality $\theta_n(\gamma) \leq c$, and (b) is equivalent to the inequality $\theta_n(\gamma) \geq c$.

THEOREM 2. *Let* $c \geq 1$ *be any real number, and let* γ *be a number between* 0 *and* 1 *with a continued fraction expansion* $\gamma = [a_1, a_2, \cdots]$. *Let* p_n/q_n *denote the* n-*th convergent of* γ. *If* γ *has the property that for each* $\varepsilon > 0$ *there exists a number* N *such that*

$$q_n^{c-1-\varepsilon} < a_{n+1} < q_n^{c-1+\varepsilon} \quad \textit{for all } n > N, \tag{2}$$

then

$$\theta_1(\gamma) = c.$$

Proof. To show that $\theta_1(\gamma) \geq c$ it is sufficient to assume that for each $\varepsilon > 0$ the inequality

$$q_n^{c-1-\varepsilon} < a_{n+1} \tag{3}$$

holds for infinitely many n. In the theory of continued fractions it is shown that

$$\left|\gamma - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}\, q_n^2}. \tag{4}$$

From the inequality (3) it follows that for each $\varepsilon > 0$ there exist infinitely many n such that

$$\left|\gamma - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{c+1-\varepsilon}},$$

that is,

$$|q_n \gamma - p_n| < \frac{1}{q_n^{c-\varepsilon}}.$$

Therefore for each $\varepsilon > 0$ there exist infinitely many polynomials $P(x) \in \mathfrak{P}(1, h)$ for which

$$|P(\gamma)| < h^{-c+\varepsilon},$$

since $q_n > p_n$ for $0 < \gamma < 1$. Hence, according to condition (b) of Definition 1,

$$\theta_1(\gamma) \geq c.$$

It remains to be proved that $\theta_1(\gamma) \leq c$. (Compare the remainder of the proof with the proof of Theorem 2 in [6].) Let ε be such that $0 < \varepsilon < c/5$. By assumption there exists an N such that for $n > N$ the inequalities (2) are satisfied. Let p/q be a rational number. Since for each positive k the equation $q_k = a_k q_{k-1} + q_{k-2}$ holds,

(5) $$a_k q_{k-1} < q_k < a_{k+1} q_k .$$

Hence, there exists an n such that

(6) $$\frac{a_n q_{n-1}}{2} \leq q < \frac{a_{n+1} q_n}{2}.$$

Suppose that the denominator of p/q is so large that $n > N + 1$. Then (2) is satisfied for n and $n - 1$. Let $r_n = |\gamma - p_n/q_n|$. By (4), the inequality

$$r_n < \frac{1}{a_{n+1} q_n^2}$$

holds, and hence it follows by (6) that

$$2qq_n r_n < \frac{2q}{a_{n+1} q_n} < 1 .$$

Similarly by (5),

$$2qq_{n+1} r_{n+1} < \frac{2q}{a_{n+2} q_{n+1}} < \frac{2q}{a_{n+1} q_n} < 1 .$$

Since $q_{n+1} > q_n$ we obtain the further inequalities

(7) $$\frac{1}{qq_n} - r_n = \frac{2}{2qq_n} - \frac{2qq_n r_n}{2qq_n} \geq \frac{1}{2qq_n} \geq \frac{1}{2qq_{n+1}}$$

and

(8) $$\frac{1}{qq_{n+1}} - r_{n+1} \geq \frac{1}{2qq_{n+1}} .$$

The inequality (2) for $n - 1$ implies that

$$q_n = a_n q_{n-1} + q_{n-2} < 2a_n q_{n-1} < 2q_{n-1}^{c+\varepsilon} .$$

Thus we find that

$$q_{n+1} < 2q_n^{c+\varepsilon} < 2(2q_{n-1}^{c+\varepsilon})^{c+\varepsilon} = 2^{1+c+\varepsilon} q_{n-1}^{(c+\varepsilon)^2} .$$

Now using the inequality (2) for $n - 1$, together with (6), we conclude

$$\frac{q_{n-1}^{c-\varepsilon}}{2} < \frac{a_n q_{n-1}}{2} \leq q .$$

Therefore,

(9) $$q_{n+1} < 2^{1+c+\varepsilon} (2q)^{\frac{(c+\varepsilon)^2}{c-\varepsilon}} < 2^{1+c+\varepsilon} (2q)^{c+4\varepsilon} = 2^{1+2c+5\varepsilon} q^{c+4\varepsilon} ,$$

since $\varepsilon < c/5$. From $p_n/q_n \neq p_{n+1}/q_{n+1}$ we conclude that one of the numbers

$$p_n q - q_n p \quad \text{and} \quad p_{n+1} q - q_{n+1} p$$

does not vanish and therefore has absolute value at least equal to 1. From the inequalities (7) and (8) we see, consequently, that at least one of the following relations holds:

$$\left|\gamma - \frac{p}{q}\right| = \left|\frac{p_n q - q_n p}{qq_n} + \gamma - \frac{p_n}{q_n}\right| \geq \frac{1}{qq_n} - r_n \geq \frac{1}{2qq_{n+1}},$$

