MEROMORPHIC FUNCTIONS WITH SMALL CHARACTERISTIC AND NO ASYMPTOTIC VALUES

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1. INTRODUCTION

By the well-known theorem of Fatou, if f(z) is holomorphic and bounded in $|z| < 1$ then f(z) possesses radial limits almost everywhere, that is, $\lim_{r \uparrow 1} f(re^{i\theta})$ exists for almost all θ. This result was extended by Nevanlinna to meromorphic functions of bounded characteristic T(r), for as Nevanlinna showed, the functions meromorphic in $|z| < 1$ with bounded T(r) are exactly those which may be obtained as quotients of bounded holomorphic functions [4, p. 189]. A natural question raised by Lohwater and Piranian [2, p. 16], is this: if the condition of boundedness of T(r) be relaxed to "T(r) doesn't grow faster than so-and-so," can one still conclude that *some* radial limits must exist? Bagemihl, Erdös and Seidel [1, Theorem 7] have given an example of a *holomorphic* function without radial limit for which $T(r) = O((1 - r)^{-8})$. Lohwater and Piranian [2] gave an example of a *meromorphic* function without radial limit for which $T(r) = O(-\log(1 - r))$. See also Noshiro [5, p. 90].

The object of the present paper is to prove (Theorem 5) that there exists a function F(z), *meromorphic* in $|z| < 1$, whose characteristic is dominated by an arbitrarily given increasing unbounded function, such that F(z) has no asymptotic value, finite or infinite, and hence no radial limit. As will be obvious, the method used to prove this result will *not* apply to *holomorphic* functions; a holomorphic function must possess at least one asymptotic value, though not necessarily a radial limit [4, p. 292].

It is of interest to note one result for holomorphic functions which follows easily from theorems of Zygmund [6, pp. 90-91]. A slight reformulation of these results of Zygmund may be stated as

THEOREM 1. *Let*

$$F(z) = \sum_{k=1}^{\infty} c_k z^{n_k},$$

where

$$(1) \qquad n_{k+1}/n_k > q > 1, \quad \limsup |c_k|^{1/n_k} = 1, \quad \lim c_k = 0, \quad \sum_{k=1}^{\infty} |c_k|^2 = \infty.$$

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Let E *denote the subset of* $|z| = 1$ *on which* F(z) *possesses finite radial limits. Then* E *is of measure zero.*

As a simple consequence of this result we obtain

THEOREM 2. *Let* $\mu(r) > 0$ *on* $[0, 1)$ *and let* $\mu(r) \uparrow \infty$ *as* $r \uparrow 1$. *Then there exists a function* F(z), *holomorphic in* $|z| < 1$, *whose maximum modulus* M(r) *satisfies*

$$M(r) \leq \mu(r), \tag{2}$$

and such that the subset E *of* $|z| = 1$ *on which* F(z) *possesses finite radial limits is of measure zero.*

Proof. Choose any sequence $\{c_k\}$ which satisfies the three conditions on c_k stated in (1); for example, let $c_k = k^{-1/2}$. Then choose the n_k by induction to satisfy the first restriction in (1), with $q = 2$, and so that (2) will be an obvious consequence. Choose the positive integer n_1 so that

$$|c_1| r^{n_1} < \frac{1}{2}\mu(r) \qquad (0 \leq r < 1),$$

which is clearly possible from the conditions on $\mu(r)$. Choose n_2 so that $n_2 > 2n_1$ and

$$|c_2| r^{n_2} < \frac{1}{4}\mu(r) \qquad (0 \leq r < 1);$$

and so on.

We conclude the introduction by stating for future reference several well-known facts on characteristic functions. Let f(z) be meromorphic in $|z| < R \leq \infty$, with characteristic

$$T(r) = T(r, f) = N(r, \infty) + m(r, \infty) = N(r, f, \infty) + m(r, f, \infty),$$

where N and m have their usual significance. According to the result of Cartan [4, p. 178],

$$T(r) = \frac{1}{2\pi}\int_0^{2\pi} N(r, e^{i\alpha})\, d\alpha + \overset{+}{\log} |f(0)| \qquad (f(0) \neq \infty). \tag{3}$$

Also, we note the following lemma.

