JORDAN DOMAINS AND ABSOLUTE CONVERGENCE OF POWER SERIES

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Let J denote a Jordan curve that separates the origin from the point at infinity, let J^{-1} denote the image of J under reflection in the unit circle, and let the functions

(1) $$f(z) = \sum_1^{\infty} a_n z^n,$$

(2) $$g(z) = b_{-1} z^{-1} + \sum_0^{\infty} b_n z^n,$$

(3) $$h(z) = \sum_1^{\infty} c_n z^n$$

provide conformal schlicht mappings of the unit disk D onto the interior of J, the exterior of J, and the interior of J^{-1}, respectively,

We know little about the relation between geometric properties of J and the absolute convergence of the series Σa_n, Σb_n, Σc_n, beyond the facts that rectifiability of J implies the absolute convergence of all three series and that (for example) absolute convergence of Σa_n implies uniformly rectifiable accessibility of J from the interior. (The boundary of a domain B is *uniformly rectifiably accessible* if to each point p in B there corresponds a finite constant $S = S(p)$ such that each boundary point of B can be joined to p by a path that lies in B and has length at most S. In the case of the function g, the relation between absolute convergence and uniformly rectifiable accessibility follows from the inequalities

$$\int_{1/2}^{1} |g'(re^{i\theta})|\, dr \leq \int_{1/2}^{1} |b_{-1}| r^{-2}\, dr + \int_{1/2}^{1} \sum_1^{\infty} n|b_n| r^{n-1} dr$$

$$\leq |b_{-1}| + \sum_1^{\infty} |b_n|$$

and the fact that g maps the circle $|z| = 1/2$ onto a rectifiable curve.)

In a recent conversation, Ch. Pommerenke asked whether absolute convergence of Σa_n implies absolute convergence of Σb_n.

THEOREM. *There exists a Jordan curve* J *such that the functions* (1), (2), *and* (3) *satisfy the conditions*

Received November 3, 1961.
This research was supported by the Air Force Office of Scientific Research.

$$\sum |a_n| < \infty, \quad \sum |b_n| = \infty, \quad \sum |c_n| = \infty, \tag{4}$$

respectively.

Before describing such a curve J, we note an implication of the theorem. Since inversion with respect to the unit circle produces only minor changes in the geometric properties of an arc, we can not hope to find a simple geometric characterization of the Jordan curves that are associated with functions (1) for which $\Sigma |a_n| < \infty$. The question of absolute convergence must essentially be decided in terms of the Jordan domain, rather than in terms of the domain's boundary viewed as a curve.

To construct our example, we first describe a certain Jordan domain; then we describe a univalent function f that maps D onto a serviceable approximation of this domain.

At the points $z = e^{\pm \pi i/4}$, we construct two horizontal jaw-like extensions of the unit disk, of length s_1, and with N_1 deeply interlocking teeth of length t_1 (see Figure 1). Either between these two jaws or elsewhere on the boundary of the unit disk, we construct further pairs of jaws, of lengths s_j, and with N_j teeth of lengths t_j ($j = 2, 3, \cdots$), in such a way that

$$\sum s_j < \infty, \quad \sum \sqrt{N_j}\, t_j < \infty, \quad N_j t_j \to \infty. \tag{5}$$