$$\left|\gamma - \frac{p}{q}\right| = \left|\frac{p_{n+1} q - q_{n+1} p}{qq_{n+1}} + \gamma - \frac{p_{n+1}}{q_{n+1}}\right|$$

$$\geq \frac{1}{qq_{n+1}} - r_{n+1} \geq \frac{1}{2qq_{n+1}}.$$

By (9), the estimate

$$|q\gamma - p| \geq \frac{1}{2q_{n+1}} \geq 2^{-2-2c-5\varepsilon} q^{-c-4\varepsilon}$$

follows. Since this holds for any fraction p/q whose denominator is sufficiently large, it is clear that for every sufficiently large h,

$$w_1(h, \gamma) \geq 2^{-2-2c-5\varepsilon} h^{-c-4\varepsilon},$$

$$-\frac{\log w_1(h, \gamma)}{\log h} \leq \frac{(2 + 2c + 5\varepsilon)\log 2}{\log h} + c + 4\varepsilon,$$

and hence

$$\theta_1(\gamma) \leq c + 4\varepsilon.$$

Since this is true for each positive ε, we obtain finally the desired conclusion

$$\theta_1(\gamma) \leq c.$$

3. THE FRACTIONAL DIMENSION OF THE SET OF ALL NUMBERS γ FOR WHICH $\theta_1(\gamma) \geq c$

We now define the fractional dimension of a linear set S (see [2]).

Definition 3. Let $U(S, r)$ be any covering of S by a countable number of open intervals $I_1, I_2, \cdots$ of lengths $d_1, d_2, \cdots$, where $d_i < r$ for $i = 1, 2, \cdots$. Let

$$|S|_t = \lim_{r \to 0} \inf_{U(S,r)} \left(\sum_{I_i \in U(S,r)} d_i^t \right)$$

where the infimum is taken over all coverings $U(S, r)$ of the kind described above. Then we say that S *has fractional dimension* α, and we write $\dim S = \alpha$ if $|S|_t = 0$ for all $t > \alpha$ and $|S|_t = \infty$ for all $t < \alpha$.

The fractional dimension has the following properties, in which $M, M_1, M_2, \cdots$ are countably many sets of real numbers.

(1) If $\dim M < 1$, then the Lebesgue measure of M is 0.

(2) If M_1 is a subset of M_2, then $\dim M_1 \leq \dim M_2$.

(3) $$\dim \bigcup_{n=1}^{\infty} M_n = \sup_{n=1}^{\infty} \dim M_n .$$

Properties (1) and (2) follow directly from the definition. For the proof of (3) see B. Volkmann [8].

V. Jarnik [4] proved the following theorem:

Let $c > 1$ *be any real number and let* $A(c)$ *be the set of all numbers* γ *with* $0 < \gamma < 1$ *for which the inequality*

(10) $$\left|\gamma - \frac{p}{q}\right| < q^{-c-1}$$

has infinitely many solutions in integers p, q *with* $q > 0$. *Then*

$$\dim A(c) = \frac{2}{c+1} .$$

It is clear that, for each $\gamma \in A(c)$, $\theta_1(\gamma) \geq c$; in other words, $A(c) \subset S_1(c)$. For since $|\gamma| < 1$ and $c > 1$, we see that $p \leq q$ for those integers p and q that satisfy the inequality (10). Therefore to every $\gamma \in A(c)$ there correspond infinitely many polynomials $P(x) \in \mathfrak{P}(1, h)$ for which

$$|P(\gamma)| < h^{-c} .$$

Hence for such γ, $\theta_1(\gamma) \geq c$, since the inequality in Definition 1, part (b) is satisfied.

We will use the relation $A(c) \subset S_1(c)$ to demonstrate

THEOREM 4. *Let* $c > 1$ *be any real number. Then the set* $S_1(c)$ *of all real numbers* γ *between* 0 *and* 1 *for which* $\theta_1(\gamma) \geq c$ *and the corresponding set* $S_1'(c)$ *of numbers* γ *for which* $\theta_1(\gamma) > c$ *have the fractional dimensions*

$$\dim S_1(c) = \dim S_1'(c) = \frac{2}{c+1} .$$

Proof. Since $S_1'(c) \subset S_1(c)$ it is sufficient to prove that

(11) $$\dim S_1'(c) \geq \frac{2}{c+1}$$

and that

(12) $$\dim S_1(c) \leq \frac{2}{c+1}$$

hold.

