SOME CHARACTERIZATIONS OF HOMOLOGICAL DIMENSION

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1. Let X be a compact Hausdorff space, and let G be an abelian group. The *homological dimension of* X *relative to* G is the largest integer n such that there exists a pair (A, B) of closed subsets of X whose n-dimensional Čech homology group $H_n(A, B: G)$ is not zero. By $D_*(X: G)$ we shall denote the homological dimension of X. We have the relation $\dim X \geq D_*(X: G)$, for each group G. The equality $\dim X = D_*(X: G)$ does not necessarily hold. For example, for any positive integer n, there exists a continuum X such that $\dim X = 2n$ and $D_*(X: G) = n$ for each finitely generated abelian group G.

Let N be a class of compact Hausdorff spaces. A countable system

$$\{T_i(G);\ i = 1, 2, \cdots\}$$

of locally compact fully normal spaces is called a *T-system for the group* G *with respect to the class* N if, for each X of N, we have the equality

$$D_*(X: G) = \operatorname{Min}\{\dim(X \times T_i(G)) - \dim T_i(G);\ i = 1, 2, \cdots\}.$$

If a T-system for G with respect to N consists of only one space, then the space is called a *test space for* G *with respect to* N (see [7]). The following notations will be used throughout this paper.

Z: the additive group of all integers.

Z_q: the cyclic group Z/qZ of order q.

R: the additive group of all rational numbers.

R_1: the additive group of all rational numbers reduced mod 1.

Q_p: the p-primary component of R_1.

$Z(\alpha_p)$: the limit group of the inverse system

$$\{Z_{p^i},\ i = 1, 2, \cdots;\ h_i^{i+1}: Z_{p^{i+1}} \to Z_{p^i}\},$$

where the homomorphism $Z_{p^{i+1}} \to Z_{p^i}$ is a natural homomorphism induced by the inclusion $p^{i+1} Z \subset p^i Z$.

L: the class of all finite-dimensional compact Hausdorff spaces.

L_n: the class of all n-dimensional compact Hausdorff spaces.

$L_n(G)$: the class of all finite-dimensional compact Hausdorff spaces X such that $\dim X - D_*(X: G) = n$.

Received September 18, 1961.

This research was supported by the U.S. Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract No. AF49(638)-774.

In the present paper, we shall prove the following theorems.

THEOREM A. (I) *Let* G *be one of the following groups:* (i) R, (ii) Z_p, (iii) Q_p, *where* p *is a prime, and* (iv) *a direct sum of groups of types* (i) *to* (iii). *Then there exists a* T-*system for* G *with respect to* L.

(II) *Let* n *be a non-negative integer, and let* G *be one of the groups in* (I). *Then there exists a test space for* G *with respect to* $\bigcup_{i=0}^{n} L_i$.

(III) *Let* G *be one of the following groups:* (i) R, (ii) Z_p, (iii) Q_p, (iv) $Z(\alpha_p)$, *where* p *is a prime, or* (v) *a direct sum of groups of types* (i) *to* (iv). *Then there exists a test space for* G *with respect to* $L_0(G) \cup L_1(G)$.

THEOREM B. (I) *If* $n \geq 2$ *and* p *is a prime, there exists no* T-*system for* $Z(\alpha_p)$ *with respect to* $L_n(Z(\alpha_p))$.

(II) *There exists no* T-*system for* Z *with respect to* L_i $(i = 1, 2, \cdots)$.

(III) *Let* G *be one of the groups in* (i) *to* (iv) *of* (III) *of* Theorem A. *Then there exists no* T-*system for* G *with respect to* L *consisting of a finite number of spaces; in particular, there exists no test space for* G *with respect to* L.

2. Let $q = (q_1, q_2, \cdots)$ be a sequence of positive integers. In Section 4 of [5] we constructed the Cantor manifold $R(q)$ for the sequence q. Choose a prime p. Let q_0 be a sequence of positive integers which contains all powers of all primes, and let q_p be the subsequence of q_0 consisting of all powers of primes which are different from the prime p. We denote the Cantor manifolds $R(q_0)$ and $R(q_p)$ by M_0 and M_p, respectively. Let $\alpha = (p_1, p_2, \cdots)$ be a sequence of positive integers such that p_i is a divisor of p_{i+1} for $i = 1, 2, \cdots$. In Section 3 of [4] we constructed the Cantor manifold $Q(\alpha)$. The Pontryagin Cantor manifold P_p mod p [11] is the Cantor manifold $Q(\alpha)$ for the sequence $\alpha = (p, p, p, \cdots)$. Let us denote by α_p the sequence $(p, p^2, p^3, \cdots)$ of powers of the prime p. The following theorem was proved in [7], but we shall give a simple proof here.

