ON A PROPERTY OF ORTHOGONAL POLYNOMIALS

J. J. Price

Let $\rho(x)\,dx$ be a finite measure on the closed interval $[-1, 1]$, and let $\{p_n(x)\}_{n=0}^{\infty}$ be the associated orthonormal polynomials. Define

$$\sigma_n(x) = \frac{1}{n+1} \sum_{j=0}^{n} p_j^2(x).$$

G. Freud has proved [2] that if

$$\rho(x) \geq \rho_0 > 0 \tag{1}$$

almost everywhere in some open subinterval (α, β), then

$$\sigma_n(x) = O(1) \quad \text{uniformly in } [\alpha + \varepsilon, \beta - \varepsilon]. \tag{2}$$

Freud used this result to prove the following theorem [2]. If condition (1) holds almost everywhere on a proper subinterval $[\alpha, \beta]$, then the expansion of any function in $L^2(\rho(x))$ in terms of the polynomials $p_n(x)$ is $(C, \alpha > 0)$-summable almost everywhere in $[\alpha, \beta]$.

In his book, G. Alexits [1, p. 43] raises the question whether (2) is true without condition (1) holding almost everywhere. He points out that if this were so, then Freud's theorem would be valid without the assumption that $\rho(x) \geq \rho_0 > 0$ almost everywhere in $[\alpha, \beta]$.

The answer to the question is negative. This may be seen from a classical work of G. Szegö [4, 4.1.6] and a recent work of V. P. Konoplev [3]. These authors study, respectively, the densities $\rho(x) = |x|^\gamma$ $(\gamma > -1)$ and

$$\rho(x) = (1 - x)^\alpha (1 + x)^\beta |x - x_0|^\gamma \quad (\alpha, \beta > -1,\ \gamma \geq 0,\ |x_0| < 1).$$

Konoplev's results show immediately that $\sigma_n(x_0) \sim cn^\gamma$, where c is a constant. Thus (2) fails if $\gamma > 0$.

The purpose of this paper is to give a short proof of the theorem that $\sigma_n(x_0) \to \infty$ if $\rho(x)$ is any density for which $\rho(x) \to 0$ as $x \to x_0$, $|x_0| < 1$, and to find estimates for the rate of growth of $\sigma_n(x_0)$.

THEOREM 1. *If at some interior point* x_0 *of the interval* $[-1, 1]$, $\lim_{x \to x_0} \rho(x) = 0$, *then*

$$\lim_{n \to \infty} \sigma_n(x_0) = \infty. \tag{3}$$

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Proof. Let $f(x)$ be a polynomial of degree n. For suitable coefficients a_j, $f(x) = \Sigma_{j=0}^n a_j p_j(x)$. Suppose $f(x_0) = 1$. By the Schwarz inequality,

$$1 = f^2(x_0) = \left(\sum_{j=0}^n a_j p_j(x_0)\right)^2 \leq \left(\sum_{j=0}^n a_j^2\right)\left(\sum_{j=0}^n p_j^2(x_0)\right).$$

Hence

$$(4) \qquad (n+1)\,\sigma_n(x_0) = \sum_{j=0}^n p_j^2(x_0) \geq \left(\sum_{j=0}^n a_j^2\right)^{-1} = \left(\int_{-1}^1 f^2(x)\,\rho(x)\,dx\right)^{-1}.$$

To prove the theorem, it suffices to construct a sequence of polynomials $\{f_n(x)\}_{n=1}^\infty$ such that the degree of $f_n(x)$ is $3n$, $f_n(x_0) = 1$, and

$$(5) \qquad \int_{-1}^1 f_n^2(x)\rho(x)\,dx = o(n^{-1}).$$

To see this, note that because of (4), the existence of such a sequence implies $\lim_{n\to\infty} \sigma_{3n}(x_0) = \infty$. But if $r = 1$ or 2,

$$\sigma_{3n+r}(x_0) \geq \frac{3n+1}{3n+r+1}\sigma_{3n}(x_0) \geq \frac{2}{3}\sigma_{3n}(x_0) \qquad (n \geq 1),$$

and (3) follows.

Let $\{q_n(x)\}$ be the normalized Legendre polynomials. Set

$$h_n(x) = \left(\sum_{j=0}^n q_j^2(x_0)\right)^{-1}\left(\sum_{j=0}^n q_j(x_0)q_j(x)\right).$$

The polynomial $h_n(x)$ is of degree n, and $h_n(x_0) = 1$. Furthermore,

$$(6) \qquad \int_{-1}^1 h_n^2(x)\,dx \sim an^{-1} \qquad (n \to \infty),$$

$$(7) \qquad h_n^2(x) < bn \qquad (|x| \leq 1),$$

where the constants a and b depend on x_0. To prove (6), note that because of orthonormality, the value of the integral in question is $(\Sigma_{j=0}^n q_j^2(x_0))^{-1}$. This quantity can be shown to be asymptotic to an^{-1} by using formula 9.3.5 of [4] with $\alpha = \beta = 0$ and $\theta = \phi = \cos^{-1} x_0$. (7) follows directly from (6) and Theorem 7.71.1 of [4].