LEMMA 1. *Let* f(z) *be holomorphic in* $|z| < R \leq \infty$, *and let* $\phi(w)$ *be meromorphic in* $|w| < \infty$, *with both* f *and* ϕ *non-constant. Set* $F(z) = \phi(f(z))$, *which is meromorphic in* $|z| < R$, *and non-constant. Let* a *be any complex number and let the* a*-points of* $\phi(w)$ *be* $\zeta_1, \zeta_2, \cdots, \zeta_k, \cdots$, *repeated according to multiplicity, as usual. Then*

$$N(r, F, a) = \sum_{k=1}^{\infty} N(r, f, \zeta_k). \tag{4}$$

This lemma is obvious, for clearly the functions $n(r, F, a)$ and $n(r, f, \zeta_k)$ satisfy a similar relation. For any given $r < R$, the apparent infinite series in (4) is a finite sum, since $|\zeta_k| \to \infty$.

It is convenient to note that Jensen's formula may be put in the form: let $f(z)$ be holomorphic in $|z| < R$, $f(0) \neq 0$; then

$$N(r, f, 0) = \frac{1}{2\pi}\int_0^{2\pi} \log |f(re^{i\theta})|\, d\theta - \log |f(0)| \qquad (0 \leq r < R). \tag{5}$$

Finally, we recall that if $\phi(z)$ is a rational function, then $T(r, \phi) \sim c \log r$ as $r \to \infty$, whereas if $\phi(z)$ is meromorphic in $|z| < \infty$ and non-rational, then [Nevanlinna 4, p. 218]

$$\lim_{r \to \infty} \frac{T(r, \phi)}{\log r} = \infty. \tag{6}$$

2. A WILD MEROMORPHIC FUNCTION

Let

$$1 < \lambda_1 < \lambda_2 < \cdots \uparrow \infty \tag{7}$$

be a given sequence of numbers. Let $\{a_n\}_1^\infty$ be a sequence of *distinct* finite points in the w-plane such that

$$a_1 \neq 0; \qquad \Im a_n \neq \Im a_{n+1} \quad (n \geq 1), \tag{8}$$

and

$$\{a_n\}_1^\infty \textit{ is dense in } |w| < \infty. \tag{9}$$

Then for any N, $\{a_n\}_N^\infty$ *is dense in* $|w| < \infty$. Now, let $w = \phi_1(z)$ map $|z| < R_1$ onto the Riemann sphere slit along (the stereographic image of) the rectilinear segment $[a_1, a_1 + h_1]$, where $h_1 > 0$ is such that $0 \notin [a_1, a_1 + h_1]$, with $\phi_1(0) = 0$, $\phi_1'(0) = 1/4$. As $h_1 \downarrow 0$, $R_1 \to \infty$, and we choose h_1 so that

$$R_1 > 4\lambda_1, \qquad h_1 < 1. \tag{10}$$

Now look at the two-sheeted covering $\mathscr{S}_2$ of the w-sphere consisting of the sheets S_1 and S_2, where S_1 is the sphere slit along $[a_1, a_1 + h_1]$ and S_2 is the sphere slit along both $[a_1, a_1 + h_1]$ and $[a_2, a_2 + h_2]$; S_1 and S_2 are joined in the usual fashion along $[a_1, a_1 + h_1]$. We note that $[a_1, a_1 + h_1]$ and $[a_2, a_2 + h_2]$ are disjoint, by (8). Let $w = \phi_2(z)$ map $|z| < R_2$ onto $\mathscr{S}_2$ with $\phi_2(0) = 0 \in S_1$, $\phi_2'(0) = 1/4$. As $h_2 \downarrow 0$, $R_2 \to \infty$. To see this, start with $h_2 = 0$ and with the map, similarly normalized, of the corresponding $\mathscr{S}_2$ onto $|\zeta| < \infty$; the $\mathscr{S}_2$ in which we are interested is the image of $|\zeta| < \infty$ less a slit γ along an arc ending at ∞. By continuity, for a given $M > 0$, $\gamma \subset \{|\zeta| > M\}$ for $0 \leq h_2 < \delta_M$. This domain D_ζ, corresponding to $\mathscr{S}_2$, is mapped onto $|z| < R_2$ by $z = g(\zeta)$ with $g(0) = 0$ and $g'(0) = 1$. On application of the Koebe 1/4-theorem [4, pp. 87-91] to the function $M^{-1}g(M\omega)$, which maps $|\omega| < 1$ *into* $|z| < R_2/M$, it follows that $R_2 \geq M/4$ for $0 \leq h_2 < \delta_M$. Thus we may choose h_2 so that