We first establish the relation

$$S_1'(c) = \bigcup_{n=1}^{\infty} S_1\left(c + \frac{1}{n}\right) \tag{13}$$

as follows. If

$$\gamma \in \bigcup_{n=1}^{\infty} S_1\left(c + \frac{1}{n}\right),$$

then there exists an integer n_0 for which $\gamma \in S_1(c + 1/n_0)$. Hence $\theta_1(\gamma) \geq (c + 1/n_0)$. Since $n_0^{-1} > 0$, the number γ belongs to $S_1'(c)$. Similarly, if $\gamma \in S_1'(c)$, there must exist a number n_0 such that

$$\gamma \in S_1\left(c + \frac{1}{n_0}\right) \quad \text{so that} \quad \gamma \in \bigcup_{n=1}^{\infty} S_1\left(c + \frac{1}{n}\right).$$

This proves (13), and this equation in connection with property (3) of fractional dimension yields the equation

$$\dim S_1'(c) = \sup_{n=1}^{\infty} \dim S_1\left(c + \frac{1}{n}\right).$$

Now using the fact that $A(c) \subset S_1(c)$ for each c, we obtain from Jarnik's theorem that

$$\dim S_1\left(c + \frac{1}{n}\right) \geq \dim A\left(c + \frac{1}{n}\right) = \frac{2}{c + 1 + 1/n}.$$

Hence

$$\dim S_1'(c) \geq \sup_{n=1}^{\infty} \frac{2}{c + 1 + 1/n} = \frac{2}{c + 1},$$

which completes the proof of (11).

On the other hand, if $\gamma \in S_1(c)$, then to every $\varepsilon > 0$ there correspond infinitely many polynomials $P(x) = qx - p$ with integral coefficients such that

$$|q\gamma - p| < q^{-c+\varepsilon},$$

or such that

$$\left|\gamma - \frac{p}{q}\right| < q^{-c-1+\varepsilon}.$$

This implies that $S_1(c) \subset A(c - \varepsilon)$, and from this relation it follows that

$$\dim S_1(c) \leq \dim A(c - \varepsilon) = \frac{2}{c + 1 - \varepsilon}.$$

This inequality is valid for every $\varepsilon > 0$. Therefore

$$\dim S_1(c) \leq \frac{2}{c + 1}.$$

Hence the inequality (12) holds also.

Since $\theta(\gamma) = \sup_{n=1}^{\infty} \theta_n(\gamma)$, it is clear from Theorem 2 or from Theorem 4 that for each $c > 1$ the set $S(c)$ is not empty. It follows in fact that

$$\dim S(c) \geq \frac{2}{c + 1}.$$

Theorem 4 has also the following consequence:

COROLLARY 5. *Let* $c > 1$ *be any real number, and let* n *be any positive integer. Then*

$$\dim S_n(c) \geq \frac{2}{nc + 1} > 0.$$

Proof. It follows from the definition of $w_n(h, \gamma)$ that the inequality

$$w_{n+1}(h, \gamma) \leq w_n(h, \gamma)$$

holds for all positive integers n and h, and that thus also

$$(n + 1)\theta_{n+1}(\gamma) \geq n\theta_n(\gamma).$$

Hence

$$n\theta_n(\gamma) \geq \theta_1(\gamma).$$

For any $\gamma \in S_1(nc)$ we therefore see that

$$n\theta_n(\gamma) \geq cn,$$

or

$$\theta_n(\gamma) \geq c.$$

This shows that $S_1(nc) \subset S_n(c)$. Thus

$$\dim S_n(c) \geq \dim S_1(nc) = \frac{2}{nc + 1} > 0.$$

Clearly, the corollary implies that the set $S_n(c)$ is not empty. The relation $S_1(nc) \subset S_n(c)$ in combination with Theorem 2 gives a means of finding elements of $S_n(c)$.

4. TWO LEMMAS ABOUT THE UNIFORM DISTRIBUTION OF THE RATIONAL NUMBERS

We introduce for every positive integer q the set B_q of intervals

$$\left(\frac{p}{q} - d_q, \frac{p}{q} + d_q\right) \qquad (p = 1, 2, \cdots, q).$$

The intervals of B_q thus have constant length $2d_q < 1/q$, which depends only on q. Let r be any natural number. We say that a set B of intervals is a *set of uniformly distributed intervals of order* r if

$$B = \bigcup_{q \in J} B_q,$$

where J is any set of positive integers at least as great as r. For any set M of not necessarily disjoint intervals let us denote by $|M|$ the sum of the lengths of the intervals of M, if it exists, and by Z(M) the number of elements of M if M is finite. Finally, for any two sets M and N of intervals, let the set which consists of those intervals of M that have points in common with some interval of N be denoted by $M|N$, and let the set consisting of those intervals of M which are completely covered by intervals of N be denoted by M^N.

LEMMA 6. *Let* q *be a positive integer. Let* C *be any open or closed set of intervals in the unit interval, each of which has length at least equal to* 8/q. *Then*

$$\frac{1}{2}q|C| < Z(B_q^C) \leq Z(B_q|C) < 2q|C|$$

for any choice of the length $2d_q$ *of the intervals of* B_q (where $2d_q < 1/q$).

