ON THE STRUCTURE OF SEMIGROUPS WITH IDENTITY ON A NONCOMPACT MANIFOLD

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This paper deals with a problem concerning the structure of a semigroup with identity whose space is a noncompact n-dimensional manifold ($n \geq 2$), and which has an $(n - 1)$-dimensional compact connected group containing the identity. This problem is analogous to the one considered by Mostert and Shields in [7].

Definition 1. A *thread* is a semigroup whose space is homeomorphic to an open interval.

Definition 2. An M-*thread* is a semigroup with identity whose space is homeomorphic to a half-open interval and such that the endpoint acts as a zero (see [6]).

The term "isomorphism" will be used to denote a function which is simultaneously an algebraic isomorphism and a homeomorphism. For further definitions and background material, the reader is referred to [7] and [10].

THEOREM. *Let* M *be an* n-*dimensional noncompact manifold* ($n \geq 2$) *which is a semigroup with identity,* 1, *and assume that* M *is not a group. Suppose there exists a compact connected* $(n - 1)$-*dimensional group* G *containing* 1 *and contained in* M. *Then there exists either a thread with identity, or else an* M-*thread,* T, *such that* M = TG *and* tg = gt *whenever* $t \in T$ *and* $g \in G$.

The proof of the theorem is divided into two main cases. If G is allowed to act on M by right multiplication, then the orbit space with respect to this action is either an open interval or a half-open interval. The cases in the proof correspond to the two kinds of orbit space. In the following twenty-two lemmas that comprise the proof, the hypotheses of the theorem are assumed to hold throughout.

LEMMA 1. *Define* θ: $M \times G \to M$ *by* $\theta(x, g) = xg$ *for* $x \in M$, $g \in G$. *Then* G *is a Lie group acting effectively on* M. *If* S *denotes the orbit space with respect to* G, *then* S *is homeomorphic to a half-open interval or an open interval.*

Proof. By assumption, M is an n-dimensional manifold and a semigroup with identity. By Mostert and Shields [8], H(1), the maximal subgroup of M containing 1, is an open subset of M and is a Lie group. G, being a closed subgroup of H(1), is therefore also a Lie group.

From the definition of θ given above it follows quite easily, since multiplication in M is continuous and associative, that G acts as a transformation group on M. To see that G acts effectively, it is sufficient to note that G acts by right multiplication, and that the identity for G is also an element of M.

This proves that G is a compact connected Lie group acting effectively on an n-dimensional manifold. To prove that the orbit space is as stated, it suffices to establish the existence of at least one $(n - 1)$-dimensional orbit. This follows

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however from the fact that G itself is an orbit, for $G = \theta(\{1\} \times G)$. Hence, by Mostert [5], the space of orbits is one of the following: (i) a circle; (ii) a closed interval; (iii) an open interval; or (iv) a half-open interval. In cases (i) and (ii), however, it follows that M is compact, and since this is contrary to the assumption that M is noncompact, the space of orbits must be as stated in the lemma.

The following notation will be used: S will denote the space of orbits under θ, and π the natural map from M to S. That is, for $x \in M$, $\pi(x) = \theta(\{x\} \times G) = xG$. It is known [4] that π is open and continuous and that S is a locally compact Hausdorff space.

For a point x in M, the isotropy group of x, denoted by G_x, is defined by $G_x = \{g \in G \mid \theta(x, g) = x\}$. Mostert shows in [5] that, in the case where S is homeomorphic to an open interval, G_x is conjugate to G_y for all x and y in M. Since G_1 is obviously the single element 1, it follows that in this case $G_x = \{1\}$ for all $x \in M$. For the other case, where S is homeomorphic to a half-open interval, Mostert shows in the same paper that there is a unique singular orbit, and that for p in this orbit, G_p is nontrivial. Also, for x and y not in the singular orbit, G_x is conjugate to G_y, so that $G_x = \{1\}$ if x is not in the singular orbit.

