TRANSFORMATION GROUPS WITH ORBITS OF UNIFORM DIMENSION

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1. INTRODUCTION

The purpose of this paper is to prove a local analogue of a result of Borel and Conner. The present theorem states that if G is a compact Lie group acting on an n-dimensional cohomology manifold (n-cm) M over the integers in such a way that all the non-fixed orbits are of the same dimension k, then either G is of rank one and all isotropy subgroups are finite, or the dimension of the fixed set F(G, M) is n - k - 1. In the latter case we also know from Chapter XV of [1] that F(G, M) is an (n - k - 1)-cm, the orbit space M/G is an (n - k)-cm with boundary F(G, M), there exist local cross-sections for the orbits of G near F(G, M), and the non-fixed orbits are k-spheres (see [2]).

The importance of this result lies in the fact that the set of those points x in B, the singular set, for which the hypotheses of the theorem are true for the action of G_x on a slice at x, is a dense open subset of B. The proof of the theorem is quite different from that of Borel's and Conner's result, the difficulty lying in the fact that, in order to obtain a result which is hereditary with respect to taking slices, we cannot make any hypotheses of simple connectivity, whereas the proofs of Borel and Conner make strong use of such a hypothesis.

We use the notation of [1], throughout. B denotes the set of points on singular orbits, that is, orbits of less than the highest dimension. E denotes the set of points on exceptional orbits of highest dimension, so that $M - (B \cup E)$ is the set of points on principal orbits. Unless otherwise specified, H denotes a principal isotropy group, T_0 is a maximal torus of G, and $T \subset T_0$ a maximal torus of H. N(K) and Z(K) denote, respectively, the normalizer and the centralizer of K in G, and K^0 denotes the identity component of K.

We assume that the reader is familiar with the theory of transformation groups, or more specifically with the material in Chapters I, V, VIII, IX, and XIII of [1]. It is also desirable, but not essential, that the reader be familiar with the author's Chapter XV of [1], particularly because it is the main result of that chapter that gives the present work its force. One of the main tools used in the present paper is the remarkable formula of Borel [1, XIII] relating the dimension of the fixed point set of a torus acting on an n-cm to the dimensions of the fixed point sets of the subtori of codimension one.

2. THE MAIN THEOREM

Our main result is the following:

THEOREM. *Let* G *be a compact Lie group acting effectively on an* n-cm M *over* Z *such that every point of* F(G, M) *has a countable system of neighborhoods in* M. *Assume that* $F(G, M) \neq \emptyset$, *and that for* $x \in M - F(G, M)$ *we have* $\dim G(x) = k$,

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where k *is some fixed positive integer. Then either* rank (G) = 1 *and* G_x *is finite for* $x \in M - F(G, M)$, *or* $\dim_Z F(G, M) = n - k - 1$.

Remark. Note that in the second case of the conclusion of the theorem, our transformation group satisfies the hypotheses of Theorem 1.4 of Chapter XV of [1], provided M is separable and of finite covering dimension, which yields some strong restrictions on the transformation group. In this connection also see [2].

We now proceed with the proof of the theorem. Note that it suffices to prove the theorem for G connected, since under the assumptions, $F(G, M) = F(G^0, M)$. Let H denote a principal isotropy group of G. By hypothesis, G_x^0 is conjugate to H^0 for any $x \in M - F(G, M)$. Note that if rank (H) $\leq$ rank (G) - 2, then

$$F(S, M) = F(T_0, M) = F(G, M)$$

for any torus S of codimension one in T_0, where T_0 is a maximal torus of G. This is impossible by [1, XIII, 4.3]. Thus rank (H) $\geq$ rank (G) - 1.

We wish to reduce the proof of the theorem to the case in which rank (H) = 1. To do this, let T be a maximal torus of H, and let $S \subset T$ be a torus of codimension one in T such that $\dim_Z F(S, M) > \dim_Z F(T, M)$. S exists by [1, XIII, 4.3].

