UNIQUENESS THEOREMS FOR ORDINARY AND HYPERBOLIC DIFFERENTIAL EQUATIONS

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1. This note is concerned with uniqueness theorems for the ordinary differential equation $x' = f(t, x)$ and for the hyperbolic partial differential equation

$$u_{xy} = f(x, y, u, p, q).$$

We first consider a uniqueness result for the ordinary differential equation, under assumptions more general than the uniqueness hypothesis of Kamke [2, pp. 48-52], and then extend this result to prove the analogous uniqueness theorem for the hyperbolic partial differential equation.

Uniqueness theorems for the partial differential equation, under what may be called Nagumo's uniqueness conditions, have been considered by Diaz and Walter [3] and Shanahan [4]. A "crude analogue" of Kamke's uniqueness assertion also appears in [4].

2. Consider the ordinary differential equation

$$x' = f(t, x), \qquad x(0) = 0, \tag{1}$$

where $f(t, x)$ is a real-valued function defined on $0 \le t \le a$, $|x| \le b$. A solution of (1) in the classical sense will mean a real-valued function $x(t)$, continuous in $0 \le t \le a$ and having a finite derivative $x'(t)$, for $0 < t < a$, that satisfies $x'(t) = f(t, x(t))$ for $0 < t < a$. Suppose $x(t)$ and $y(t)$ are solutions of (1) existing on $0 \le t \le a$; then the requirement

$$\lim_{t \to 0+} \frac{|x(t) - y(t)|}{t} = 0,$$

which is satisfied when f is continuous at $(0, 0)$, is a necessary condition for the uniqueness of solutions. This requirement can be generalized. Suppose

$$\lim_{t \to 0+} \frac{|x(t) - y(t)|}{B(t)} = 0, \tag{2}$$

where the function $B(t)$ is continuous, positive on $0 < t \le a$ and such that $B(0+) = 0$. This condition is necessary for uniqueness, but not sufficient. As a matter of fact, we prove

LEMMA 1. *Suppose the function* $B(t)$ *is continuous, positive on* $0 < t \le a$, *with* $B(0+) = 0$. *Then there exists an infinity of functions* f *such that* (1) *has more than one solution satisfying the condition* (2).

Proof. We first construct a function $A(t)$ having a non-negative derivative on $0 \le t \le a$ and such that $\lim_{t \to 0+} A(t)/B(t) = 0$. We proceed as follows:

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Divide the interval $0 \leq t \leq a$ into subintervals I_n such that

$$I_1 = \left(\frac{a}{2}, a\right), \quad I_2 = \left(\frac{a}{4}, \frac{a}{2}\right), \quad \cdots .$$

Suppose $b_n = \inf_{I_n} B(t)$. Find a positive linear function L_1 in I_1 such that

$$L_1(a) = b \quad \text{and} \quad L_1\left(\frac{a}{2}\right) \leq \frac{1}{2} L_1(a).$$

Then find L_2 in I_2 such that

$$L_2\left(\frac{a}{2}\right) = \frac{b_2}{2} \quad \text{if } \frac{b_2}{2} \leq L_1\left(\frac{a}{2}\right),$$

$$L_2\left(\frac{a}{2}\right) \leq \frac{1}{2} L_1\left(\frac{a}{2}\right) \quad \text{and} \quad L_2\left(\frac{a}{4}\right) \leq \frac{1}{2} L_2\left(\frac{a}{2}\right), \quad \text{otherwise.}$$

Continue this process. We then connect the linear functions near the points $a/2^n$ by suitable functions having non-negative derivatives (for example, by arcs of parabolas). This modification gives us the function with the required properties.

When we have constructed the function $A(t)$, it is easy to define $f(t, x)$. For example, if we take $f(t, x) = x^{\alpha} A'(t)$ $(0 < \alpha < 1)$, then

$$x(t) \equiv 0 \quad \text{and} \quad x(t) = (1 - \alpha)^{\frac{1}{1-\alpha}} A^{\frac{1}{1-\alpha}}$$

are solutions of (1) with $x(0) = 0$. It is clear that any two solutions satisfy the condition (2). Hence the proof is complete.