LEMMA 2. *For* x *and* y *in* M, *define a relation* $\prec$ *by* $x \prec y$ *if and only if* $x \notin yG$ *and* $\pi(x) < \pi(y)$, *where* $<$ *is the natural order of* S. *Then* $x \prec y$ *for* x *and* y *in* M *implies that* $xg \prec yg$ *and* $gx \prec gy$ *for each* g *in* G.

Proof. Since $x \notin yG$, $xg \notin (yg)G$ for each g in G. Hence, if $\pi(x) < \pi(y)$, it follows that $\pi(xg) = \pi(x) < \pi(y) = \pi(yg)$, so that $xg \prec yg$ for each g in G.

To prove that $x \prec y$ implies $gx \prec gy$ for each $g \in G$, define $\delta(g) = \pi(gx)$ and $\alpha(g) = \pi(gy)$ for $g \in G$. Then δ and α are continuous functions from G to S, which is either a half-open interval or an open interval. By assumption, $\pi(x) < \pi(y)$, hence $\delta(1) = \pi(x) < \pi(y) = \alpha(1)$. Also, $\delta(g) \neq \alpha(g)$ for each g, as is easily seen from the fact that $x \notin yG$. But G is a connected space, and hence $\delta(g) < \alpha(g)$ for each g in G, since this relation holds for one element of G. This completes the proof of Lemma 2.

LEMMA 3. *For* $x \in M$, $xG = Gx$.

Proof. Let x be an element of M. In this proof it will be shown that $x \prec gx$ and $gx \prec x$ cannot hold for any g in G. This will prove that $gx \in xG$ for each $g \in G$, in other words, that $Gx \subset xG$. A dual argument proves that $xG \subset Gx$.

To show that $x \prec gx$ and $gx \prec x$ cannot hold, let us assume, to the contrary, that there exists an element g_0 in G such that $x \prec g_0 x$. By Lemma 2, it follows that $g_0 x \prec g_0^2 x \prec \cdots \prec g_0^n x$, for all positive integers n. If $\Gamma(g_0)$ denotes the smallest closed subsemigroup of G containing g_0, then for each $a \in \Gamma(g_0)$ it is true that $x \prec ax$. Since G is compact, however, it follows from [2] that $1 \in \Gamma(g_0)$, contrary to the fact that $1x = x$ and $x \not\prec x$. This contradiction shows that $x \prec gx$ cannot hold for any $g \in G$. The proof that $gx \prec x$ cannot hold is similar. Thus it is true that $gx \in xG$ for each g in G, and by the remarks at the beginning, the lemma is established.

LEMMA 4. *Define* μ: $S \times S \to S$ *by* $\mu(\pi(x), \pi(y)) = \pi(xy)$ *for* x *and* y *in* M. *Then* S *is a semigroup under* μ.

Proof. The fact that μ is continuous and associative follows immediately from Lemma 3. Since S is a Hausdorff space, S is a semigroup.

LEMMA 5. $\pi(H(1)) = H(\pi(1))$. $H(\pi(1))$ *is isomorphic to* $(-\infty, 0) \cup (0, \infty)$ *or to* $(0, \infty)$ *under real multiplication.*

Proof. It is clear from the definition of μ that π is a homomorphism from M onto S. Since this is true and since H(1) is a group, $\pi(H(1)) \subset H(\pi(1))$, for the latter is the maximal subgroup of S containing $\pi(1)$.

Conversely, let $\bar{x} \in H(\pi(1))$. Then $\bar{x} = \pi(x)$ for some $x \in M$, and there exists $\bar{y} \in S$ such that $\bar{x}\bar{y} = \pi(1) = \bar{y}\bar{x}$. Let $y \in M$ be such that $\pi(y) = \bar{y}$. Then

$$\pi(xy) = \pi(x)\,\pi(y) = \bar{x}\bar{y} = \pi(1)$$

and, similarly, $\pi(yx) = \pi(1)$. From this it follows that $xy \in G$ and $yx \in G$, so that, for some g and g_0 in G, $xy = g$ and $yx = g_0$. Thus $x(yg^{-1}) = (xy)g^{-1} = 1$, and also $(g_0^{-1}y)x = 1$, so that $x \in H(1)$. This shows that $\bar{x} = \pi(x) \in \pi(H(1))$, and since $\bar{x}$ was arbitrary in $H(\pi(1))$, that $H(\pi(1)) \subset \pi(H(1))$. This completes the proof of the claim that $H(\pi(1)) = \pi(H(1))$.