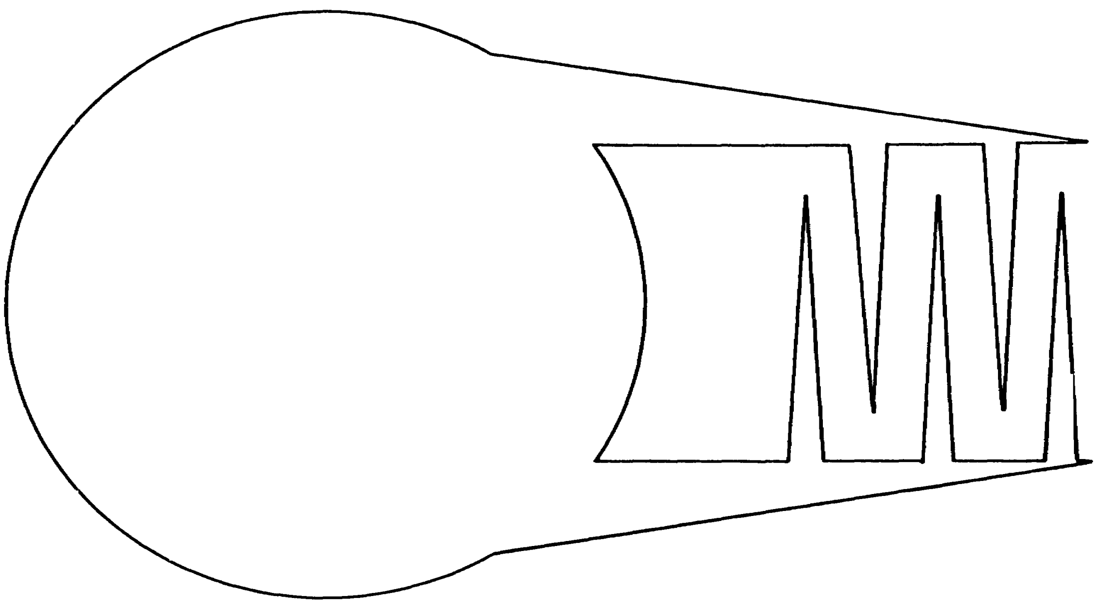

Figure 1.

In order to achieve absolute convergence of Σa_n, we now modify our domain so that we can write a simple analytic expression for the corresponding mapping function f. We define the function

$$f(z) = z + \sum_{m=1}^{\infty} L_m \{1 - (1 - z/z_m)^{k_m}\}, \tag{6}$$

where the L_m are appropriate complex numbers, where the z_m are certain points on the unit circle, with the property that for each m the powers z_m^n $(n = 1, 2, \cdots)$ are uniformly distributed on the circle, and where the k_m are small positive numbers; the expression in braces denotes a function which is holomorphic in D and vanishes at $z = 0$.

The function f provides a univalent mapping of the unit disk onto a domain which resembles the unit disk, except that each term in the infinite series in the right-hand member of (6) represents a tooth-like extension, of length $|L_m|$ and pointing in the direction arg L_m (for details, see [1, Section 3] or [2, Lemma A]). It is obvious that the L_m, z_m, and k_m can be chosen so that for our purposes the image of D under f is interchangeable with the Jordan domain described earlier in our construction. The image of the unit circle under the mapping (6) will serve as our curve J.

Those terms in the right-hand member of (6) which represent jaws contribute to $\Sigma |a_n|$ a quantity not greater than $2\Sigma s_j$. The contribution from the terms representing teeth is infinite if k_{m+1}/k_m tends to zero fast enough; but we can keep this contribution finite by using the same exponent k_m for all teeth pertaining to the j th pair of jaws. For if N_j of the exponents k_m are equal, and if the corresponding lengths $|L_m|$ have a common value t_j, then many of the contributions from the N_j power series of the corresponding terms in (6) cancel each other in part, and, for all sufficiently small values of the common exponent k_m, the total contributions to $\Sigma |a_n|$ from the N_j teeth is less than $Ct_j\sqrt{N_j}$, where C is a universal constant (for details, see the proof of Theorem 3 in [1]). The first of conditions (4) now follows from the first two conditions in (5).

The second condition in (4) follows from the last condition in (5) and the fact that every path in the outer domain which leads from a point "outside" of the j th pair of jaws to a point lying "behind the teeth" of that pair of jaws has length at least $(N_j - 1)t_j/2$.

The third condition in (4) follows from the fact that if J is not uniformly rectifiably accessible from its exterior, then J^{-1} is not uniformly rectifiably accessible from its interior. This completes the proof.

We conclude with a pessimistic remark. Let $\{k_m\}$ be a sequence such that k_{m+1}/k_m tends to zero very rapidly. Let k_1 and k_2 be assigned as exponents in the two terms of (6) that represent the first pair of jaws of our domain; let K_1, the set of the next N_1 of the constants k_m, be associated with the teeth in the first pair of jaws; and so forth. As was pointed out earlier, the series $\Sigma|a_n|$ then diverges. However, if for the j th set of teeth we simply use the greatest of the k_m in K_j, or the least of the k_m in K_j (or, for that matter, any one of the k_m in K_j), convergence of $\Sigma|a_n|$ is achieved. Now the only effect of these changes in the choice of the k_m is this, that the teeth in the j th pair of jaws have different degrees of sharpness in the one case, but not in the other cases; that is, absolute convergence is achieved not by making the teeth of the j th set rather blunt, or very sharp, but by making them equally sharp. This points to the proposition that, in cases where J is not rectifiable, absolute convergence of Σa_n must be regarded as an arithmetic accident.

REFERENCES

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