Proof. Let $C = \bigcup_{j=1}^m C_j$, where C_j are the intervals of C $(j = 1, 2, \cdots, m)$. Any interval of C_j covers at least $[|C_j|q] - 1$ midpoints of intervals of B_q, and hence covers at least $[|C_j|q] - 3$ intervals of B_q completely. Since the number m of intervals of C is bounded above by $[q|C|/8]$, we obtain the following estimate:

$$Z(B_q^C) \geq \sum_{j=1}^m ([|C_j|q] - 3) > \sum_{j=1}^m (|C_j|q - 4)$$

$$\geq q\sum_{j=1}^m |C_j| - 4\frac{q|C|}{8} = \frac{1}{2}q|C|.$$

To show the right-hand inequality of the lemma, we notice that an interval C_j of C intersects at most

$$[|C_j|q] + 2 \leq |C_j|q + 2$$

intervals of B_q. Therefore, C intersects at most

$$\sum_{j=1}^m (|C_j|q + 2) \leq q|C| + 2\frac{q|C|}{8} < 2q|C|$$

intervals of B_q. This proves Lemma 6.

LEMMA 7. *Let* B *be a set of uniformly distributed intervals of order* r. *Let* C *be a set of open or closed intervals in* (0, 1), *each of which has length at least equal to* 8/r. *Then*

$$\frac{1}{2}Z(B)|C| < Z(B^C) \leq Z(B|C) < 2Z(B)|C|.$$

Proof. Let $B = \bigcup_{q \in J} B_q$ $(q \geq r)$. Since the length of the intervals of C is at least equal to $8/r \geq 8/q$, we can, for each q separately, apply Lemma 6. Therefore,

$$Z(B^C) = \sum_{q \in J} Z(B_q^C) \geq \sum_{q \in J} |C|\frac{q}{2} = \frac{|C|}{2} \sum_{q \in J} q = \frac{|C|}{2} Z(B),$$

since $q = Z(B_q)$. In the same way one verifies the right-hand inequality.

This lemma indicates in which way the "uniform distribution" of the rational numbers in the unit interval will be used.

5. THE FRACTIONAL DIMENSION OF THE SET OF ALL NUMBERS γ FOR WHICH $\theta_1(\gamma) = c$

We have shown in Theorem 4 that the set $S_1(c)$ of all real numbers γ for which $\theta_1(\gamma) \geq c$ has the fractional dimension $2(c+1)^{-1}$. The set $R_1(c)$ of all numbers γ for which $\theta_1(\gamma) = c$ can be written as

$$R_1(c) = S_1(c) - \bigcup_{n=1}^{\infty} S_1\left(c + \frac{1}{n}\right).$$

Here

$$\dim S_1(c) = \frac{2}{c+1} \quad \text{and} \quad \dim S_1\left(c + \frac{1}{n}\right) = \frac{2}{c + \frac{1}{n}} < \frac{2}{c+1}.$$

We shall prove that

$$\dim R_1(c) = \dim S_1(c) = \frac{2}{c+1}.$$

The question arises whether this is a general property of fractional dimension; whether it is true that if M is given and if M_n is a sequence of sets for which $\dim M_n < \dim M$ for each n, then

$$\dim M = \dim\left(M - \bigcup_{n=1}^{\infty} M_n\right).$$

If there is a constant k such that $\dim M_n < k < \dim M$ for each n, then this equation is a direct consequence of property (3) of fractional dimension. However, if $\lim_{n \to \infty} \dim M_n = \dim M$, then it may be that

$$\dim\left(M - \bigcup_{n=1}^{\infty} M_n\right) < \dim M.$$

Thus, for instance, for any $c > 1$

$$\emptyset = S_1'(c) - \bigcup_{n=1}^{\infty} S_1'\left(c + \frac{1}{n}\right)$$

with

$$0 = \dim\left[S_1'(c) - \bigcup_{n=1}^{\infty} S_1'\left(c + \frac{1}{n}\right)\right] < \dim S_1'(c) = \frac{2}{c+1}.$$

Thus the following theorem is not trivial.

THEOREM 8. *Let* $c > 1$ *be any real number. Then*

$$\dim R_1(c) = \frac{2}{c+1}.$$

A. S. Besicovitch [1] gave a second proof of the theorem of Jarnik. We will extend his ideas to obtain a proof of Theorem 8.

Proof. We shall establish the inequalities

$$\dim R_1(c) \leq \frac{2}{c+1}$$

and

$$\dim R_1(c) \geq \frac{2}{c+1}. \tag{14}$$

The first follows directly from Theorem 4: since

$$R_1(c) \subset S_1(c),$$

$$\dim R_1(c) \leq \dim S_1(c) = \frac{2}{c+1}.$$

The second is much more difficult to prove, and the remainder of this section will be devoted to its proof. Let r be a real number such that $0 < r \leq 1$. Let $S(r, s)$ be a countable system of open intervals with lengths $d \leq r$ for which

$$\sum_{S(r,s)} d^s < 1 \qquad \left(s = \frac{2}{c+1}\right) \tag{15}$$

where the sum is taken over all members of $S(r, s)$. We shall show that if r is sufficiently small, then $S(r, s)$ does not cover the set $R_1(c)$ completely. Since $S(r, s)$ is otherwise arbitrary, this means that there exists no system $S(r, s)$, with the given properties, that covers $R_1(c)$ completely. Hence, for all systems $U(R_1(c), r)$,

$$\sum_{U(R_1(c),r)} d^s \geq 1;$$

and, as a consequence,

$$|R_1(c)|_s = \lim_{r\to 0} \inf_{U(R_1(c),r)} \Big(\sum_{I\in U(R_1(c),r)} d^s \Big) \geq 1$$