By Mostert and Shields [8], H(1) is open in M, hence $H(\pi(1))$ is open in S, since the function π is open. By Storey [9], $H(\pi(1)) = S$, $H(\pi(1)) = \pi(1)$, $H(\pi(1))$ consists of exactly two elements, $H(\pi(1))$ is isomorphic to $(0, \infty)$ under real multiplication, or $H(\pi(1))$ is isomorphic to $(-\infty, 0) \cup (0, \infty)$ under real multiplication. By assumption, M is not a group, hence $H(\pi(1))$ is not all of S. By the preceeding, $H(\pi(1))$ is open in S, so that $H(\pi(1))$ is one of the last two cases stated, and the proof of the lemma is complete.

LEMMA 6. K *is the minimal ideal of* M *if and only if* $\pi(K)$ *is the minimal ideal of* S.

Proof. It is easily seen that the image of an ideal of M is an ideal of S, and conversely, that if I is an ideal of S, then $\pi^{-1}(I)$ is an ideal in M.

First assume that K is the minimal ideal of M. Then by the above remark, $\pi(K)$ is an ideal of S. To see that $\pi(K)$ is the minimal ideal of S, suppose I is an ideal and $I \subset \pi(K)$. The claim is now made that $\pi^{-1}(I) = K$. To see this, suppose $x \in \pi^{-1}(I)$. Then $\pi(x) \in I \subset \pi(K)$, so that there exists k in K with $\pi(x) = \pi(k)$. Hence $x \in kG \subset KM \subset K$, so that $x \in K$, and therefore $\pi^{-1}(I) \subset K$. But K is the minimal ideal of M, hence $\pi^{-1}(I) = K$, and the claim is true. From this it follows that $\pi(\pi^{-1}(I) = \pi(K)$, that is, $I = \pi(K)$, which shows that $\pi(K)$ is the minimal ideal of S.

For the converse, let $\bar{K}$ be the minimal ideal of S, and let $K = \pi^{-1}(\bar{K})$. Then K is an ideal of M, by the above remarks. Now suppose that I is an ideal of M contained in K. Then $\pi(I) \subset \pi(K) = \bar{K}$, so that $\pi(I) = \bar{K}$, since $\bar{K}$ is minimal. By an argument similar to the one given above, it can be shown that $\pi^{-1}(\bar{K}) \subset I$, so that $K = I$ is the minimal ideal of M.

The remainder of the proof is divided into the two cases that occur in the orbit space S.

CASE 1. S is homeomorphic to $[0, \infty)$

LEMMA 7. *Let* $K = \pi^{-1}(0)$. *Then* K *is the minimal ideal of* M, *and it is a group. If* e *is the idempotent in* K, *then the function* $\alpha : G \to eG$ *defined by* $\alpha(g) = eg$ *is a homeomorphism.*

Proof. The proof of this lemma is given by Mostert and Shields [7], for an analgous situation.

LEMMA 8. S *is an* M-*thread.*

Proof. By Lemma 7, $\pi^{-1}(0)$ is the minimal ideal of M, hence by Lemma 6, $\pi(\pi^{-1}(0)) = 0$ is the minimal ideal of S. Thus 0 is a zero for S, and since $\pi(1)$ is the identity for S, S is an M-thread.

LEMMA 9. *There exists in* M *an* M-*thread* T *such that* $M = TG$ *and* T *is isomorphic to* S. *Furthermore, for* x *in* T *and* g *in* G, $xg = gx$.

Proof. The construction of T is given by Mostert and Shields in [6].

Let us note that with Lemma 9, the proof of the theorem in Case 1 is complete.