(11) $$R_2 > 4\lambda_2, \quad 0 < h_2 < \frac{1}{2}.$$

And so on by induction. $\mathscr{S}_n$ is the n-sheeted covering of the w-sphere consisting of the sheets $S_1, S_2, \cdots, S_n$. S_1 is as above; S_p $(1 < p \leq n)$ is the smooth w-sphere slit along the segments

$$[a_{p-1}, a_{p-1} + h_{p-1}] \quad \text{and} \quad [a_p, a_p + h_p].$$

S_p and S_{p-1} are joined along the slit $[a_{p-1}, a_{p-1} + h_{p-1}]$. The slit $[a_n, a_n + h_n]$ in S_n is left open. Let $w = \phi_n(z)$ map $|z| < R_n$ onto $\mathscr{S}_n$ with $\phi_n(0) = 0 \in S_1$, $\phi_n'(0) = 1/4$. By the type of argument used previously, $R_n \to \infty$ as $h_n \downarrow 0$, and therefore we may choose h_n so that

(12) $$R_n > 4\lambda_n, \quad 0 < h_n < \frac{1}{n} \quad (n \geq 1).$$

Now let $\mathscr{S} = \lim \mathscr{S}_n$ denote the simply connected surface consisting of all the sheets $S_1, \cdots, S_n, \cdots$; then $\mathscr{S}_n \subset \mathscr{S}$. Let $w = \phi(z)$ map $|z| < R \leq \infty$ onto $\mathscr{S}$ with $\phi(0) = 0 \in S_1$ and $\phi'(0) = 1/4$. The sheets S_n correspond to regions D_n in $|z| < R$ that are bounded by elements of an expanding sequence of analytic Jordan curves C_n; C_n is the image of the doubled segment $[a_n, a_n + h_n]$ where S_n and S_{n+1} are joined. D_1 is the interior of C_1; D_n $(n > 1)$ is the doubly-connected region between C_{n-1} and C_n. Let Δ_n denote the interior of C_n. Then $g_n(z) \equiv \phi^{-1}(\phi_n(z))$ is a schlicht map of $|z| < R_n$ onto Δ_n, with $g_n(0) = 0$, $g_n'(0) = 1$. Thus Δ_n contains the disc $|z| < R_n/4$. Then, by (12),

(13) $$C_n \subset \{|z| > \lambda_n\}.$$

By (7), $\lambda_n \to \infty$, and hence $\mathscr{S}$ is parabolic and $w = \phi(z)$ maps $|z| < \infty$ onto $\mathscr{S}$.