It is also easy to prove the following fact.

LEMMA 2. *Suppose* $f(t, x)$ *is a real-valued function, defined on* $0 \leq t \leq a$, $|x| < b$ *and continuous at* $(0, 0)$. *Then there exists a function* $B(t)$ *on* $0 \leq t \leq 1$ *such that* (2) *is satisfied.*

Proof. Let $x(t)$ and $y(t)$ be the maximal and minimal solutions of (1) on $0 < t < a$. Define $m(t) = |x(t) - y(t)|$. Then $m(0) = 0$, and because of the assured continuity of f at $(0, 0)$, $\lim_{t \to 0+} m(t)/t = 0$. If we now take

$$B(t) = \operatorname*{Sup}_{s \leq t \leq 1} \frac{m(s)}{s},$$

it is easy to verify that $\lim_{t \to 0+} m(t)/B(t) = 0$. This can be done if $m(s) \not\equiv 0$ in some neighborhood of the origin; otherwise, the existence of $B(t)$ is trivial. This implies the stated result.

Remark. We note that the continuity requirement of f at $(0, 0)$ is stronger than condition (2). This can be seen from the following example. Define $f(t, x)$ for $0 \leq t \leq a$ by

$$f(t, x) = \begin{cases} 1 & (x > t), \\ \frac{x}{t} & (0 < x \leq t), \\ 0 & (x \leq 0). \end{cases}$$

The solutions of (1) are $x = ct$, where $0 \leq c \leq 1$. Taking $B(t) = t^{1/2}$, we see that condition (2) is satisfied, even though f is not continuous at $(0, 0)$.

Thus the considerations above show that condition (2) is natural. If (2) is not satisfied, then the solution is clearly not unique.

Now we state our uniqueness result.

THEOREM 1. *Suppose two solutions* $x(t)$ *and* $y(t)$ *of* (1) *satisfy the condition* (2), *where* $B(t)$ *is positive, continuous on* $0 \leq t \leq a$, *with* $B(0) = 0$. *Let the function* $h(t, r) \geq 0$ *be continuous on* $0 < t \leq a$, $r \geq 0$. *Suppose the only solution* $r(t)$ *of*

$$r' = h(t, r) \tag{3}$$

on $0 \leq t \leq a$ *such that* $\lim_{t \to 0+} r(t)/B(t) = 0$ *is the trivial solution. Suppose further that the function* $f(t, x)$ *of* (1) *satisfies the condition*

$$|f(t, x_1) - f(t, x_2)| \leq h(t, |x_1 - x_2|) \tag{4}$$

for (t, x_1) *and* (t, x_2) *in a region* $0 < t \leq a$, $|x| < b$. *Then there exists at most one solution of* (1) *on* $0 \leq t \leq a$.

Proof. (The proof follows, with minor modifications, the general outline of that of Kamke's uniqueness theorem. We include it, so that we can use it in the proof of our uniqueness theorem for partial differential equations.) Suppose there exist two solutions $x(t)$ and $y(t)$ of (1) on $0 \leq t \leq a$, and let $m(t) = |x(t) - y(t)|$. Then $m(0) = 0$ and

$$|m'(t)| \leq |f(t, x(t)) - f(t, y(t))| \leq h(t, m(t)) \tag{5}$$

almost everywhere. Suppose $m(\sigma) > 0$ for some σ $(0 < \sigma < a)$. Consider a solution $r(t)$ of (3) through $(\sigma, m(\sigma))$, existing on some interval to the left of σ. As far as the left of σ as $r(t)$ exists, it satisfies the inequality

$$r(t) \leq m(t). \tag{6}$$

To prove this, we observe that

$$r' = h(t, r) + \varepsilon, \qquad r(\sigma) = m(\sigma), \tag{7}$$

has solutions $r(t, \varepsilon)$ for all sufficiently small $\varepsilon > 0$, existing as far to the left of σ as $r(t)$ exists, and with $\lim_{\varepsilon \to 0} r(t, \varepsilon) = r(t)$ [2]. Thus it is enough to prove