CASE 2. S is homeomorphic to $(-\infty, \infty)$.

Here two subcases will be considered, first that where $H(\pi(1))$ is isomorphic to $(-\infty, 0) \cup (0, \infty)$, then that where $H(\pi(1))$ is isomorphic to $(0, \infty)$.

Case 2.1. $H(\pi(1))$ is isomorphic to $(-\infty, 0) \cup (0, \infty)$ under real multiplication.

By Storey [9], there exists an element $\theta < \pi(1)$ such that, if $T_0 = \{s \in S \mid s \geq \theta\}$, then T_0 is an M-thread, θ is a zero for all of S, and the structure of S is as follows:

There exists an idempotent $e \in [\theta, \pi(1))$ such that $S \setminus [\theta, e] = H(\pi(1))$ and $[\theta, e]$ is an I-semigroup.

LEMMA 10. *Let* $\bar{f}$ *be an idempotent in* S. *Then* $\pi^{-1}(\bar{f}) = fG$ *for some idempotent* f *in* M. *Furthermore,* fG *is isomorphic to* G *under the mapping* $g \to fg$.

Proof. Let x and y in M be such that $\pi(x) = \bar{f} = \pi(y)$. Then

$$\pi(xy) = \pi(x)\,\pi(y) = \bar{f}\bar{f} = \bar{f},$$

so that xy belongs to $\pi^{-1}(\bar{f})$. This shows that $\pi^{-1}(\bar{f})$ is a compact subsemigroup of M and therefore (see [2]) contains an idempotent, say f. Since $xG = fG = Gf = Gx$ for all x in $\pi^{-1}(\bar{f})$, it follows that f is a two-sided identity for $\pi^{-1}(\bar{f}) = fG$. This, however, holds for every idempotent in $\pi^{-1}(\bar{f})$; hence f is the unique idempotent. By Wallace [10], it follows that $\pi^{-1}(\bar{f})$ is a group. Clearly, the mapping $g \to fg$ is continuous and is a homomorphism. Since $G_f = \{1\}$, this mapping is also one-to-one, and therefore it is an isomorphism, since both G and fG are compact.

LEMMA 11. $K = \pi^{-1}(\theta)$ *is the minimal ideal of* M, *and* K *is a group and is isomorphic to* G *under the mapping* $g \to eg$, *where* e *is the identity for the group* K.

Proof. This lemma follows immediately from Lemmas 6 and 10, since θ is the zero for S and hence the minimal ideal and an idempotent.

LEMMA 12. *For an idempotent* f *in* M, *define* $T(f) = \{x \in M \mid fx = x\}$. *Then* T(f) *is a closed subsemigroup of* M. *If* $\pi(f)$ *is a left (or right) zero for* S, *then* T(f) *meets each orbit of* G *in exactly one point. Furthermore, if* $fg = gf$ *for each* g *in* G, *then* $xg = gx$ *for* x *in* T(f) *and* g *in* G.

Proof. Clearly T(f) is a closed subsemigroup of M. Assuming that $\pi(f)$ is a left zero for S, let x belong to M. Then $\pi(f) = \pi(f)\,\pi(x) = \pi(fx)$, which implies that $fG = (fx)G$, that is, $f = (fx)g$ for some $g \in G$. From this equality it follows that $xg \in T(f)$, so that T(f) meets each orbit of G. To see that T(f) can meet an orbit of G in at most one point, let us assume that x and y belong to T(f) and that $xG = yG$. Then $x = yg$ for some g in G and, therefore, $f = fx = f(yg) = (fy)g = fg$, so that $g = 1$, since $G_f = \{1\}$. Hence $x = y1 = y$.

To complete the proof of this lemma, assume that $fg = gf$ for all g in G, and let $x \in T(f)$. Then for g in G, $xg = g_0 x$ for some $g_0 \in G$, since $xG = Gx$. But then

$$fg = (fx)g = f(xg) = f(g_0 x) = (fg_0)x = (g_0 f)x = g_0(fx) = g_0 f = fg_0,$$

so that $g = g_0$ and $xg = gx$.

