THE GEOMETRY OF EXTREMAL QUASICONFORMAL MAPPINGS

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1. In this paper we consider Teichmüller mappings between orientable C^2 surfaces immersed (without singularities) in E^3. We investigate the geometric consequences of the assumption that certain standard differential geometric correspondences are Teichmüller mappings. Since conformal mappings are the simplest Teichmüller mappings, it is not surprising that our results tend to generalize known facts about the conformal mapping of surfaces in E^3.

In order to describe the extremal problems which lend importance to Teichmüller mappings, and in order to define these mappings in an appropriate setting, we begin with a brief review of some material from the theory of quasiconformal mapping. A thorough explanation of the subject matter outlined in Section 2 can be found in [1] or [2]; but for a quick reading of the theorems proved here, it is enough to assume that Teichmüller mappings satisfy the conclusion of Lemma 1 in Section 3.

2. Quasiconformal mappings of plane domains may be defined as follows. Let $\mathscr{R}$ denote a *topological rectangle* in E^2, that is, a closed Jordan region with four distinguished boundary points. Then there exists a unique value m $(m \geq 1)$ such that some homeomorphism, *conformal in the interior of* $\mathscr{R}$, carries $\mathscr{R}$ onto the classical rectangle

$$0 \leq x \leq m, \qquad 0 \leq y \leq 1$$

in E^2 in such a way that the four distinguished boundary points are mapped onto the four vertices of the classical rectangle. The value m is called the *modulus* of $\mathscr{R}$ (notation: $m = \bmod \mathscr{R}$).

An arbitrary (sense-preserving) homeomorphism $w: D \to E^2$ of a plane domain D carries each topological rectangle $\mathscr{R} \subset D$ onto a topological rectangle $w(\mathscr{R})$. Such a w is k-*quasiconformal* if and only if

$$\text{l.u.b.}\ \frac{\bmod w(\mathscr{R})}{\bmod \mathscr{R}} = K = \frac{1+k}{1-k} < \infty ,$$

where $\mathscr{R}$ ranges over the class of all topological rectangles $\mathscr{R} \subset D$.

The constant K is called the *maximal dilatation* of w, a name it receives because of the special case in which w is in C^1 and has a positive Jacobian. For here, w carries an infinitesimal circle at any p in D onto an infinitesimal ellipse at w(p) whose major and minor axes are in the ratio $K(p) \geq 1$. In this context, K(p) is called the *dilatation of* w *at* p, and w is k-quasiconformal (by our previous definition) if and only if

$$\sup_{p \in D} K(p) = K = \frac{1+k}{1-k} < \infty .$$

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We note that, in general, $K \geq 1$, so that $0 \leq k < 1$. A mapping is conformal if and only if it is 0-quasiconformal. The inverse of a k-quasiconformal mapping is k-quasiconformal. And, finally, the composition (in either order) of a k-quasiconformal mapping with a conformal mapping is still k-quasiconformal.

The definition of k-quasiconformality can be extended as follows to cover (sense-preserving) homeomorphisms $f: R \to \hat{R}$ between Riemann surfaces. If $f(p) = \hat{p}$, then f maps some coordinate patch U on R containing p onto a coordinate patch $\hat{U}$ on $\hat{R}$ containing $\hat{p}$. Moreover, f induces a homeomorphism $\tilde{f}$ of the conformal preimage of U in E^2 onto the conformal preimage of $\hat{U}$ in E^2. If there exists a $\hat{k}$ $(0 \leq \hat{k} < 1)$ such that, for all choices of p, $\tilde{f}$ is $\tilde{k}$-quasiconformal $(\tilde{k} \leq \hat{k})$, then f is said to be k-quasiconformal, where k is the least such $\hat{k}$. Once again, $K = (1 + k)/(1 - k)$ is called the maximal dilatation of f.

Every k-quasiconformal mapping $f: R \to \hat{R}$ between Riemann surfaces is characterized by a uniquely determined Beltrami differential

$$m = \mu(z)\frac{d\bar{z}}{dz}$$

on R with $\mu(z)$ measurable and $|\mu(z)| \leq k < 1$. This Beltrami differential has the following property. Let $z = x + iy$ designate a conformal parameter on R. Let w(z) be a homeomorphic solution to the Beltrami equation

$$w_{\bar{z}} = \mu(z)\, w_z\,. \tag{1}$$

(By this we mean that w is a homeomorphism with generalized derivatives w_x and w_y, and that, if we set $w_z = (w_x - iw_y)/2$ and $w_{\bar{z}} = (w_x + iw_y)/2$, (1) is satisfied almost everywhere.) Then the assignment of w(z) to the image under f of the point with coordinate z on R yields a conformal parameter w on $\hat{R}$.

