A PROOF OF THE ABIAN-BROWN FIXED POINT THEOREM

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In [1] the following generalization of the Brouwer Fixed Point Theorem is proved.

THEOREM 1. *If* C^n *is an* n-*cell in euclidean* n-*space* E^n, *and if* f *is a map of* C^n *into* E^n *such that* Bd C^n, *the boundary of* C^n, *is carried by* f *into* C^n, *then* f *has a fixed point.*

We shall present a new proof of Theorem 1 which establishes an equivalence between Theorem 1 and the Brouwer Fixed Point Theorem.

LEMMA. *Let* C^n *be an* n-*cell in a* T_2-*space* M (*so that* $C^n - \mathrm{Bd}\, C^n = \mathrm{int}\, C^n$ *is a component of* M - Bd C^n). *Then, if* f *is a map of* C^n *into* M *while* $f(\mathrm{Bd}\, C^n) \subset C^n$ *and* $f(C^n) \cap \mathrm{Bd}\, C^n \neq \mathrm{Bd}\, C^n$, f *has a fixed point.*

Proof. We observe that under the assumptions of the lemma, $f(C^n)$ is a compact metric space. Since $P = f(C^n) \cap \mathrm{Bd}\, C^n \neq \mathrm{Bd}\, C^n$, $P \subset C^{n-1}$, where C^{n-1} is an (n - 1)-cell in Bd C^n. Let r be the identity map on P. The (n - 1)-cell C^{n-1} is an absolute retract, and therefore the map r can be extended to a map defined on $f(C^n) - C^n$ as well. Denote by $\tilde{r}$ the map r so extended, with its domain enlarged by the condition that $\tilde{r} \,|\, [f(C^n) \cap C^n]$ is the identity map. The map $\tilde{r}f$ carries C^n into C^n and has a fixed point if and only if f has a fixed point. By the Brouwer Fixed Point Theorem, $\tilde{r}f$ does have a fixed point, and thus f has a fixed point. This completes the proof of the lemma.

We proceed to the direct proof of Theorem 1. Let us assume that the map f of this theorem has no fixed point and that $E^n \subset E^{n+1}$. We consider the single suspension

$$C^{n+1} = \omega_1 \wedge C^n \cup \omega_2 \wedge C^n$$

of C^n in E^{n+1}. The map f is to be extended to a map $\tilde{f}$ of C^{n+1} into E^{n+1} by the following definition. Let E_1^n be a hyperplane in E^{n+1} which is parallel to E^n; we observe that $E_1^n \cap C^{n+1}$ is the empty set, an n-cell, or a point. If

$$E_1^n \cap C^{n+1} \subset \omega_1 \wedge C^n,$$

let x be any point in C^n, and let $x' = (\omega_1 \wedge x) \cap E_1^n$. Then

$$\tilde{f}(x') = [\omega_1 \wedge f(x)] \cap E_1^n .$$

We extend f in a similar manner to $\omega_2 \wedge C^n$. One can see that f is a map of C^{n+1} into E^{n+1}, $\tilde{f}(\mathrm{Bd}\, C^{n+1}) \subset C^{n+1}$, and $\tilde{f}$ has exactly two fixed points, ω_1 and ω_2.

We imagine that the image $\tilde{f}(C^{n+1})$ is symmetrically cut by E^n. It is then possible to follow $\tilde{f}$ by a reflection h through E^n which interchanges the two pieces of $\tilde{f}(C^{n+1})$. Let $g = h\tilde{f}$. The map g has no fixed point and $g(\mathrm{Bd}\, C^{n+1}) \subset C^{n+1}$.

Let E_2^n be a plane which is parallel to E^n and which intersects $\omega_1 \wedge C^n$ in an n-cell C_2^n. There is in E_2^n an n-cell P which contains both C_2^n and $B = g(C^{n+1}) \cap E_2^n$

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in its interior. Define a map r on $\omega_1 \wedge C_2^n \cup B$ as follows. The map r is pointwise fixed on $C_2^n \cup B$ and carries $\omega_1 \wedge C_2^n$ onto C_2^n. Since P is in an absolute retract, r may be extended to a map $\tilde{r}$ into P of the closure of those parts of C^{n+1} and $g(C^{n+1})$ which lie on the same side of E_2^n as ω_1. Defining $\tilde{r}$ to be the identity map on the rest of $C^{n+1} \cup g(C^{n+1})$, we note that $\tilde{r}g$ is a map of C^{n+1} meeting the conditions $\tilde{r}g(\text{Bd } C^{n+1}) \subset C^{n+1}$, and $\tilde{r}g$ has no fixed point, $\tilde{r}g(\text{Bd } C^{n+1}) \cap C^{n+1} \neq \text{Bd } C^{n+1}$. But this contradicts the lemma. Hence, the map f of Theorem 1 has a fixed point.

REFERENCE

1. S. Abian and A. B. Brown, *A new fixed point theorem for continuous maps of the closed* n-*cell* (to appear).

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