Clearly $\phi(z)$ takes each value, including ∞, just once in each D_n, with the usual interpretation for values assumed on some C_ν. Since we shall be interested in points where $\phi(z)$ assumes the value $e^{i\alpha}$ (α real), we restrict a_n and h_n by

(14) $$a_1 > 1, \quad |a_n| \neq 1, \quad [a_n, a_n + h_n] \cap \{|w| = 1\} = \emptyset,$$

which is compatible with (8), (9), and (12). Then all the $e^{i\alpha}$-points of ϕ are simple; none occur on any C_n. The condition $a_1 > 1$ implies that S_1 contains the schlicht disc $|w| < 1$. By the normalization $\phi'(0) = 1/4$, ϕ^{-1} maps $|w| < 1$ onto a domain including $|z| < 1$. Hence by (13), $|\phi(z)| < 1$ in $|z| < 1$. Thus if $\zeta_n(\alpha)$ $(n \geq 1)$ denotes the single simple $e^{i\alpha}$-point of $\phi(z)$ in D_n, then $|\zeta_n(\alpha)| \geq \lambda_{n-1}$, where we define $\lambda_0 = 1$.

Now if K is any unbounded curve:

$$z = \gamma(t) \quad (0 \leq t < \infty, \ \gamma \text{ continuous}, \ \limsup_{t \to \infty} |\gamma(t)| = \infty),$$

then the image $\phi(K)$ is dense on the sphere. This is clear from (9), (12), and the fact that any such K intersects all but a finite number of the curves C_n. Collecting these results, we have

THEOREM 3. *Let* $1 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \uparrow \infty$ *be a given sequence. There exists a function* $\phi(z)$, *meromorphic in* $|z| < \infty$, *with* $\phi(0) = 0$ *and with the properties*

(15) *the image* $\phi(K)$ *of any unbounded curve is dense on the sphere,* and

(16) $\begin{cases} \textit{if } \zeta_n = \zeta_n(\alpha) \ (n \geq 1) \textit{ are the } e^{i\alpha}\textit{-points of } \phi(z) \ (\alpha \textit{ real}), \textit{ then each} \\ \zeta_n \textit{ is a simple } e^{i\alpha}\textit{-point, and } |\zeta_n(\alpha)| \geq \lambda_{n-1} \ (n \geq 1). \end{cases}$

A consequence of (15) is that ϕ possesses no asymptotic values.

Before going on, we note the following corollary of Theorem 3, which is the analog of Theorem 5 (our principal objective) for functions in $|z| < \infty$. The result is as strong as possible, in view of (6).

COROLLARY. *Let* $\mu(r)$ *be given for* $r \geq 0$ *and let it satisfy* $0 < \mu(r) \uparrow \infty$. *Then there exists* $\phi(z)$, *meromorphic in* $|z| < \infty$, *and satisfying* (15) *and also*

(17) $$T(r, \phi) = O(\mu(r) \log r).$$

Proof. By (16): $n(r, \phi, e^{i\alpha}) \leq \nu$, for $0 \leq r \leq \lambda_\nu$ and $\nu = 0, 1, \cdots$; in particular, $n = 0$ for $r \leq \lambda_0 = 1$. Therefore $N(r, \phi, e^{i\alpha}) = 0$ for $r \leq 1$, and

$$N(r, \phi, e^{i\alpha}) \leq \int_1^r \frac{\nu\, dt}{t} = \nu \log r \qquad (1 \leq r \leq \lambda_\nu,\ \nu \geq 1).$$

Then by (3), since $\phi(0) = 0$, $T(r, \phi) = 0$ for $0 \leq r < 1$ and

$$T(r, \phi) \leq \nu \log r \qquad (1 \leq r \leq \lambda_\nu,\ \nu \geq 1).$$

Finally, if the initial sequence $\{\lambda_n\}$ was chosen so that $\mu(\lambda_{\nu-1}) \geq \nu$ $(\nu > 1)$ which is obviously possible, then for $\lambda_{\nu-1} \leq r \leq \lambda_\nu$ and $\nu > 1$,

$$T(r, \phi) \leq \nu \log r \leq \mu(\lambda_{\nu-1}) \log r \leq \mu(r) \log r.$$

3. AN UNSAVORY HOLOMORPHIC FUNCTION

THEOREM 4 (Bagemihl, Erdös, and Seidel). *Let* $\mu(r)$ *be given for* $0 \leq r < 1$, *and let it satisfy* $0 < \mu(r) \uparrow \infty$ *as* $r \uparrow 1$. *Then there exists a function* $f(z)$, *holomorphic in* $|z| < 1$, *whose maximum modulus* $M(r)$ *satisfies*