A Teichmüller mapping $f: R \to \hat{R}$ is either a conformal mapping, or else it is the k-quasiconformal mapping characterized by a Beltrami differential of the form

$$\mu(z)\frac{d\bar{z}}{dz} = k\frac{\overline{\psi(z)}}{|\psi(z)|}\frac{d\bar{z}}{dz},$$

where $0 \leq k < 1$, and where $\Omega = \psi(z)\,dz^2$ is a meromorphic quadratic differential on R. We note that in any neighborhood not containing zeros and poles of the defining quadratic differential Ω, special conformal parameters $z = x + iy$ may be chosen so that the assignment of $w(z) = Kx + iy$ to the image under f of the point with coordinate z on R yields a conformal parameter w on $\hat{R}$. (This leads directly to Lemma 1 of Section 3, which will therefore be stated without proof.)

Thus, except at the zeros and poles of Ω, a Teichmüller mapping is real analytic and has constant dilatation equal to its maximal dilatation K. It can be shown that the inverse mapping f^{-1} of a Teichmüller mapping is again a Teichmüller mapping defined by a quadratic differential whose zeros and poles correspond under f to those of Ω, and with the same maximal dilatation K.

An *extremal quasiconformal mapping* is one which minimizes the maximal dilatation within some given class of quasiconformal mappings. Thus any quasiconformal mapping is extremal in the narrow class containing only itself. Teichmüller mappings arise as the solutions to more significant extremal problems.

Consider, for instance, the problem of finding the extremal mapping among all the quasiconformal homeomorphisms between two closed Riemann surfaces of the same finite genus $g \geq 2$ that are homotopic to some fixed homeomorphism between R and $\hat{R}$. It is known that a unique solution to this problem always exists. Moreover, the extremal mapping is a Teichmüller mapping defined by a holomorphic quadratic differential on R.

It is thus ensured, for instance, that a Teichmüller mapping exists (usually, there are several), between any two closed, oriented C^2 surfaces immersed in E^3. And each such mapping yields, among all homeomorphisms homotopic to it, the "most nearly conformal" mapping between the two surfaces in question.

Although Teichmüller mappings defined by meromorphic quadratic differentials with only simple poles are extremal among mappings that attain prescribed values at preassigned points, it is not clear that Teichmüller mappings defined by arbitrary meromorphic quadratic differentials are automatically extremal in any significant sense. The title of this paper is not meant to exclude such Teichmüller mappings from the considerations of Section 3.

3. Let S and $\hat{S}$ be oriented C^2 surfaces immersed in E^3. The Euclidean metric imposes a specific conformal structure upon S and $\hat{S}$. Conformal parameters $z = x + iy$ may be introduced on S, for instance, by means of isothermal coordinates x, y. A mapping $f: S \to \hat{S}$ which yields a Teichmüller mapping between the Riemann surfaces represented by S and $\hat{S}$ is called a Teichmüller mapping between S and $\hat{S}$. Lemma 1 is a direct consequence of remarks made in Section 2.

LEMMA 1. *If* $f: S \to \hat{S}$ *is a Teichmüller mapping, then special isothermal coordinates* x, y *may be chosen, in the neighborhood of all but a discrete set of exceptional points on* S, *so that the first fundamental forms at corresponding points of* S *and* $\hat{S}$ *respectively are given by*

$$(2)\qquad \begin{aligned} ds^2 &= \lambda(x, y)\,(dx^2 + dy^2)\,, \\ d\hat{s}^2 &= \hat{\lambda}(x, y)\,(K^2\,dx^2 + dy^2)\,, \end{aligned}$$

where K *is the maximal dilatation of* f.

The exceptional points referred to in Lemma 1 are the zeros and poles of the defining quadratic differential Ω of f. Lemma 1 leads immediately to our first result.

LEMMA 2. *If* $f: S \to \hat{S}$ *is a non-conformal Teichmüller mapping which preserves a net of lines of curvature, then* S *and* $\hat{S}$ *are isothermal except at irremovable umbilics.*

(We note in passing that closed isothermal surfaces of arbitrary genus may be formed by smoothly joining, by appropriate surfaces of revolution, spheres from which holes have been removed. This recalls the problem (see Section 7 of [7]) of finding surfaces of genus $g \geq 2$ free of spherical portions, and yet isothermal away from irremovable umbilics.)