$$r(t, \varepsilon) \leq m(t) \tag{8}$$

for all $\varepsilon > 0$ and all solutions $r(t, \varepsilon)$ of (7). If this inequality does not hold, there exists a least upper bound s of numbers $t \leq \sigma$ for which (8) is false. Since $r(\sigma) = m(\sigma) = r(\sigma, \varepsilon)$ and the functions $r(t)$ and $r(t, \varepsilon)$ are continuous,

$$m(s) = r(s, \varepsilon), \qquad m'(s) \geq r'(s, \varepsilon). \tag{9}$$

It then follows that

$$h(s, m(s)) \geq h(s, r(s, \varepsilon) + \varepsilon,$$

by (7) and (9). This contradiction proves (8), which implies (6).

The solution $r(t)$ can be continued to $t = 0$. If $r(c) = 0$ for some c $(0 < c < \sigma)$, the continuation can be effected by defining $r(t) = 0$ for $0 < t < c$; otherwise, (6) ensures the possibility of continuation. Since $m(0) = 0$, $\lim_{t \to 0+} r(0) = 0$, and we define $r(0) = 0$. Now we have a non-trivial solution $r(t)$ of (3) on $0 \leq t \leq \sigma$ such that $r(\sigma) = m(\sigma)$, and $0 \leq r(t) \leq m(t)$. Then, because of the assumed condition,

$$0 \leq \lim_{t \to 0+} \frac{r(t)}{B(t)} \leq \lim_{t \to 0+} \frac{m(t)}{B(t)} = 0.$$

which, by hypothesis, implies that $r(t) \equiv 0$. This contradicts $r(\sigma) = m(\sigma) > 0$, and hence $m(t) \equiv 0$ on $0 \leq t \leq a$. This completes the proof.

Remark. If $B(t) = t$, this theorem reduces to Kamke's uniqueness theorem. This result also contains the result of Brauer [1], since the extra conditions of Brauer imply the assumption (2).

3. Consider the initial value problem of the partial differential equation

$$u_{xy} = f(x, y, u, p, q); \quad u(x, 0) = E(x); \quad u(0, y) = F(y) \quad \text{and } F(0) = E(0), \tag{10}$$

where the functions $E(x)$ and $F(y)$ are defined on $0 \leq x \leq a$, $0 \leq y \leq b$ respectively. Suppose the real-valued function $f(x, y, h, p, q)$ is defined for $0 \leq x \leq a$, $0 \leq y \leq b$, $-\infty < u, p, q, < \infty$. By a solution of (10), we mean a continuous function $u(x, y)$ having partial derivatives $u_x(x, y)$, $u_y(x, y)$, and $u_{xy}(x, y)$ in the domain $0 \leq x \leq a$, $0 \leq y \leq b$.

Now we have the analogue of Theorem 1 for partial differential equations.

THEOREM 2. *Suppose that for any two solutions* $u(x, y)$ *and* $v(x, y)$ *of* (1),

$$\lim_{y \to 0+} \frac{|u_x(x, y) - v_x(x, y)|}{L(y)} = 0 \quad \text{and} \quad \lim_{x \to 0+} \frac{|u_y(x, y) - v_y(x, y)|}{K(x)} = 0, \tag{11}$$

where the functions $K(x)$ *and* $L(y)$ *are positive and continuous on* $0 < x \leq a$, $0 < y \leq b$, *with* $K(0) = 0$, $L(0) = 0$ *respectively. Suppose the function* $f(x, y, u, p, q)$ *of* (10) *satisfies the condition*

$$|f(x, y, u_1, p_1, q_1) - f(x, y, u_2, p_2, q_2)| \leq h(x, y, |u_1 - u_2|, |p_1 - p_2|, |q_1 - q_2|) \tag{12}$$

for $x, y \neq 0$, *where the function* $h(x, y, r, r_x, r_y) \geq 0$ *is continuous on* $0 < x \leq a$, $0 < y \leq b$ *and* $r, r_x, r_y \geq 0$. *Suppose that for all non-negative functions* m, n, *and* θ, *the only solution* $r(x, \cdot)$ *of*