THEOREM 1. *Let* X *be an* n-*dimensional compact Hausdorff space. Then*

(i) $D_*(X: R) = n$ *if and only if* $\dim(X \times M_0) = n + 2$.

(ii) $D_*(X: Z_p) = n$ *if and only if* $\dim(X \times P_p) = n + 2$.

(iii) $D_*(X: Q_p) = n$ *if and only if* $\dim(X \times M_p) = n + 2$.

(iv) $D_*(X: Z(\alpha_p)) = n$ *if and only if* $\dim(X \times Q(\alpha_p)) = n + 2$.

Proof. We shall prove only (iii) and (iv). The cases (i) and (ii) are treated similarly. We begin with the relations

$$D_*(M_p: R) = D_*(M_p: Q_p) = D_*(M_p: Q_q) = D_*(M_p: Z(\alpha_p)) = D_*(M_p: Z_p) = 2,$$

$$D_*(M_p: Z(\alpha_q)) = D_*(M_p: Z_q) = 1,$$

where q is a prime not equal to p. By the theorem of [8], at least one of the relations

$$D_*(X: R) = n, \quad D_*(X: Q_p) = n, \quad D_*(X: Z_p) = n, \quad D_*(X: Z(\alpha_q)) = n$$

holds for some prime q. But each of these relations means that $D_*(X: Q_p) = n$. To prove (iv) we observe that $D_*(Q(\alpha_p): Q_p) = 2$ and

$$D_*(Q(\alpha_p)\colon Q_q) = D_*(Q(\alpha_p)\colon Z_p) = D_*(Q(\alpha_p)\colon Z_q) = D_*(Q(\alpha_p)\colon Z(\alpha_p))$$
$$= D_*(Q(\alpha_p)\colon Z(\alpha_q)) = D_*(Q(\alpha_p)\colon R) = 1,$$

where q is a prime not equal to p. Therefore, by the theorem of [8],

$$D_*(X\colon Z(\alpha_p)) = n.$$

The "only if" parts are obvious.

The following lemma is proved easily in a way analogous to the proof of the theorem of [8].

LEMMA 1. *Let* X *be a compact Hausdorff space. Then*

(i) $D_*(X\colon R) + 2 = D_*(X \times M_0\colon R)$,

(ii) $D_*(X\colon Z_p) + 2 = D_*(X \times P_p\colon Z_p)$ *and*

(iii) $D_*(X\colon Q_p) + 2 = D_*(X \times M_p\colon Q_p)$.

THEOREM 2. *Let* k *be a non-negative integer, and let* $j > k$. *Then*

(i) $(M_0)^j$ *is a test space for* R *with respect to* $L_k(R)$,

(ii) $(P_p)^j$ *is a test space for* Z_p *with respect to* $L_k(Z_p)$,

(iii) $(M_p)^j$ *is a test space for* Q_p *with respect to* $L_k(Q_p)$.

Here we mean by $(Y)^j$ *the* j*-fold product space* $Y \times Y \times \cdots \times Y$ *of a space* Y.

Proof. We shall prove only (iii). The other cases are treated similarly. Let $X \in L_k(Q_p)$, and let $h = D_*(X\colon Q_p)$. By (iii) of Lemma 1, it follows that

$$D_*(X \times (M_p)^j\colon Q_p) = h + 2j.$$

Therefore, $\dim(X \times (M_p)^j) \geq h + 2j$ $(j = 1, 2, \cdots)$. We must prove the equality

$$\dim(X \times (M_p)^j) = h + 2j \qquad \text{for } j > k.$$

Assume that $\dim(X \times (M_p)^j) = m > h + 2j$ for some $j > k$. Then either

$$\dim(X \times (M_p)^{j-1}) = m - 2 \qquad \text{or} \qquad m - 1.$$

If $\dim(X \times (M_p)^{j-1}) = m - 2$, then $D_*(X \times (M_p)^{j-1}\colon Q_p) = m - 2$ by (iii) of Theorem 1. By (iii) of Lemma 1, $D_*(X\colon Q_p) = m - 2j > h$. Thus

$$\dim(X \times (M_p)^{j-1}) = m - 1.$$

If $\dim(X \times (M_p)^{j-2}) = m - 3$, then

$$D_*(X \times (M_p)^{j-2}\colon Q_p) = m - 3 \qquad \text{and} \qquad D_*(X\colon Q_p) = m - 3 - 2(j - 2) > h.$$

Therefore $\dim(X \times (M_p)^{j-2}) = m - 2$. In general,

$$\dim(X \times (M_p)^{j-i}) = m - i \qquad (i = 1, 2, \cdots, j).$$

But this means that

$$\dim X = m - j > h + 2j - j = h + j > h + k = \dim X.$$

This contradiction implies the truth of the equality $\dim (X \times (M_p)^j) = h + 2j$.