Before proving Lemma 2, let us clarify some terminology. A *line of curvature* is a curve along which the formula of Rodrigues holds. In Lemma 2 we assume that some fixed choice of a net of lines of curvature on S and $\hat{S}$ is preserved by f. Thus, in case S has spherical portions, lines of curvature not part of the chosen net need not be preserved.

An *umbilic* is a point at which normal curvature is independent of direction. A *removable umbilic* is an umbilic in the neighborhood of which lines-of-curvature coordinates (corresponding to the chosen net) are "good" coordinates. That is, an umbilic is removable if it is not a singular point in the chosen net of lines of curvature. Thus, if one chooses meridians and parallels as a net of lines of curvature on a surface of revolution, umbilics at the poles are irremovable, and any other umbilics are removable.

It is our assumption, throughout, that a net of curves on a surface is regular on a dense subset. Thus, the closed set of irremovable umbilics in any net of lines of curvature never covers a neighborhood on the surface.

Proof of Lemma 2. Since f is not conformal, $K > 1$. But then, in some neighborhood of any point which is not exceptional, (2) holds. Wherever (2) holds, the only orthogonal net of curves which can be preserved is the net determined by setting $x =$ constant, $y =$ constant.

If a net of lines of curvature is preserved, then, except at irremovable umbilics and exceptional points, x and y must be lines-of-curvature coordinates on S, so that Kx and y are lines-of-curvature coordinates on $\hat{S}$. But every exceptional point must be an irremovable umbilic. For every exceptional point p is either a zero (of order m) or a pole (of order $-m$) of Ω. If every neighborhood of p contains an irremovable umbilic $q \neq p$, then p must be an irremovable umbilic. If some neighborhood of p contains no irremovable umbilic $q \neq p$, then the index of p in the chosen net of lines of curvature must be $-m/2$ (see p. 82 of [6]). Here again, p must be an irremovable umbilic. Thus, except at irremovable umbilics, x, y and Kx, y constitute isothermal lines-of-curvature coordinates on S and $\hat{S}$ respectively.

The conclusion of Lemma 2 does not hold for conformal mappings, since translations, rotations and magnifications of arbitrary surfaces yield counterexamples. It should be remarked that mappings $f: S \to \hat{S}$ between isothermal surfaces preserving lines of curvature need not be Teichmüller mappings. One has only to consider a projection from one surface of revolution onto another in the direction of their common axis of revolution. In most cases this is not a Teichmüller mapping.

We turn now to the standard mapping between parallel surfaces S and $\hat{S}$ which associates with each point p on S the point on $\hat{S}$ a fixed distance $t \neq 0$ from S along the normal to S at p. It is well known that the standard mapping preserves lines of curvature and normals (see p. 272 of [5]).

Moreover, this standard mapping is conformal if and only if S and $\hat{S}$ are pieces of spheres or planes, or are surfaces of constant mean curvature $1/t$ and $-1/t$, respectively. In the last case, $\hat{S}$ has singularities at points corresponding to umbilics on S (see p. 273 of [5]). As a generalization, we prove the following.

THEOREM 1. *If the standard mapping* f *of* S *onto a parallel surface* $\hat{S}$ *is a Teichmüller mapping, then* S *and* $\hat{S}$ *are Weingarten surfaces, isothermal except at irremovable umbilics. Moreover, if* f *composed with the immersion of* S *in* E^3 *yields an immersion (without singularities) of* $\hat{S}$ *in* E^3, *then* S *and* $\hat{S}$ *have no umbilics unless they are pieces of spheres or planes.*

Proof of Theorem 1. In order to prove the first portion of Theorem 1, we need only show that if f is not conformal, then S and $\hat{S}$ are Weingarten .rfaces. For if f is conformal, S and $\hat{S}$ have constant mean curvature and are therefore isothermal Weingarten surfaces. On the other hand, if f is not conformal, Lemma 2 insures the isothermality of S and $\hat{S}$.

Take g_{ij} and $\hat{g}_{ij}$ as coefficients of the first fundamental forms on S and $\hat{S}$, respectively. Take l_{ij} as the coefficients of the second fundamental form on S. It follows that

$$\hat{g}_{ij} = g_{ij} - 2t\,l_{ij} + t^2(2\mathscr{H}l_{ij} - \mathscr{K}g_{ij}), \tag{3}$$

where $\mathscr{H}$ and $\mathscr{K}$ are mean and Gaussian curvature on S, respectively (see p. 272 of [5]).