(see Definition 3 of Section 3 for the exact meaning of the summation). Hence, for each $t < s = 2(c+1)^{-1}$, $|R_1(c)|_t = \infty$, and thus $\dim R_1(c) \geq s = 2(c+1)^{-1}$.

We now put $k = c + 1$. Let ε_i $(i = 0, 1, \cdots)$ be a null sequence such that

$$\frac{k-2}{2} > \varepsilon_0 > \varepsilon_1 > \cdots > 0;$$

this is possible since $k = c + 1 > 2$. Further, set

$$k_i = c + 1 - \varepsilon_i, \quad k_i' = c + 1 + \varepsilon_i .$$

Then, evidently, $k_i' > k_i > 2$ $(i = 0, 1, \cdots)$. We choose an increasing sequence of positive integers

$$n_0, n_1, n_2, \cdots$$

which has the properties that n_0 is sufficiently large and that the sequence n_i is so rapidly increasing that certain inequalities hold between each two succeeding terms. These inequalities will be specified later. Let $S(r, s)$ be an arbitrary system of intervals such that $r = n_0^{-k}$ and $s = 2/k$. For $i = 1, 2, \cdots$, let F_i be the set of intervals

$$\left(\frac{p}{q} - \frac{1}{q^{k_i}}, \frac{p}{q} + \frac{1}{q^{k_i}}\right), \tag{16}$$

where the values of q are all prime numbers in the interval $n_i \leq q < 2n_i$ and for each q, p runs through all nonnegative integers no greater than q. We know, then, by well-known prime number theorems that for sufficiently large n_i the inequality

$$Z(F_i) = \sum_{n_i \leq q \leq 2n_i} (q+1) > \frac{n_i^2}{2 \log n_i} \tag{17}$$

is valid.

For $i = 1, 2, \cdots$, let $D_{q,i}$ denote the set of $q + 1$ intervals of the form

$$\left(\frac{p}{q} - \frac{1}{q^{k_i'}}, \frac{p}{q} + \frac{1}{q^{k_i'}}\right) \qquad (p = 0, 1, \cdots, q),$$

and let

$$D_i = \overline{\bigcup_{q=n_i}^{\infty}} D_{q,i},$$

$$D^i = \bigcup_{q=n_i}^{n_{i+1}-1} D_{q,i},$$

$$D = \bigcup_{i=1}^{\infty} D^i.$$

Since $D_{q,i} \subset D_{q,j}$ for $i \leq j$, it follows that

$$D_i = \bigcup_{q=n_i}^{\infty} D_{q,i} = \bigcup_{j=i}^{\infty} \bigcup_{q=n_j}^{n_{j+1}-1} D_{q,i} \subset \bigcup_{j=1}^{\infty} \bigcup_{q=n_j}^{n_{j+1}-1} D_{q,j} = \bigcup_{j=1}^{\infty} D^j = D.$$

On the other hand

$$\bigcup_{i=1}^{\infty} D_i = \bigcup_{i=1}^{\infty} \bigcup_{q=n_i}^{\infty} D_{q,i} \supset \bigcup_{i=1}^{\infty} \bigcup_{q=n_i}^{n_{i+1}-1} D_{q,i} = D.$$

Hence,

$$D = \bigcup_{i=1}^{\infty} D_i. \tag{18}$$

We want to find an estimate for $\Sigma_D d^s$. Obviously,

$$\sum_{D_i} d^s = \sum_{q=n_i}^{\infty} (q+1)\left(\frac{2}{q^{k_i'}}\right)^s < 2^{s+1} \sum_{q=n_i}^{\infty} q^{1-sk_i'}.$$

Since

$$1 - sk_i' = 1 - \frac{2}{c+1}(c + 1 + \varepsilon_i) = -1 - \frac{2\varepsilon_i}{c+1},$$

we can choose n_i so large that

$$\sum_{q=n_i}^{\infty} q^{1-sk_i'} < 2^{-s-1-i}.$$

As a consequence, we obtain the inequality $\Sigma_{D_i} d^s < 2^{-i}$, and by (18) it follows that

$$\sum_{D} d^s = \sum_{\bigcup_{i=1}^{\infty} D_i} d^s < \sum_{i=1}^{\infty} 2^{-i} = 1. \tag{19}$$

We consider now the set of intervals T consisting of the intervals of S(r, s) together with those of D. It may be that some or even infinitely many of the intervals of $S(r, s)$ overlap with intervals of D. Such overlapping intervals still constitute different elements of T and occur separately for instance in the inequality

$$(20) \qquad \sum_T d^s < 2,$$

which follows from (15) and (19). Also suppose that n_1 is large enough so that

$$\frac{2}{n_1^{c+1+\varepsilon_1}} < \frac{1}{n_1^{c+1}}.$$

Then, since the maximal length of the intervals of D equals $2n_1^{-(c+1+\varepsilon_1)}$, it is clear that the lengths of the intervals of T will not exceed $r = n_0^{-k}$. We shall show that even the set T does not cover the set $R_1(c)$ completely.