We claim that it suffices to consider the action of $\frac{Z(S)}{S}$ on F(S, M). First we must show that this action satisfies the hypotheses of the theorem. Let $x \in M$ be on a principal orbit with $G_x = H$, and let K be a slice at x. Then $G(K) \approx K \times G/H$ in a natural way. $F(S, G(K)) \approx K \times F(S, G/H)$ is an open subset of F(S, M). But, as shown in [2, Section 2], F(S, G/H) consists of a finite collection of orbits of Z(S) on G/H, and each of these orbits is of the form $\frac{Z(S)\,gH}{H}$, where $g \in N(T_0)$ and $g^{-1}Sg \subset T$. Since $K \times F(S, G/H)$ is open in F(S, M), each of these orbits is principal for the action of Z(S) on F(S, M). Also

$$\dim_Z \frac{F(S, M)}{Z(S)} = \dim_Z K = \dim_Z \frac{M}{G} = n - k.$$

Now, given $x \in F(S, M) - F(G, M)$ with $G_x^0 = H^0$, we see by [2] that every orbit of Z(S) on $F(S, G(x)) \approx F(S, G/G_x)$ is of the form

$$\frac{Z(S)\,gG_x}{G_x}, \quad g \in N(T_0), \quad g^{-1}Sg \subset T.$$

The isotropy group of this orbit is $g^{-1}Z(S)g \cap G_x$, which is of the same dimension as $g^{-1}Z(S)g \cap H$, the isotropy group of a principal orbit of Z(S) on F(S, M). Thus we have seen that the action of Z(S), when made effective, satisfies the hypotheses of the theorem, and also F(Z(S), F(S, M)) = F(G, M) by the choice of S. Also, a principal isotropy group of Z(S) is $Z(S) \cap H$, which is effectively of rank at most one, since S acts trivially on F(S, M) and rank S = rank $(Z(S) \cap H) - 1$ by definition. But also, $(Z(S) \cap H)^0 \supset T$ does not act trivially on F(S, M), since

$$\dim_Z F(S, M) > \dim_Z F(T, M).$$

Thus the principal isotropy group of the action of Z(S) on F(S, M) is effectively of rank one.

Moreover, if the theorem is true for the action of Z(S), then

$$\dim_Z F(G, M) = \dim_Z F(Z(S), F(S, M)) = \dim_Z \frac{F(S, M)}{Z(S)} - 1 = n - k - 1,$$

as was to be shown. This completes the reduction to the case in which rank (H) = 1.

Before proceeding with the proof of the theorem, we shall establish the following lemma. See [1, XV, 1.1] for the notation $\dim_L (X, x)$.

LEMMA. *Let* G *be a compact Lie group acting on a locally compact space* M. *Let* K *be a finite group in* G. *Let* $x \in F(K, M)$ *be such that for each* $y \in F(K, M)$ *sufficiently close to* G(x) *we have* $G_y \sim G_x$. *Then*

$$\dim_L ((F(K, M))^*, x^*) = \dim_L (F(K, M), x) - \dim \frac{N(K)}{N(K) \cap G_x}$$

for any coefficient domain L.

Proof. Let S be a slice at x so small that $G_y \sim G_x$ for each $y \in G(S) \cap F(K, M)$. Note that $F(K, S) = F(G_x, S)$, since $G_y = G_x$ for $y \in F(K, S)$. Say $y = g(s)$, $s \in S$ and $y \in F(K, M)$. Then $G_y \sim G_x$ and hence $G_s \sim G_x$. But $G_s \subset G_x$, and hence $G_s = G_x \supset K$. Thus $s \in F(K, S)$. If $y = g'(s')$ also, then $g^{-1}g'(s') = s$, and hence $g^{-1} g' \in G_x = G_{s'}$. Thus $s = s'$, and we obtain a well-defined projection

$$F(K, G(S)) \to F(K, S).$$

Note also that if $y = g(s)$ is as above, then $K \subset G_y = gG_s g^{-1} = gG_x g^{-1} = G_{g(x)}$. It follows that we have in a natural way $F(K, G(S)) \approx F(K, S) \times F(K, G(x))$, and also that $F(K, G(S))^* \approx F(K, S)$.

Since at most a finite number of non-conjugate subgroups of G_x are isomorphic to K (see [1, VII, 3.3]), we can find a finite number $K_1, K_2, \cdots, K_m$ of conjugates of K in G, $K_i \subset G_x$, such that if $K' \sim K$ in G and $K' \subset G_x$, then $K' \sim K_i$ in G_x for some i. Let $g_i \in G$ be such that $g_i^{-1} K g_i = K_i$. Now say that $g(x) \in F(K, G(x))$. Then $K \subset G_{g(x)} = gG_x g^{-1}$; that is, $g^{-1} K g \subset G_x$. Thus for some $h \in G_x$, $h^{-1} g^{-1} K g h = K_i$ for some i. Then $g_i h^{-1} g^{-1} K g h g_i^{-1} = K$, so that $ghg_i^{-1} \in N(K)$; that is, $g \in N(K) g_i G_x$. Thus

$$F(K, G(x)) = \bigcup N(K) g_i(x).$$

This implies that N(K) F(K, S) is open in F(K, G(S)), and also that

$$N(K)\, F(K, S) \approx F(K, S) \times \frac{N(K)}{N(K) \cap G_x}.$$

The lemma follows immediately.