Conversely, let $\dim (X \times (M_p)^j) = h + 2j$ $(j > k)$. We shall prove that $\dim (X \times (M_p)^{j-1}) = h + 2(j - 1)$. Assume that $\dim (X \times (M_p)^{j-1}) = h + 2j - 1$. If $\dim (X \times (M_p)^{j-2}) = h + 2j - 3$, it follows that

$$D_*(X \times (M_p)^{j-2}: Q_p) = h + 2j - 3,$$

$$D_*(X \times (M_p)^j: Q_p) = h + 2j + 1 > \dim (X \times (M_p)^j).$$

Therefore $\dim (X \times (M_p)^{j-2}) = h + 2j - 2$. In general,

$$\dim (X \times (M_p)^{j-i}) = h + 2j - i \quad (i = 1, 2, \cdots, j).$$

But this means that $\dim X = h + j > \dim X$. Thus we can conclude that

$$\dim (X \times (M_p)^{j-1}) = h + 2(j - 1).$$

By (iii) of Theorem 1 and (iii) of Lemma 1, $D_*(X: Q_p) = h$. This completes the proof.

Proof of Theorem A. (I) Put

$$T_i(R) = (M_0)^i, \quad T_i(Z_p) = (P_p)^i, \quad T_i(Q_p) = (M_p)^i \qquad (i = 1, 2, \cdots).$$

Let G be one of the groups R, Z_p and Q_p. By the proof of Theorem 2, if $X \in L_0(G) \cup L_1(G)$, then $D_*(X: G) = \dim (X \times T_i(G)) - \dim T_i(G)$ $(i = 1, 2, \cdots)$. If $X \in L_i(G)$ for $i > 1$, then

$$D_*(X: G) = \dim (X \times T_j(G)) - \dim T_j(G) \qquad (j \geq i),$$

$$D_*(X: G) = \dim (X \times T_j(G)) - \dim T_j(G) - i + j \qquad (j > i).$$

Therefore $\{\dim (X \times T_i(G)) - \dim T\ (G);\ i = 1, 2, \cdots\}$ is a nonincreasing sequence. Thus, it is obvious that the system $\{T_i(G)\}$ forms a T-system for G with respect to L. Next let $G = \Sigma_\alpha G_\alpha$, where G_α is one of the groups R, Z_p and Q_p. Put $T_i(G) = \bigcup_\alpha T_i(G_\alpha)$ $(i = 1, 2, \cdots)$. Then $T_i(G)$ is locally compact and fully normal. Let X be a finite-dimensional compact Hausdorff space. Since

$$D_*(X: G_\alpha) = \dim (X \times T_j(G_\alpha)) = 2j = \operatorname{Min} \{\dim (X \times T_i(G_\alpha)) - 2i;\ i = 1, 2, \cdots\}$$

for $j > \dim X$ and each α, and since $D_*(X: G) = \operatorname{Max}_\alpha \{D_*(X: G_\alpha)\}$ [2] and

$$\dim (X \times T_i(G)) = \operatorname*{Max}_\alpha \{\dim (X \times T_i(G_\alpha))\},$$

we have the equality

$$D_*(X: G) = \operatorname{Min} \{\dim (X \times T_i(G)) - \dim T_i(G);\ i = 1, 2, \cdots\}.$$

(II) If G is one of the groups in (i) to (iv) of (I) of Theorem A, then the space $T_{n+1}(G)$, which is the $(n + 1)$st member of the T-system constructed in the proof of (I), is a test space for G with respect to $\bigcup_{i=0}^n L_i$.

(III) This is a consequence of Theorem 1.

Proof of Theorem B. (I) Let $\{T_i;\ i = 1, 2, \cdots\}$ be a T-system for $Z(\alpha_p)$ with respect to $L_n(Z(\alpha_p))$ for $n \geq 2$. Put $X = (M_0)^n$. Now $\dim X = 2n$, and

$$D_*(X\colon Z(\alpha_p)) = n\,.$$

Therefore X belongs to $L_n(Z(\alpha_p))$. By the definition of a T-system, there exists an integer i such that

$$D_*(X\colon Z(\alpha_p)) = \dim (X \times T_i) = \dim T_i\,.$$

But, for every $Y \in L$, we have the relation $\dim (X \times Y) \geq 2n - 1 + \dim Y$. Therefore, $n = \dim (X \times T_i) = \dim T_i \geq 2n - 1$. This contradicts the fact that $n \geq 2$.