We now classify the intervals of F_1. An interval belongs to F_1' if at least one fourth of its length is covered by an interval of the set T. We put all the other intervals of the set F_1 in the set F_1''. Then $F_1 = F_1' \cup F_1''$, $F_1' \cap F_1'' = \emptyset$, where $\emptyset$ is the nullset. To give an estimate for the number $Z(F_1'')$, we denote by H the subset of intervals of T each of which covers at least one fourth of an interval of F_1', and put

$$H = H' \cup H'' \cup H''',$$

where the lengths d', d'', d''' of the intervals of H', H'', H''' satisfy the inequalities

$$(21) \qquad r = \frac{1}{n_0^{k_0}} \geq d' > \frac{4}{n_1} \geq d'' > \frac{1}{8n_1^2} \geq d'''$$

and n_1 is supposed to be large enough to satisfy the inequality $n_1 > 2n_0^k$. Since $H' \subset H \subset T$, we also see that, according to the inequality (20),

$$\sum_{H'} d^s < 2;$$

and hence by the first inequality of (21),

$$(22) \qquad |H'| = \sum_{H'} d' = \sum_{H'} d'^s d'^{1-s} < 2r^{1-s} = 2n_0^{-k_0(1-s)}.$$

Each set F_i is, according to (16), a set of uniformly distributed intervals of order n_i. Since the order of F_1 equals n_1, and since the lengths of the intervals belonging to H' are at least $4/n_1$, it follows by Lemma 6 that

$$Z(F_1' \mid H') \leq 2Z(F_1)|H'|,$$

from which, by use of (22), we obtain the estimate

$$(23) \qquad Z(F_1' \mid H') < 4\, n_0^{-k_0(1-s)}\, Z(F_1).$$

The distance between the midpoints p/q and $\bar{p}/\bar{q}$ of two different intervals of F_1 is at least $1/(q\bar{q})$, and consequently at least equal to $1/(4n_1^2)$, since q, $\bar{q}$ are both in the interval $[n_1, 2n_1)$. So the number h of intervals of F_1 that are covered at least to a fourth of their lengths by a single interval of H'', cannot be larger than

$$\left|\frac{d''}{1/(4n_1^2)}\right| + 2 \leq d''\, 4n_1^2 + 2 \leq 20\, d''\, n_1^2$$

since $2 \leq 16\, n_1^2\, d''$ by (21). Hence we find for $Z(F_1' \mid H'')$ the estimate

$$Z(F_1' \mid H'') \leq \sum_{H''} h \leq 20\, n_1^2 \sum_{H''} d''\,.$$

Using again (20) for the subset H'' of T, we find, in view of (21),

$$|H''| = \sum_{H''} d''^{s}\, d''^{1-s} \leq \left(\frac{4}{n_1}\right)^{1-s} \sum_{H''} d''^{s} < 2\left(\frac{4}{n_1}\right)^{1-s} \leq 8 n_1^{-(1-s)}\,.$$

So it follows that

$$Z(F_1' \mid H'') < 160\, n_1^{2-(1-s)} = 160\, n_1^{1+s}. \tag{24}$$

The distance between two intervals of F_1 with midpoints p/q and $\bar{p}/\bar{q}$ is at least equal to

$$\frac{1}{q\bar{q}} - \frac{1}{q^{k_1}} - \frac{1}{\bar{q}^{k_1}} \geq \frac{1}{4n_1^2} - \frac{2}{n_1^{k_1}} > \frac{1}{8n_1^2}\,,$$

the last inequality holding if n_1 is sufficiently large. Since the intervals of the system H''' have lengths at most $1/(8n_1^2)$, each interval of H''' can intersect at most one interval of F_1. $Z(F_1' \mid H''')$ has its maximal value if the various intervals of H''' cover at least the fourth parts of different intervals of F_1. Thus we find for the number $Z(H''')$:

$$Z(H''') \geq Z(F_1' \mid H''')\,.$$

The length d''' is, according to the definition of H''', at least as great as one-fourth of the minimal length of the intervals of F_1, and therefore

$$d''' \geq \frac{1}{4} \frac{2}{(2n_1)^{k_1}}\,.$$

Since $H''' \subset T$, it follows from (20) that

$$2 > \sum_{H'''} d'''^{s} \geq Z(H''') \frac{1}{2^s (2n_1)^{k_1 s}}\,;$$

hence

$$Z(F_1' \mid H''') \leq Z(H''') < 2^{s+1} (2n_1)^{k_1 s} < 4(2n_1)^{2k_1/k} \tag{25}$$