Resuming the proof of the theorem, we first take up the case in which rank (G) = rank (H) = 1 and hence G = SO(3) and H = SO(2) (since G/H must be orientable). Let $Z_2 \approx K \subset H$. Then, since F(K, M) touches principal orbits, we calculate

$$\dim_{Z_2} F(K, M) = \dim_{Z_2} M^* + \dim F(K, G/H) = n - 2.$$

Let $Z_2 \oplus Z_2 \approx J \subset G$. Then, since the subgroups of J are conjugate to K, we obtain from Borel's formula ([1, XIII, 4.3])

$$n - \dim_{Z_2} F(J, M) = 3(n - 2 - \dim_{Z_2} F(J, M)),$$

which yields $\dim_{Z_2} F(J, M) = n - 3$.

If $E \neq \emptyset$, then $\dim_{Z_2} E^* \geq \dim_{Z_2} (F(J, M))^* = n - 3$, by the lemma. But then $\dim_{Z_2} E \geq n - 3 + 2 = n - 1$, contrary to [1, IX, 5.4]. Thus $F(J, M) = F(G, M)$, and $\dim_{Z_2} F(G, M) = n - 3 = n - k - 1$ (from which the same fact for coefficients in Z follows).

We shall now take up the case in which rank (G) = 2, rank (H) = 1. Here $F(G, M) = F(T_0, M)$. Let t be the number of conjugates of T in G which lie in T_0. The possible situations are listed in Table 1, which can easily be verified by consulting the infinitesimal diagram of G. Locally isomorphic groups are not distinguished. (The following argument is similar to one used in [3].)

Table 1

G	T	t
$D_1 \times A_1$	regular	1 2
$A_1 \times A_1$	singular	1
	regular	2
A_2	singular	3
	regular	3 6
B_2	singular	2
	regular	4
G_2	singular	3
	regular	6

Note that if $T' \subset T_0$ is a circle group such that $\dim_Z F(T', M) > \dim_Z F(T_0, M)$, then by the hypotheses of the theorem, T' is conjugate to a subgroup of H^0 and hence conjugate to T. Letting $r_0 = \dim_Z F(T_0, M)$ and $r = \dim_Z F(T, M)$, we obtain from Borel's formula [1, XIII, 4.3]

(1) $$n - r_0 = t(r - r_0).$$

But also (see [1, XV, proof of Lemma 2.7])

(2) $$r = n - k + \dim \frac{N(T)}{T} = n - k + j,$$

where j = 1 if T is regular and j = 3 if T is singular. Combining (1) and (2), we obtain

(3) $$r_0 = n - k - s, \qquad \text{where } s = \frac{k - tj}{t - 1}.$$

(If $t = 1$, then, by (1), $n = r$, which contradicts the assumption that H is effectively of rank one.)

By dimensional parity, $k + s$ must be even, and hence s must be odd. Also, by [1, IX, 2.2, Corollary], $s \geq 1$. Each of the cases in Table 1 splits into two parts, according as dim (H) = 1 or 3. The value of s for each case is given in Table 2.

Table 2

G	j	t	k	s
$D_1 \times A_1$	1	1	3	∞
		2	3	1
$A_1 \times A_1$	3	1	5	∞
			3	∞
	1	2	5	3
			3	1
A_2	3	3	7	-1
			5	-2
	1	3	7	2
			5	1
		6	7	1/5
			5	-1/5
B_2	3	2	9	3
			7	1
	1	4	9	5/3
			7	1
G_2	3	3	13	2
			11	1
	1	6	13	7/5
			11	1

Only the cases in which $s = 1$ or 3 satisfy our requirements on s, and the cases in which $s = 1$ are precisely those for which the theorem is true, since

$$r_0 = n - k - s = n - k - 1$$

for these cases. Thus it remains to eliminate the two cases for which $s = 3$.

We now take up the proof for the two exceptional cases. In the first of these, $G \approx A_1 \times A_1$ locally, and $H^0 = T$ is some regular circle subgroup. Also, $k = 5$,