(II) If for $i = 1, 2, \cdots$, the X_i are replicas of the unit circle in the complex plane, and if $f_i\colon X_{i+1} \to X_i$ is defined by $f_i(z) = z^2$, then the inverse limit space $X = \varprojlim X_i$ is the dyadic solenoid. Also, $X \in L_1$, $D_*(X\colon Z) = 0$, and $\dim (X \times Y) = \dim Y + 1$ for every Y in L. This shows that there exists no T-system for Z with respect to L_1. Let $n \geq 2$. Let M be the continuum constructed in Lemma 17 of [5], and let E^i be the i-cube. Put $X = M \times E^{n-2}$. Then $X \in L_n$, $D_*(X\colon Z) = n - 1$, and

$$\dim (X \times Y) = \dim Y + n$$

for every Y in L. This completes the proof of (II).

(III) Let $\{T_i;\ i = 1, 2, \cdots, m\}$ be a T-system for G with respect to L. Take an integer $n > \max_i \{\dim T_i\}$. There exists a continuum X which belongs to $L_n(G) \cap L_{2n}$. Then it follows that

$$n = D_*(X\colon G) = \min_i \{\dim (X \times T_i) - \dim T_i\} \geq 2n - \max_i \{\dim T_i\} > n\,.$$

This completes the proof of the theorem.

The following corollary, which is a generalization of Theorem 4.1 of [1] or Theorem 5 of [6], is easily proved in analogy with the proof of Theorem 5 of [6].

COROLLARY 1. *Let X and Y be finite-dimensional compact Hausdorff spaces. Then, if $\dim (X \times Y) = k$, there exists a prime p such that*

$$D_*(X\colon Q_p) + D_*(Y\colon Q_p) \geq k\,.$$

Proof. Since $R_1 \cong \Sigma_p Q_p$, there exists a prime p such that $D_*(X \times Y\colon Q_p) = k$, by [2] and [8]. Take an integer $i_0 > \max\{\dim X, \dim Y\}$. We know, by Theorem 2, that $\dim (X \times Y \times (M_p)^{2i_0}) = k + 4i_0$. By Theorem 4 of [10] it follows that

$$\dim (X \times (M_p)^{i_0}) + \dim (Y \times (M_p)^{i_0}) \geq k + 4i_0\,.$$

Since $\dim (X \times (M_p)^{i_0}) = D_*(X\colon Q_p) + 2i_0$ and since $\dim (Y \times (M_p)^{i_0}) = D_*(Y\colon Q_p) + 2i_0$ by Theorem 2, it follows that $D_*(X\colon Q_p) + D_*(Y\colon Q_p) \geq k$.

The following corollary was proved by Cohen [2].

COROLLARY 2. *Let X be a finite-dimensional compact Hausdorff space which is a union of countable number of compact subsets X_i, and let G be one of the groups* *in* (i) *to* (iv) *of* (I) *of* Theorem A. *Then* $D_*(X: G) = \mathrm{Max}\{D_*(X_i: G);\ i = 1, 2, \cdots\}$.

Proof. Let $\dim X = n$. There exists a test space T for G with respect to $\bigcup_{i=0}^{n} L_i$, by (II) of Theorem A. By the classical sum theorem of dimension theory, we know that $\dim(X \times T) = \mathrm{Max}\{\dim(X_i \times T);\ i = 1, 2, \cdots\}$. Since T is a test space for G and $\dim X_i \leq n$, it follows that

$$D_*(X: G) = \dim(X \times T) - \dim T = \mathrm{Max}\{\dim(X_i \times T);\ i = 1, 2, \cdots\} - \dim T$$

$$= \mathrm{Max}\{D_*(X_i: G);\ i = 1, 2, \cdots\}.$$

COROLLARY 3. *Let* X *and* Y *be finite-dimensional, compact metric spaces,* f *a continuous mapping of* X *into* Y, *and* G *one of the groups in Corollary* 2.

(i) *If* $\dim f^{-1}(y) \leq n$ *for every point* y *of* Y, *then* $D^*(X: G) \leq D^*(Y: G) + n$.

(ii) *If* $\mathrm{ord}\, f \leq n$, *then* $D^*(Y: G) \leq D^*(X: G) + n$.

(iii) *If* f *is open and has finite order, then* $D^*(X: G) = D^*(Y: G)$.

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