(18) $$M(r) \leq \mu(r) \qquad (0 \leq r < 1),$$

and such that on each curve Γ *in* $|z| < 1$ *tending to* $|z| = 1$, $f(z)$ *assumes arbitrarily large values.*

We remark that the curve Γ in this theorem need not tend to a point on $|z| = 1$; it is actually sufficient that it have points in every neighborhood of the unit circle. The theorem is due to Bagemihl, Erdös, and Seidel [1, Theorems 3 and 5], who construct f as an ingenious infinite product. The first example of one such function seems to be that of Lusin and Priwaloff [3, pp. 147-150], who use a gap Taylor series; the formulation used by Lusin and Priwaloff is not best adapted to an arbitrary μ, but it is interesting to note that the basic idea in their construction may be used to prove Theorem 4 as follows:

Set $a_1 = 1$ and $a_n = n! - (n-1)!$ $(n > 1)$. Then

$$a_n = (n-1) \sum_{k=1}^{n-1} a_k \quad (n > 1). \tag{19}$$

There exists a sequence $\{\beta_n\}$ of positive integers such that

$$a_n r^{\beta_n} < 2^{-n} \mu(r) \quad (0 \leq r < 1,\ n \geq 1). \tag{20}$$

If $\{\lambda_n\}$ is any sequence of integers such that

$$\lambda_n \geq \beta_n, \tag{21}$$

the function

$$f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n} \tag{22}$$

is holomorphic in $|z| < 1$ and satisfies (18). Now, by induction, we choose the λ_n and a sequence $\{r_n\}$ $(0 < r_n \uparrow 1)$ so as to satisfy (21) and

$$r_{n-1}^{\lambda_n} \leq \frac{1}{2^n n!} \quad (n > 1), \tag{23}$$

and

$$r_{n-1} < r_n < 1, \quad r_n^{\lambda_n} \geq \frac{1}{2} \quad (n \geq 1). \tag{24}$$

To get started, set $\lambda_1 = \beta_1$ and choose r_1 so that (24) is valid for $n = 1$ (r_0 is zero and (23) is vacuous for $n = 1$). Now, when λ_n and r_n have been chosen so that (21), (23), and (24) are satisfied for $1 \leq n \leq m$, choose λ_{m+1} large enough so that (21) and (23) are satisfied for $n = m + 1$; then choose r_{m+1} so that (24) is satisfied for $n = m + 1$.

It follows from (20) and (21) that $f(z)$ satisfies (18). The remaining part of the theorem follows if we show that

$$\lim_{n \to \infty} \left(\min_{|z| = r_n} |f(z)| \right) = \infty. \tag{25}$$

But, for $|z| = r_n$,

$$|f(z)| = \left| a_n z^{\lambda_n} + \sum_{k=1}^{n-1} a_k z^{\lambda_k} + \sum_{k=n+1}^{\infty} a_k z^{\lambda_k} \right|$$

$$\geq a_n r_n^{\lambda_n} - \sum_{k=1}^{n-1} a_k - \sum_{k=n+1}^{\infty} a_k r_n^{\lambda_k}.$$

Then, by (19), (23), and (24), for $n > 1$:

$$|f(z)| \geq \frac{1}{2} a_n - \frac{1}{n-1} a_n - \sum_{k-n+1}^{\infty} k! \, r_{k-1}^{\lambda_k}$$

$$\geq \frac{n-3}{2(n-1)} a_n - \sum_{n+1}^{\infty} 2^{-k} = \frac{n-3}{2} (n-1)! - 2^{-n}.$$

Thus (25) follows and Theorem 4 is proved.

Theorem 4 can also be proved in a more geometrical fashion by starting with the Riemann surface onto which f(z) maps $|z| < 1$. This will be the subject of a forthcoming paper. The advantage of the geometric approach is that it makes f(z) seem less outlandish; the disadvantage is that the proof is considerably longer than that given here.