We choose the special isothermal coordinates referred to in Lemma 1. Then, using the proof of Lemma 2, we obtain

$$g_{11} = g_{22} = \lambda, \qquad g_{12} = l_{12} = 0,$$

$$l_{11} = k_1\lambda, \qquad l_{22} = k_2\lambda,$$

where k_1 and k_2 are the principal curvatures in the directions determined by setting y = constant and x = constant, respectively. Since $\hat{g}_{11} = K^2\hat{g}_{22}$, (3) yields

$$\{1 - 2tk_1 + t^2(2\mathscr{H}k_1 - \mathscr{K})\} = K^2\{1 - 2tk_2 + t^2(2\mathscr{H}k_2 - \mathscr{K})\},$$

or

$$(1 - tk_1) = \pm K(1 - tk_2). \tag{4}$$

Since f^{-1} is a Teichmüller mapping with the same maximal dilatation K as f, we have the corresponding Weingarten relation for $\hat{S}$. (Note that (4) is satisfied even in the conformal case. For here, $K = 1$, and either $t = a$ with $k_1 + k_2 = 2/a$, or else $k_1 \equiv k_2$.)

But more can be said. By continuity, (4) holds even at irremovable umbilics. Therefore, unless $K = 1$, we obtain

$$(1 - tk_1) = (1 - tk_2) = 0,$$

that is, $k_1 = k_2 = 1/t$ at every umbilic. But then, (3) yields

$$\hat{g}_{11} = \hat{g}_{22} = \hat{g}_{12} = 0$$

at points on $\hat{S}$ corresponding to umbilics on S. This last fact and the remark just preceding the statement of Theorem 1 are sufficient to prove the second portion of Theorem 1. The following corollary is an immediate consequence of Theorem 1.

COROLLARY TO THEOREM 1. *If the standard mapping* $f: S \to \hat{S}$ *between closed parallel surfaces of genus* $g > 1$ *composed with the immersion of* S *in* E^3 *yields an immersion (without singularities) of* $\hat{S}$ *in* E^3, *then* f *is not a Teichmüller mapping.*

Theorem 1 suggests an investigation of Teichmüller mappings which preserve normals. A study of conformal mappings which preserve normals can be found in Chapter 11 of [4].

In Theorem 2, we obtain a restriction on Teichmüller mappings that preserve both normals and a net of lines of curvature. The restriction in question is an equation $W(k_1, k_2; \hat{k}_1, \hat{k}_2) = 0$ relating the principal curvatures k_1, k_2, $\hat{k}_1$ and $\hat{k}_2$ at points of S and $\hat{S}$ in correspondence under the mapping. It seems natural to refer to such an equation as a joint Weierstrass condition. Note, incidentally, that such a joint Weierstrass condition holds automatically for the standard mapping between arbitrary parallel surfaces (see p. 82 of [3]). This implies, for instance, that surfaces parallel to Weingarten surfaces are Weingarten surfaces.

THEOREM 2. *If* $f: S \to \hat{S}$ *is a Teichmüller mapping which preserves lines of curvature and normals, then the principal curvatures* k_1, k_2, $\hat{k}_1$ *and* $\hat{k}_2$ *at points of* S *and* $\hat{S}$ *in correspondence under* f *satisfy*

$$k_1 \hat{k}_2 = \pm K k_2 \hat{k}_1 ,$$

where K *is the maximal dilatation of* f.

Proof of Theorem 2. In the conformal case, choose lines-of-curvature coordinates in the neighborhood of any point on S which is not an irremovable umbilic. Then

$$ds^2 = g_{11}\,dx^2 + g_{22}\,dy^2 ,$$

$$d\hat{s}^2 = \lambda(g_{11}\,dx^2 + g_{22}\,dy^2) ,$$

and

$$l_{12} = \hat{l}_{12} = 0 ,$$

$$l_{11} = k_1 g_{11}, \qquad l_{22} = k_2 g_{22} ,$$

$$\hat{l}_{11} = \lambda g_{11} \hat{k}_1, \qquad \hat{l}_{22} = \lambda g_{22} \hat{k}_2 .$$

The coefficients h_{ij} of the first fundamental form of the spherical image mappings of S and $\hat{S}$ must be equal, since normals are preserved, and thus (see p. 253 of [5])

$$h_{ij} = 2\mathscr{H} l_{ij} - \mathscr{K} g_{ij} = 2\hat{\mathscr{H}} \hat{l}_{ij} - \hat{\mathscr{K}} \hat{g}_{ij}. \tag{5}$$

But then

$$2\mathscr{H} k_1 - \mathscr{K} = 2\hat{\mathscr{H}} \lambda \hat{k}_1 - \hat{\mathscr{K}} ,$$

$$2\mathscr{H} k_2 - \mathscr{K} = 2\hat{\mathscr{H}} \lambda \hat{k}_2 - \hat{\mathscr{K}} ,$$

or

$$k_1 \hat{k}_2 = \pm k_2 \hat{k}_1 .$$

By continuity, this last relation holds on all of S.