From (23), (24), and (25) we now obtain the inequalities

$$
(26)\qquad \begin{aligned} Z(F_1') &\le Z(F_1' \mid H') + Z(F_1' \mid H'') + Z(F_1' \mid H''') \\ &< 4n_0^{-k_0(1-s)} Z(F_1) + 160\, n_1^{1+s} + 4(2n_1)^{2k_1/k}. \end{aligned}
$$

For the last two terms we find that

$$
(27)\qquad \begin{aligned} 160\, n_1^{1+s} + 4(2n_1)^{2k_1/k} &< 160\, n_1^{1+2/(c+1)} + 4(2n_1)^{2(c+1-\varepsilon_1)/(c+1)} \\ &= 160\, n_1^{2-(c-1)/(c+1)} + 4(2n_1)^{2-2\varepsilon_1/(c+1)} \\ &\le 176\, n_1^{2-2\varepsilon_1/(c+1)}. \end{aligned}
$$

Suppose that n_1 was chosen so large that

$$
n_1^{2\varepsilon_1/(c+1)} > 176 (\log n_1)\, n_0^{k_0(1-s)},
$$

which we can assume since $\log n_1 = o(n_1^{2\varepsilon_1/(c+1)})$. Then conclude that

$$
(28)\qquad 176\, n_1^{2-2\varepsilon_1/(c+1)} < \frac{n_1^2}{n_0^{k_0(1-s)} \log n_1} < Z(F_1)\, 2n_0^{-k_0(1-s)},
$$

where the last inequality results from (17). The estimates (26), (27), and (28) together now yield the inequality

$$
(29)\qquad Z(F_1') < 6\, n_0^{-k_0(1-s)} Z(F_1).
$$

From the equation $Z(F_1) = Z(F_1') + Z(F_1'')$ and the relations (17) and (29) it follows, for sufficiently large n_0, that

$$
(30)\qquad Z(F_1'') > (1 - 6\, n_0^{-k_0(1-s)})\, Z(F_1) > \frac{n_1^2}{4 \log n_1}.
$$

By definition, the set F_1'' consists of those intervals I of F_1 for which there exists no interval of T that covers a fourth or more of I. The formula (30) shows that F_1'' is not empty and gives a lower bound for the number of intervals belonging to it.

We now shorten each interval of F_1'' at both ends by a fourth of its length and denote the set of the remaining middle halves, whose endpoints we include, by G_1. It is clear that the intervals of G_1 have lengths at least half that of those intervals of F_1 which have smallest length, and from (30) we find an inequality for the sum of the lengths of the intervals of G_1:

$$
(31)\qquad |G_1| > (2n_1)^{-k_1} Z(F_1'') > \frac{1}{4(2n_1)^{k_1}} \frac{n_1^2}{\log n_1}.
$$

Further, the lengths of the associated intervals of T that cover any part of an interval of G_1 are by construction of G_1 smaller than the fourth part of the maximal length of the intervals of F_1''. This maximal length equals $2n_1^{-k_1}$. So G_1 has no common point with any interval of T of length $d \geq \frac{1}{2} n_1^{-k_1}$, hence also with no interval of T of length $d \geq n_1^{-k_1}$. We want to show that certain points which belong to intervals of G_1 are not covered by any interval of T. So it is evident that for the following considerations we can restrict ourselves to the subset T_1 of T consisting of those intervals of T whose lengths are less than $n_1^{-k_1}$. From (20) it also follows that

$$\sum_{T_1} d^s < 2 .$$

Let us now look at the set F_2. Let F_2' denote the set of intervals of F_2 each of which is covered at least to a fourth of its length by an interval of T_1; let F_2'' be the set of the remaining intervals of F_2. We find then in the same way as we found (29) the inequality

$$Z(F_2') < 6 n_1^{-k_1(1-s)} Z(F_2) . \tag{32}$$

We are interested in those intervals of F_2'' that are completely covered by intervals of G_1. We want to show that they contain certain points which are not covered by intervals of T_1. Therefore we investigate $F_2''^{G_1}$. First, by Lemma 7, we find a lower bound for $Z(F_2^{G_1})$: each interval of G_1 has at least the length $(2n_1)^{-k_1}$, and F_2 is of order n_2, so for $n_2 > 8(2n_1)^{k_1}$ we obtain the inequality

$$Z(F_2^{G_1}) > \frac{1}{2} Z(F_2) |G_1| .$$

It follows then from (32) that

$$\begin{aligned} Z(F_2''^{G_1}) &= Z(F_2^{G_1}) - Z(F_2'^{G_1}) > Z(F_2^{G_1}) - Z(F_2') \\ &> Z(F_2)\left(\frac{1}{2}|G_1| - 6 n_1^{k_1(s-1)}\right) , \end{aligned}$$

and from (31) that

$$\begin{aligned} Z(F_2''^{G_1}) &> Z(F_2)\left(\frac{1}{8}(2n_1)^{-k_1}\frac{n_1^2}{\log n_1} - 6 n_1^{k_1(s-1)}\right) \\ &> Z(F_2)\frac{1}{9}(2n_1)^{-k_1}\frac{n_1^2}{\log n_1} , \end{aligned} \tag{33}$$

if n_1 is sufficiently large. This follows from the fact that $2 - k_1 > k_1(s - 1)$, an inequality which is easily checked.