Now define

$$g_n(x) = \left[1 - \frac{1}{4}(x - x_0)^2\right]^n, \quad f_n(x) = g_n(x)\,h_n(x).$$

Clearly, $f_n(x)$ is of degree 3n, and $f_n(x_0) = 1$. Thus, the proof of Theorem 1 reduces to the verification of (5) for the polynomials $f_n(x)$ just defined.

Suppose $0 < \alpha < 1/2$. Let $\Delta_n = \{x: |x - x_0| < n^{-\alpha}\} \cap [-1, 1]$, and let Δ_n' be its complement relative to $[-1, 1]$. Let

$$\int_{-1}^{1} f_n^2(x)\, \rho(x)\, dx = \int_{\Delta_n} + \int_{\Delta_n'} = I_n + I_n'. \tag{8}$$

It will be shown that both I_n and I_n' are $o(n^{-1})$. Obviously,

$$I_n = \int_{\Delta_n} g_n^2(x)\, h_n^2(x)\, \rho(x)\, dx \le \int_{\Delta_n} h_n^2(x)\, \rho(x)\, dx$$

$$\le \left[\sup_{\Delta_n} \rho(x)\right] \int_{\Delta_n} h_n^2(x)\, dx \le \left[\sup_{\Delta_n} \rho(x)\right] \int_{-1}^{1} h_n^2(x)\, dx.$$

From (6),

$$I_n \le \left[\sup_{\Delta_n} \rho(x)\right] O(n^{-1}) \tag{9}$$

But $\rho(x) \to 0$ as $x \to x_0$. Therefore, $I_n = o(n^{-1})$.

$$I_n' = \int_{\Delta_n'} g_n^2(x)\, h_n^2(x)\, \rho(x)\, dx \le \left[\max_{\Delta_n'} g_n^2(x)\right] \left[\max_{\Delta_n'} h_n^2(x)\right] \int_{-1}^{1} \rho(x)\, dx.$$

From (7),

$$I_n' = O\left(n \max_{\Delta_n'} g_n^2(x)\right). \tag{10}$$

But $\max_{\Delta_n'} g_n^2(x) = g_n^2(x_0 \pm n^{-\alpha})$ since $1 - (1/4)(x - x_0)^2 > 0$ on $[-1, 1]$. Thus

$$I_n = O\left(n\left[1 - \frac{1}{4n^{2\alpha}}\right]^{2n}\right) = O\left(n \exp\left[-\tfrac{1}{2} n^{1-2\alpha}\right]\right) = o(n^{-1})$$

for $\alpha < 1/2$. This completes the proof of Theorem 1.

Some estimates of the size of $\sigma_n(x_0)$ can be obtained from the preceding proof. For example, take the density $\rho(x) = |x - x_0|^\gamma$ $(\gamma > 0)$. For a fixed $\alpha < 1/2$, I_n' is negligible compared to I_n, and one obtains the inequality

$$\sigma_n(x_0) > c\left(\sup_{\Delta_n} \rho(x)\right)^{-1} = cn^{\alpha\gamma}.$$

In other words

$$\sigma_n(x_0) \ne 0\left(n^{\frac{1}{2}\gamma - \varepsilon}\right)$$

for $\varepsilon > 0$. Since in fact, $\sigma_n(x_0) \sim cn^{\gamma}$, the estimate is very crude.

Better estimates are proved by Theorem 2 below. Roughly speaking, this theorem is obtained by using the proof of Theorem 1, replacing $g_n(x)$ by a polynomial more sharply peaked at x_0.

As a matter of terminology, a "constant" will always mean a quantity independent of n, the symbol used exclusively to index orthogonal polynomials. Some constants will depend on certain parameters. These will sometimes be indicated, but not always. In general c_α, c_β; c, c_1, c_2, $\cdots$ will denote constants whose exact values are not needed.

For brevity, the following notation will be used.

$$s(y) = \sup_{|x-x_0|<y} \rho(x).$$

THEOREM 2. *Suppose, as in Theorem* 1, $\lim_{x \to x_0} \rho(x) = 0$. *Let* p *be any given positive number. Then, for any* $\alpha < 1$,

$$\sigma_n(x_0) > \frac{c_{\alpha,p}}{s(n^{-\alpha}) + n^{-p}}. \tag{11}$$

Proof. For simplicity, let $x_0 = 0$. It will suffice to prove that given any positive numbers r and α ($\alpha < 1$), there exists a sequence of polynomials $\{g_n(x)\}_{n=1}^{\infty}$, with $g_n(x)$ of degree 2n, $g_n(0) = 1$, and $|g_n(x)| \leq 1$ on $[-1, 1]$, such that

$$g_n^2(x) < \frac{c_{\alpha,r}}{n^r} \qquad (n^{-\alpha} \leq |x| \leq 1). \tag{12}$$

Assuming for a moment the existence of such sequences, choose one corresponding to the given α and $r = p + 2$. As in the proof of Theorem 1, define $f_n(x) = g_n(x)h_n(x)$. Consider the analogue of (8), this time allowing $\alpha < 1$. Since the estimates (9) and (10) are quite general,

$$I_n + I_n' \leq \left[\sup_{\Delta_n} \rho(x)\right] O(n^{-1}) + O\left[n \max_{\Delta_n'} g_n^2(x)\right].$$

Using (12), we see that

$$\int_{-1}^{1} f_n^2(x)\,\rho(x)\,dx = I_n + I_n' \leq s(n^{-\alpha})\,O(n^{-1}) + O(n^{-p-1}). \tag{13}$$

The assertion of Theorem 2 now follows from (4) and (13).

