A GENERALIZATION OF A CONTINUOUS CHOICE FUNCTION THEOREM

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1. Many applications of the Vietoris Mapping Theorem are known (see, for example, [2] and references in that paper). We give another such application concerning multi-valued functions defined on sets of hyperplanes in Euclidean space. Our theorem generalizes a certain known theorem on continuous choice functions. A special case of the theorem of the present paper was first proved in a different way by A. Kosinski, who subsequently proved a still more general theorem which will be published in [3].

We recall some of the simplest concepts concerning multi-valued functions. A *multi-valued function* $F: X \to Y$, where X and Y are topological spaces, is a function which assigns to each point $x \in X$ a non-empty subset $F(x)$ of Y. The set

$$\Gamma(F) = \{(x, y): y \in F(x)\} \subset X \times Y$$

is called the *graph* of F.

Let X be a compact space. Then a multi-valued function $F: X \to Y$ is said to be *continuous* if the graph $\Gamma(F)$ is compact. If Y is compact, this condition is equivalent to the upper semi-continuity of F, when F is regarded as a single-valued mapping of X into the hyperspace of non-empty compact subsets of Y. If F is single-valued, then the above notion of continuity is equivalent to the continuity of a single-valued function in the ordinary sense.

2. We shall use the reduced Čech homology groups $H_k(X)$ of a compact space X with coefficients modulo 2. The space X is called *acyclic* if it is non-empty and if $H_k(X) = 0$ for every $k > 0$. A multi-valued function $F: X \to Y$ is said to be *acyclic* if the sets $F(x)$ are acyclic for every $x \in X$.

VIETORIS MAPPING THEOREM. Let X and Y be compact spaces, and let $f: X \to Y$ be a continuous (single-valued) mapping such that $f^{-1}(y)$ is acyclic for every $y \in Y$. Then f induces an isomorphism $f_*: H_k(X) \approx H_k(Y)$ for every $k = 0, 1, 2, \cdots$ (see [1] and [2], Nr. 3).

3. Let $\mathscr{H}^n_m$ denote the set of all m-dimensional hyperplanes in Euclidean n-dimensional space E^n $(m < n)$ that pass through a fixed point (the origin). For each m-dimensional hyperplane $H \in \mathscr{H}^n_m$, denote by H^* the $(n - m)$-dimensional hyperplane that passes through the origin and is orthogonal to H.

THEOREM. *Let* F *be a multi-valued continuous acyclic function which assigns to each hyperplane* $H \in \mathscr{H}^n_m$ *a subset* $F(H)$ *of* E^n. *Then there exists a hyperplane* $H_0 \in \mathscr{H}^n_m$ *such that* $F(H_0)$ *intersects* H_0^*.

Proof. Evidently it is sufficient to prove this theorem for the case when $m = n - 1$. Then the set $\mathscr{H}^n_{n-1}$ is the $(n - 1)$-dimensional projective space P^{n-1}. Let S^{n-1} be the unit $(n - 1)$-dimensional sphere in E^n with its center at the origin,

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and let ϕ be the natural mapping of S^{n-1} onto P^{n-1}; then ϕ assigns to each point $x \in S^{n-1}$ the hyperplane orthogonal to the vector $\vec{x}$.

Consider the multi-valued mapping $F: P^{n-1} \to E^n$ and the mapping

$$G = F\phi: S^{n-1} \to E^n .$$

Consider the graphs of F and G:

$$\Gamma(G) = \{ (x, u): u \in G(x)\} \subset S^{n-1} \times E^n,$$

$$\Gamma(F) = \{ (y, u): u \in F(y)\} \subset P^{n-1} \times E^n.$$

The mapping ϕ induces a mapping $\Phi: \Gamma(G) \to \Gamma(F)$ defined by

$$\Phi(x, u) = (\phi(x), u) .$$

Let $r: \Gamma(G) \to S^{n-1}$ be the projection of the graph $\Gamma(G)$ defined by $r(x, u) = x$, and let $p: \Gamma(F) \to S^{n-1}$ be the projection of $\Gamma(F)$ defined by $p(y, u) = y$.

Let f be the mapping of $\Gamma(F)$ into S^{n-1} defined by the condition that if $(y, u) \in \Gamma(F)$, then u' is the orthogonal projection of the point $u \in E^n$ onto the $(n - 1)$-dimensional hyperplane represented by y, that is, onto the hyperplane orthogonal to the vector $\vec{x}$. If the theorem is false, then the point u' never falls at the origin, and therefore we can consider its central projection u'' from the origin onto the sphere S^{n-1}. Thus we obtain a point u'' which we denote by $f(y, u)$. Evidently f is a continuous (single-valued) mapping of $\Gamma(F)$ into S^{n-1}. Finally, let us put $g = f\Phi: \Gamma(G) \to S^{n-1}$.

Consider the commutative diagram

$$\begin{tikzcd} S^{n-1} \arrow[dd, "\phi"'] & \Gamma(G) \arrow[l, "r"'] \arrow[dd, "\Phi"'] \arrow[dr, "g"] & \\ & & S^{n-1} \\ P^{n-1} & \Gamma(F) \arrow[l, "p"] \arrow[ur, "f"'] & \end{tikzcd}$$

Since the multi-valued mappings F and G are continuous and acyclic, it follows that the single-valued mappings r and p satisfy the assumptions of the Vietoris Mapping Theorem. Therefore r_* and p_* are isomorphisms. Moreover, observe that if (x, u) is a point of $\Gamma(G)$, then the points $r(x, u) = x$ and $g(x, u)$ are never antipodal, for $g(x, u)$ lies on the hyperplane orthogonal to the vector $\vec{x}$. It follows that the mappings r and g are homotopic. Therefore $r_* = g_*$ and g_* is an isomorphism.

On the other hand, the homomorphism ϕ_* is zero (for the homology groups are taken modulo 2), and, since p_* is an isomorphism, g_* must be zero, which is impossible since $H_{n-1}(S^{n-1}) \neq 0$. This completes the proof.

In the special case where the mapping F is single-valued and where, for each $H \in \mathscr{H}^n_{n-1}$, the point $F(H)$ lies in the hyperplane H, our theorem reduces to the following corollary:

Let f *be a continuous function which selects from each* (n - 1)*-dimensional hyperplane* $H \in \mathscr{H}^n_{n-1}$ *a point* f(H). *Then there exists a hyperplane* $H_0 \in \mathscr{H}^n_{n-1}$ *such that the selected point* $f(H_0)$ *of* H_0 *is the origin.*

This fact was used by K. S. Stein to prove that if V is a convex body in E^n and p is an arbitrary point of V, then there exists an (n - 1)-dimensional cross-section through p whose center of gravity is at p (see [4]). A special case of it was also proved by H. Steinhaus in [5]. Evidently, the assumptions that V is convex or that the point p belongs to V are not essential here. Stein's theorem is true for any set $V \subset E^n$ and $p \in E^n$ which satisfy the condition that each (n - 1)-dimensional cross-section through p has a center of gravity and that the centers of gravity depend continuously on the hyperplanes of cross-section.

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