We now shorten each interval of F_2'' at both ends by a fourth of its length. Let G_2 be the set of middle halves of the intervals composing $F_2''^{G_1}$; the number of intervals in G_2 is equal to $Z(F''^{G_1})$ and is estimated by the right-hand side of (33). We find for G_2, in the same way as for G_1, that the intervals of G_2 have no point in common with any interval of T whose length is at least $n_2^{-k_2}$. The lengths of the intervals of G_2 are at least equal to $(2n_2)^{-k_2}$, and thus it follows from the relations (20) and (33) that

$$(34) \qquad |G_2| > Z(F_2)\frac{n_1^2}{9(2n_1)^{k_1}\log n_1}\frac{1}{(2n_2)^{k_2}} > \frac{(n_1 n_2)^2}{18(2n_1)^{k_1}(2n_2)^{k_2}\log n_1 \log n_2}.$$

We continue with the set F_3. We denote by T_2 the set of intervals of T whose lengths are less than $n_2^{-k_2}$. We further take F_3' as the set of intervals of F_3 each of which is covered, at least to a fourth of its length, by a single interval of T_2, and put $F_3'' = F_3 - F_3'$. The inequality corresponding to (29) holds. One obtains then, just as before, the set G_3 of the shortened intervals of $F_3''(G_2)$ that have no common point with any interval of T whose length is at least $n_3^{-k_3}$, and one finds an inequality similar to (34). In this way one obtains a nested sequence of sets $G_1, G_2, G_3, \cdots$, each consisting of closed intervals, with the following properties: Each G_i is closed and not empty; hence the pointset $G = \bigcap_{i=1}^{\infty} G_i$ is not empty. Further, since G_i has no point in common with any interval of the system T whose length is at least $n_i^{-k_i}$, the set G has no common point with T. Finally $G_i \subset \bigcap_{j=1}^{i} F_j$ $(i = 1, 2, 3, \cdots)$, and hence $G \subset F = \bigcap_{i=1}^{\infty} F_i$. Therefore, if γ is an arbitrary element of G, then $\gamma \notin T$, and hence $\gamma \notin S(r, s)$. Moreover, since $D \subset T$ the point γ is not contained in any set D_i. In other words, to each $\varepsilon > 0$ there corresponds an n such that for all $q > n$

$$\left|\gamma - \frac{p}{q}\right| > \frac{1}{q^{c+1+\varepsilon}}.$$

For if i is the index for which $\varepsilon_{i-1} \geq \varepsilon > \varepsilon_i$, one can take $n = n_i$, since then γ does not belong to any interval

$$\left|\gamma - \frac{p}{q}\right| < \frac{1}{q^{c+1+\varepsilon}} \qquad (q \geq n).$$

On the other hand, since $G \subset F$, there exist, for each $\varepsilon > 0$, infinitely many pairs of integers p, q $(q > 0)$ such that

$$\left|\gamma - \frac{p}{q}\right| \leq \frac{1}{q^{c+1-\varepsilon}}.$$

This means that each $\gamma \in G$ has properties (a) and (b) of Definition 1 for $n = 1$, and hence $\theta_1(\gamma) = c$. The fact that $G \cap S(r, s) = \emptyset$ shows that $S(r, s)$ does not cover all points γ for which $\theta_1(\gamma) = c$. This completes the proof of Theorem 8.

REFERENCES

1. A. S. Besicovitch, *Sets of fractional dimensions (IV): On rational approximations to real numbers*, J. London Math. Soc. 9 (1934), 126-131.

2. H. G. Eggleston, *The fractional dimension of a set defined by decimal properties*, Quart. J. of Math. Oxford Ser. (2) 20 (1949), 31-36.

3. R. Güting, *Approximation of algebraic numbers by algebraic numbers*, Michigan Math. J. 8 (1961), 149-159.

4. V. Jarnik, *Diophantische Approximationen und Hausdorffsches Mass*, Mat. Sb. 36 (1929), 371-382.

5. K. Mahler, *Zur Approximation der Exponentialfunktion und des Logarithmus*, I, J. Reine Angew. Math. 166 (1932), 118-136.

6. ———, *Arithmetische Eigenschaften einer Klasse von Dezimalbrüchen*, Proc. Akad. Wetensch. Amsterdam 40 (1937), 421-428.

7. T. Schneider, *Einführung in die transzendenten Zahlen*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957.

8. B. Volkmann, *Über Klassen von Mengen natürlicher Zahlen*, J. Reine Angew. Math. 190 (1952), 199-230.

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