To construct the polynomials $g_n(x)$, let k be a positive integer and $P_n^{(k,0)}(x)$ be the Jacobi polynomial of degree n, (for definitions, see [3, Chapter IV]). According to a result of Szegö [3, 8.21.18],

$$P_n^{(k,0)}(\cos\theta) = \left(\pi n \cos\frac{\theta}{2}\right)^{-\frac{1}{2}} \left(\sin\frac{\theta}{2}\right)^{-k-\frac{1}{2}} [\cos\{(n + c_1)\theta + c_2\} + O(1)(n \sin\theta)^{-1}],$$

for $n^{-1} \leq \theta \leq \pi - n^{-1}$. If θ is restricted to the range $n^{-1} \leq \theta \leq \pi/2$, then $\left(\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}$ and $(n \sin\theta)^{-1}$ are bounded. Hence,

$$(14) \qquad |P_n^{(k,0)}(\cos\theta)| \leq c_3 n^{-\frac{1}{2}} \left(\sin\frac{\theta}{2}\right)^{-k-\frac{1}{2}} \qquad (n^{-1} \leq \theta \leq \pi/2).$$

Now make the substitution $1 - x^2 = \cos\theta$, where $|x| \leq 1$ and $0 \leq \theta \leq \pi/2$. Then $\sin\frac{\theta}{2} = 2^{-\frac{1}{2}}|x|$, and (14) takes the form

$$(15) \qquad |P_n^{(k,0)}(1 - x^2)| \leq c_4 n^{-\frac{1}{2}} |x|^{-k-\frac{1}{2}},$$

which is valid for $0 \leq 1 - x^2 = \cos\theta \leq \cos\frac{1}{n}$. It is easy to see that this condition is satisfied if $|x| \geq n^{-1}$.

Define

$$g_n(x) = \frac{P_n^{(k,0)}(1 - x^2)}{P_n^{(k,0)}(1)}.$$

The polynomial $g_n(x)$ is of degree $2n$, and $g_n(0) = 1$. Since $P_n^{(k,0)}(1) = \binom{n+k}{n}$, it follows from [3; (7.32.2)] that $|g_n(x)| \leq 1$ in $[-1, 1]$, and from (15) that

$$|g_n(x)| \leq c_5 |nx|^{-k-\frac{1}{2}} \qquad (|x| \geq n^{-1}).$$

In particular, if $|x| \geq n^{-\alpha}$ and $\alpha < 1$,

$$(16) \qquad g_n^2(x) \leq c_5^2 n^{-(2k+1)(1-\alpha)}.$$

Now let r be any given positive number. Since $1 - \alpha > 0$, k may be chosen so large that, by (16), $g_n^2(x) < c_6 n^{-r}$ for $|x| \geq n^{-\alpha}$. Thus (12) holds for the sequence $\{g_n(x)\}$.

If $x_0 \neq 0$, the same arguments apply with $g_n\left(\frac{1}{2}[x - x_0]\right)$ in place of $g_n(x)$. This completes the proof of Theorem 2.

The following corollary lists certain special cases of Theorem 2 which seem worthy of mention.

COROLLARY. *Suppose* $\lim_{x \to x_0} \rho(x) = 0$ *and* $|x_0| < 1$.

(a) *If* $\rho(x) < c|x - x_0|^\gamma$ *as* $x \to x_0$ *for some positive* γ, *then*

$$\sigma_n(x_0) > c_\varepsilon n^{\gamma - \varepsilon}$$

for every positive ε.

(b) *If* $\rho(x) = o(|x - x_0|^\gamma)$ *for every positive* γ, *then*

$$\sigma_n(x_0) > c_\gamma n^\gamma .$$

(c) *If* $\rho(x) > c\,|x - x_0|^\gamma$ *for some positive* γ, *then*

$$\sigma_n(x_0) > \frac{c_\alpha}{s(n^{-\alpha})}$$

for every $\alpha < 1$.

Proof. (a) $s(n^{-\alpha}) < cn^{-\alpha\gamma}$. Applying Theorem 2 with $p = \alpha\gamma$, we find that

$$\sigma_n(x_0) > c_\alpha n^{\alpha\gamma} \qquad (\alpha < 1).$$

For $\alpha = 1 - \varepsilon/\gamma$,

$$\sigma_n(x_0) > c_\varepsilon n^{\gamma-\varepsilon} .$$

(b) Follows directly from (a).

(c) Since p can be taken arbitrarily large in (11), the assumption on $\rho(x)$ guarantees that $s(n^{-\alpha})$ is the dominant term in the denominator if $p > \gamma$.

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Cornell University