$r = n - 4$ and $r_0 = n - 8$. If G were simply connected, then T would pass through a central element of G, since in this case center (G) consists of exactly the elements of order two, and hence G would not be effective. Thus G cannot be simply connected, and there must be a subgroup of G isomorphic to SO(3). Let $J \approx Z_2 \oplus Z_2$ be in this subgroup. Then by Borel's formula [1, XII, 4.3]

$$n - \dim_{Z_2} F(J, M) = 3(\dim_{Z_2} F(K, M) - \dim_{Z_2} F(J, M)),$$

where $Z_2 \approx K \subset J$. Since $\dim_{Z_2} F(J, M) \geq \dim F(G, M) = n - 8$, we see that $\dim_{Z_2} F(J, M)$ is either $n - 3$ or $n - 6$. Select $x \in F(J, M)$ such that $G_y \sim G_x$ for any $y \in F(J, M)$ close to G(x). (For example, take x such that the number of components of G_x is minimal.) Then, by the lemma,

$$\dim_{Z_2} (F(J, M))^* = \dim_{Z_2} F(J, M) - \dim \frac{N(J)}{N(J) \cap G_x}$$

near x^*. Note that $G_x^0 \sim T$ and that hence J is conjugate to a subgroup of N(T). Thus J cannot be contained in a factor of G, since T is regular. It follows that N(J) is finite and hence, near x^*,

$$\dim_{Z_2} (F(J, M))^* = \dim_{Z_2} F(J, M) = n - 3 \text{ or } n - 6.$$

But $\dim M^* = n - 5$, and hence $F(J, M)^* \subset E^* \cup B^*$. But $F(J, M) \not\subset F(G, M) = B$, and hence

$$\dim_{Z_2} E \geq \dim_{Z_2} E^* + k \geq n - 6 + 5 = n - 1,$$

contrary to [1, IX, 5.4].

Now consider the second exceptional case. Here $G \approx B_2$ locally, and hence $G \approx SO(5)$ or $G \approx Sp(2)$. $H^0 = T$ is singular, and $r = n - 6$, $r_0 = n - 12$, $k = 9$. We use the following easily verified fact about B_2: there exist two conjugacy classes of singular circle groups in B_2, representatives of which can be viewed in SO(5), respectively, as a circle subgroup of $SO(3) \subset SO(5)$ and as a circle subgroup of $Sp(1) \subset SO(4) \subset SO(5)$. They can also be represented, respectively, in SO(5) by the matrices

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In case T is of the first type, $G \approx SO(5)$, since if $G \approx Sp(2)$, then $H^0 = T$ contains the center of G. Let $Z_2 \approx J \subset T$, that is, let the non-trivial element of J be the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that $\dim_{Z_2} F(J, M) = \dim_{Z_2} M/G + \dim \frac{N(J)}{N(J) \cap H}$, by the lemma. Thus $\dim_{Z_2} F(J, M) = n - k + 3 = n - 6$. Let $Z_2 \oplus Z_2 \approx K \subset SO(3) \subset G$. By Borel's formula,

$$n - \dim_{Z_2} F(K, M) = 3(n - 6 - \dim_{Z_2} F(K, M)),$$

so that $\dim_{Z_2} F(K, M) = n - 9$. The lemma may be applied near some point $x \in F(K, M) - F(G, M)$, and since $\dim N(K) = 1$ and $\dim (N(K) \cap H) = \dim (N(K) \cap T) = 0$, we obtain

$$\dim_{Z_2} (F(K, M))^* = n - 9 - 1 = n - k - 1$$

near x^*. Thus, near x^*, $(F(K, M))^* \subset E^*$, and hence

$$\dim_{Z_2} E \geq \dim_{Z_2} E^* + k \geq n - k - 1 + k = n - 1,$$

contrary to [1, IX, 5.4].

If T is of the second type, then for either case $G \approx SO(5)$ or $G \approx Sp(2)$, $T \subset Q \subset G$ with $Q \approx Sp(1)$. Let $Z_2 \approx L \subset T$, so that L is the center of the group $Q \supset T$ and may be represented in $SO(5)$ by the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By the lemma, we obtain

$$\dim_{Z_2} F(L, M) = \dim_{Z_2} M^* + \dim \frac{N(L)}{N(L) \cap H} = n - k + 5 = n - 4.$$

Let $Z_4 \approx J \subset T$. As with L, we compute

$$\dim_{Z_2} F(J, M) = n - k + 3 = n - 6.$$

Note also that $F(J, M) = F(J/L, F(L, M))$. Let

$$Z_2 \oplus Z_2 \approx K \subset \frac{Q}{L} \approx SO(3),$$

and let K' be the inverse image of K in Q. Then $F(K', M) = F(K, F(L, M))$; thus, by Borel's formula,

$$n - 4 - \dim_{Z_2} F(K', M) = 3(n - 6 - \dim_{Z_2} F(K', M)),$$

and hence $\dim_{Z_2} F(K', M) = n - 7$. Applying the lemma to some point

$x \in F(K', M) - F(G, M)$ with $G_x \supset H$, we see that near x^*

$$\dim_{Z_2} (F(K', M))^* = n - 7 - 3 = n - 10 = n - k - 1,$$

since $\dim N(K') = 3$ and $\dim (N(K') \cap G_x) = \dim (N(K') \cap T) = 0$.