$$\frac{dr}{dx} = h(x, \cdot, m(x, \cdot), n(x, \cdot), r) \tag{13}$$

on $0 \leq x \leq a$ *such that*

$$\lim_{x \to 0+} \frac{r(x, \cdot)}{K(x)} = 0 \tag{14}$$

is the trivial solution, and that the only solution $R(\cdot, y)$ *of*

$$\frac{dR}{dy} = h(\cdot, y, m(\cdot, y), R, \theta(\cdot, y)) \tag{15}$$

on $0 \leq y \leq b$ *such that*

$$\lim_{y \to 0+} \frac{R(\cdot, y)}{L(y)} = 0 \tag{16}$$

is also the trivial solution. Then there exists a unique solution for the problem (10) *on* $0 \leq x \leq a$, $0 \leq y \leq b$.

Remark. If $K(x) = x$, $L(y) = y$, the theorem above reduces to an analogue of Kamke's uniqueness theorem for ordinary differential equations.

Proof. Suppose that there exist two solutions $u(x, y)$ and $v(x, y)$ of (10) on $0 \leq x \leq a$, $0 \leq y \leq b$. Define

$$\begin{aligned} A(x, y) &= |u(x, y) - v(x, y)|, \\ B(x, y) &= |u_x(x, y) - v_x(x, y)|, \\ C(x, y) &= |u_y(x, y) - v_y(x, y)|. \end{aligned} \tag{17}$$

Since

$$\begin{aligned} &u(x, 0) = v(x, 0) = E(x); \qquad u_x(x, 0) = v_x(x, 0) = E_x(x) \text{ on } 0 \leq x \leq a, \\ &u(0, y) = v(o, y) = F(y); \qquad u_y(0, y) = v_y(0, y) = F_y(y) \text{ on } 0 \leq y \leq b, \end{aligned} \tag{18}$$

it follows that

$$A(0, 0) = 0, \qquad B(x, 0) = 0, \qquad c(0, y) = 0. \tag{19}$$

To prove that the solution is unique, we have to show that $A(x, y) \equiv 0$, $B(x, y) \equiv 0$, and $c(x, y) \equiv 0$ on $0 \leq x \leq a$, $0 \leq y \leq b$. Since $A(0, 0) = 0$, it is enough to show that $B(x, y) \equiv 0$ and $c(x, y) \equiv 0$ on $0 \leq x \leq a$, $0 \leq y \leq b$. We see from (12) and (17) that

$$\begin{aligned} \frac{|dc(x, \cdot)|}{dx} &\leq |f(x, \cdot, u(x, \cdot), u_x(x, \cdot), u_y(x, \cdot)) - f(x, \cdot, v(x, \cdot), v_x(x, \cdot), v_y(x, \cdot))| \\ &\leq h(x, \cdot, A(x, \cdot), B(x, \cdot), c(x, \cdot)). \end{aligned}$$

Suppose that for some σ $(0 < \sigma < x)$, we have $c(\sigma, \cdot) > 0$. Then, proceeding as in the proof of Theorem 1, we can conclude that there exists a solution $r(x, \cdot)$ of (13) on $0 \leq x \leq \sigma$, satisfying

$$r(\sigma, \cdot) = c(\sigma, \cdot) > 0, \qquad 0 \leq r(x, \cdot) \leq c(x, \cdot), \qquad r(0, \cdot) = 0.$$

Then, by condition (11),

$$0 \leq \lim_{x \to 0+} \frac{r(x, \cdot)}{k(x)} \leq \lim_{x \to 0+} \frac{c(x, \cdot)}{k(x)} = 0,$$

which implies $r(x, \cdot) \equiv 0$. This contradicts the fact that $r(\sigma, \cdot) = c(\sigma, \cdot) > 0$, and hence it follows that $c(x, \cdot) \equiv 0$ on $0 \leq x \leq a$.

A similar argument shows that $B(\cdot, y) \equiv 0$ on $0 \leq y \leq b$, and this completes the proof.

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