4. A REPULSIVE MEROMORPHIC FUNCTION

THEOREM 5. *Let* $p(r)$ *be a given function in* $[0, 1)$ *satisfying* $0 < p(r) \uparrow \infty$. *Then there exists a function* $F(z)$, *meromorphic in* $|z| < 1$, *with the properties*

(26) $$T(r, F) \leq p(r) \qquad (0 \leq r < 1)$$

and

(27) $\begin{cases} \text{if } \Gamma \text{ is any curve in } |z| < 1 \text{ such that } \sup_{z \in \Gamma} |z| = 1, \text{ then its image} \\ F(\Gamma) \text{ is dense on the sphere (in particular, } F(z) \text{ has no asymptotic values).} \end{cases}$

Proof. Let $f(z)$ and $\phi(z)$ be the functions of Theorems 4 and 3, respectively. We choose the sequence $\{\lambda_n\}$ in Theorem 3 to be $\lambda_n = n + 1$. Set

(28) $$F(z) = \phi(f(z)).$$

Then if Γ satisfies the conditions in (27), it follows from Theorem 4 that $f(\Gamma)$ is an unbounded curve in the plane; hence (27) follows immediately from (15).

To prove (26), we show that if $\mu(r)$ in Theorem 4 is appropriately chosen, then (26) follows. Now, by (4),

(29) $$N(r, F, e^{i\alpha}) = \sum_{n=1}^{\infty} N(r, f, \zeta_n(\alpha)),$$

and the problem comes down to making $N(r, f, \zeta_n(\alpha))$ small. Since $\phi(0) = 0$, $\zeta \neq 0$ (we suppress the subscript n, temporarily); also, by (22) $f(0) = 0$, hence $f(0) - \zeta = -\zeta \neq 0$. Applying (5) to $f(z) - \zeta$, we obtain

$$N(r, f, \zeta) = N(r, f - \zeta, 0) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta}) - \zeta| \, d\theta - \log |\zeta|.$$

Now clearly $N(r, f, \zeta) = 0$ if $M(r) \leq |\zeta|$; whereas for $M(r) > |\zeta|$, $|f(re^{i\theta}) - \zeta| < 2M(r)$ and

$$N(r, f, \zeta) < \log (2M(r)) - \log |\zeta| .$$

By (16), with our specification $\lambda_n = n + 1$, $\log |\zeta_n(\alpha)| \geq 0$, so that

$$N(r, f, \zeta_n(\alpha)) = 0 \qquad \text{if } M(r) \leq |\zeta_n(\alpha)| ,$$

$$N(r, f, \zeta_n(\alpha)) < \log [2M(r)] \qquad \text{if } M(r) > |\zeta_n(\alpha)| .$$

By (16) and (18),

$$\left.\begin{array}{ll} N(r, f, \zeta_n(\alpha)) = 0 & (\mu(r) \leq n) \\ N(r, f, \zeta_n(\alpha)) < \log (2\mu(r)) & (\mu(r) > n) \end{array}\right\} \quad (n \geq 1) .$$

Then, by (29),

$$N(r, F, e^{i\alpha}) \leq \sum_{1 \leq n < \mu(r)} \log (2\mu(r)) ,$$

and hence

$$N(r, F, e^{i\alpha}) = 0 \qquad (\mu(r) < 1) ,$$

$$N(r, F, e^{i\alpha}) \leq \mu(r) \log 2(\mu(r)) < \mu^2(r) \qquad (\mu(r) \geq 1) .$$

or, more simply,

$$N(r, F, e^{i\alpha}) < \mu^2(r) \qquad (0 \leq r < 1) .$$

It follows from (3), since $F(0) = 0$, that

$$T(r, F) < \mu^2(r) \qquad (0 \leq r < 1) .$$

Therefore, if we choose $\mu(r) = \sqrt{p(r)}$, which is permissible, then (26) follows and Theorem 5 is proved.

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