Thus, near x^*, $(F(K', M))^* \subset E^*$, and hence

$$\dim_{Z_2} E \geq \dim_{Z_2} E^* + k \geq n - k - 1 + k = n - 1,$$

contrary to [1, IX, 5.4]. This completes the proof of our theorem.

3. APPLICATIONS

Our first corollary is an application of the theorem to the action of G_x on a slice at x. The additional assumptions on M allow us to apply the results of [1, XV].

COROLLARY 1. *Let* G *be a compact Lie group acting on a separable* n-cm_Z M *of finite covering dimension. Say that* $x \in B$ *is such that* $G_y \sim G_x$ *for all* $y \in B$ *sufficiently close to* x. *As usual, let* $H \subset G_x$ *be a principal isotropy group, and let* $k = \dim G/H$. *Then one of the following situations must occur:*

(1) H^0 *is a normal subgroup of* G_x, G_x/H^0 *is of rank one, and hence* $\dim G(x)$ *is* $k - 1$ *or* $k - 3$.

(2) M/G *is an* $(n - k)$-cm_Z *with boundary* B/G *near* x^*, *there exists a local cross-section at* x *for the action of* G, *and* G_x/H *is a sphere.*

Proof. Let S be a slice at x. Then, by hypothesis, $B \cap S = F(G_x, S) = F(G_x^0, S)$ if S is sufficiently small. Hence the action of G_x on S satisfies the hypotheses of the theorem. The first case of the conclusion of the theorem implies conclusion (1) of Corollary 1. The second case of the conclusion of the theorem implies that the hypotheses of [1, XV, 1.4] are satisfied, which implies conclusion (2) of Corollary 1 with the clause "and G_x/H is a sphere" replaced by "and G_x/H is an integral cohomology sphere." By [2], G_x/H is a sphere unless it is effectively of the form $\frac{SO(3)}{I}$, where I is the icosahedral subgroup of SO(3). But in this latter case, conclusion (1) of Corollary 1 holds, and this completes the proof of the corollary.

Remark. The importance of Corollary 1 is evident from the fact, easily seen, that the set of points $x \in B$ for which the hypotheses of the corollary hold, is a dense open subset of B.

It follows from our results that the part of Conner's paper [4] devoted to the localization of the main result of that paper is valid for cohomology manifolds over Z without the homotopy assumption, which is very hard to handle (see the Note Added in Proof in [4]). In particular, we restate Theorem 3 of [4] as

COROLLARY 2. *Let* G *be a compact Lie group acting on a separable* n-cm_Z M, *and assume that* B^* *consists of isolated points. Then one of the following conclusions holds:*

(1) $k = n - 1$;

(2) *at each point* $x \in B$, *there exists a slice* S *on which* G_x *acts effectively as a group of rank one.*

Proof. As in the proof of Corollary 1, the action of G_x on a slice S at a point $x \in B$ satisfies the hypotheses of the theorem. Thus either (2) holds, or $0 = \dim_Z B^* = \dim_Z F(G_x^0, S) = n - k - 1$, from which we conclude that (1) holds.

We can also globalize our theorem and obtain

COROLLARY 3. *Say that the compact Lie group* G *acts effectively on an* $n\text{-cm}_Z$ M, *with orbits all of the same dimension* k. *Assume further that* $H^i(M, Z) \approx H^i(S^n, Z)$. *Then either* rank (G) = 1 *and all isotropy groups are finite, or* G *is transitive on* M *and* $M \approx S^n$.

Proof. We apply the theorem to the action of G on the cone cM over M. Thus if the first part of the conclusion of the corollary is not satisfied, then the theorem implies that $0 = \dim_Z F(G, cM) = (n + 1) - k - 1 = n - k$. Thus $k = n$, and hence G is transitive. Thus $M \approx G/G_x$ is a cohomology sphere, and by [2] it is a sphere unless $G = SO(3)$ and G_x is the icosahedral subgroup.

Remark. Note that our assumptions in Corollary 3 are weaker than the assumptions in Borel's and Conner's theorems [1, XIV, 1.1 and 1.3] in that we do not assume that $\pi_1(M) = 0$, but stronger in that we must assume that M is an n-cm and that the coefficient ring is the ring of integers.

Remark. It is also possible to use our theorem for extending Yang's theorem on singular orbits [5] to the nondifferentiable case. Since the proof parallels that of Yang closely, we shall not give it here.

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