In the nonconformal case, we use the coordinates described in Lemma 1 in the neighborhood of any point on S which is not an irremovable umbilic. Then the arguments proving Lemma 2 imply that

$$g_{12} = l_{12} = \hat{g}_{12} = \hat{l}_{12} = 0 ,$$

$$g_{11} = g_{22} = \lambda, \qquad \hat{g}_{11} = K^2 \lambda, \qquad \hat{g}_{22} = \lambda .$$

As a result,

$$l_{11} = k_1 \lambda, \qquad l_{22} = k_2 \lambda ,$$

$$\hat{l}_{11} = \hat{k}_1 \hat{\lambda} K^2, \qquad \hat{l}_{22} = \hat{k}_2 \hat{\lambda} .$$

Thus, using (5), we obtain

$$2\hat{\mathscr{H}}\hat{\lambda}K^2\hat{k}_1 - \hat{\mathscr{K}}\hat{\lambda}K^2 = 2\mathscr{H}\lambda k_1 - \mathscr{K}\lambda,$$

$$2\hat{\mathscr{H}}\hat{\lambda}\hat{k}_2 - \hat{\mathscr{K}}\hat{\lambda} = 2\mathscr{H}\lambda k_2 - \mathscr{K}\lambda,$$

or

$$k_1\hat{k}_2 = \pm Kk_2\hat{k}_1. \tag{6}$$

By continuity, (6) holds everywhere on S.

At nonplanar umbilics, $K = 1$ must hold, so that f must be conformal unless S and $\hat{S}$ are free of nonplanar umbilics. The mapping f must also be conformal if there are corresponding nonplanar points on S and $\hat{S}$ at which $\mathscr{H} = \hat{\mathscr{H}} = 0$. In particular, f must be conformal if S and $\hat{S}$ are pieces of spheres or of minimal surfaces. But this was clear from the outset, since the spherical image mappings of such surfaces are conformal. As a generalization of the old fact just mentioned, we have the following.

THEOREM 3. *If the spherical image mapping* f *of* S *is a Teichmüller mapping, then* S *is an isothermal Weingarten surface on which*

$$k_1 = \pm Kk_2,$$

where K *is the maximal dilatation of* f.

The theorem is known in the conformal case. If f is a nonconformal Teichmüller mapping, choose the coordinates described in Lemma 1 in the neighborhood of any point on S which is not an exceptional point. Then, since $h_{ij} = 2\mathscr{H}l_{ij} - \mathscr{K}g_{ij}$, we obtain

$$2\mathscr{H}l_{11} - \mathscr{K}\lambda = K^2(2\mathscr{H}l_{22} - \mathscr{K}\lambda),$$

$$2\mathscr{H}l_{12} = 0.$$

But then, $l_{12} \equiv 0$. Otherwise, $\mathscr{H} \equiv 0$ in some neighborhood on S, and f would be conformal. Since $l_{12} \equiv 0$, f preserves a net of lines of curvature.

But now we can apply Theorem 2. If we take for $\hat{S}$ the unit sphere and for f the spherical image mapping of S, then (6) becomes

$$k_1 = \pm Kk_2, \tag{7}$$

since $\hat{k}_1 \equiv \hat{k}_2$ on the sphere $\hat{S}$. In the conformal case, (7) holds with $K = 1$. The + sign yields all spheres, the - sign all minimal surfaces. It follows from (7) that S can have no removable umbilics, nor nonplanar points at which $H = 0$, unless f is conformal. (Much more can be said, naturally, about a surface which satisfies (7).) As a trivial consequence of Theorem 3, we have the following slightly broader statement.

COROLLARY TO THEOREM 3. *If* $f\colon S \to \hat{S}$ *is a Teichmüller mapping which preserves normals, and* $\hat{S}$ *is either a sphere or a minimal surface, then* S *is an isothermal Weingarten surface satisfying* $k_1 = \pm Kk_2$, *where* K *is the maximal dilatation of* f.

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