LEMMA 13. *If* T *is a closed subsemigroup of* M *meeting each orbit of* G *in exactly one point, then* π *restricted to* T *is an isomorphism from* T *onto* S.

Proof. Let π_1 denote the restriction of π to T. Since π is continuous and is a homomorphism, and since T is a subsemigroup of M, π_1 is also continuous and is a homomorphism. π_1 is clearly one-to-one and onto, since T meets each orbit of G in exactly one point.

It remains only to show that π_1 is open. For this, let p belong to T, and suppose that U is open in T and contains p. Then there exists an open set O in M such that $U = T \cap O$. Also, since $G = \{1\}$ and G is a compact Lie group, there exists a compact set F containing p and a neighborhood V of p such that V is homeomorphic to FG [1]. Hence $\pi(F)$ is a compact neighborhood of $\pi(p)$, so that there exists an open interval $I_1 \subset S$ and $I_1^* \subset \pi(F)$. Now $A = \pi^{-1}(I_1^*) \cap T$ is a closed subset of $FG \cap T$, and since F and G are compact and T is closed, A is a compact subset of T. By the statements above, π_1 restricted to A is one-to-one and continuous, hence a homeomorphism. The claim is now made that $\pi_1(A) = I_1^*$. This follows immediately from the definition of A and the fact that T meets each orbit of G. For the remainder, let $U_1 = O \cap A$. U_1 is open in A, and thus $\pi_1(U_1)$ is open in I_1^*. Hence $\pi(p)$ belongs to the interior of $\pi_1(U_1)$, and therefore, by the definition of U_1, to the interior of $\pi_1(U)$. This completes the proof of the claim that π_1 is open, which proves that π_1 is, in fact, an isomorphism.

LEMMA 14. *There exists in* M *a thread* T *such that* $M = TG$ *and* T *is isomorphic to* S *under the restriction of* π *to* T. *Furthermore, for* t *in* T *and* g *in* G, $tg = gt$.

Proof. Let e be the identity of $K = \pi^{-1}(\theta)$. Then, by Lemma 12, $T(e)$ is a closed subsemigroup of M meeting each orbit of G in exactly one point. By Lemma 13, T is isomorphic to S under the restriction of π to T. By Koch [2], since K is a group and the minimal ideal of M, $ex = xe$ for each x in M and, in particular, for each x in G. Thus, by Lemma 12, $tg = gt$ for $t \in T$ and $g \in G$. T is clearly a thread, since it is isomorphic to the thread S.

With Lemma 14, the proof for Case 2.1 is complete.

Case 2.2. In this case the assumption is made that $H(\pi(1))$ is isomorphic to the real interval $(0, \infty)$ under multiplication. Here, however, two further subcases arise: 1) the minimal ideal $\bar{K}$ of S is not empty, and 2) $\bar{K}$ is empty.

Case 2.2.1. The general assumptions for this case are as follows: S is homeomorphic to an open interval, $H(\pi(1))$ is isomorphic to the real interval $(0, \infty)$ under multiplication, and $\bar{K}$, the minimal ideal of S, is not empty.

In [9], Storey proves that, for some idempotent $\bar{e}$ in S, $(H(\pi(1))^* = [\bar{e}, \infty)$ is an M-thread, also that $\bar{K}$ consists entirely of left (or right) zeros for S, and that there exists an element $\bar{k}_1 \in \bar{K}$ such that $\bar{K} \subset (-\infty, \bar{k}_1]$, $\bar{k}_1 \leq \bar{e} < \pi(1)$, and $[\bar{k}_1, \infty)$ is an M-thread. Let us assume in this argument that $\bar{K}$ consists of left zeros.

LEMMA 15. *There exists a thread* T *in* M *satisfying the conditions that* T *is closed,* T *meets each orbit of* G *in exactly one point,* T *is isomorphic to* S, *and* $tg = gt$ *for* $t \in T$ *and* $g \in G$.

Proof. Let $T = T(k_1)$, where k_1 is the idempotent in $\pi^{-1}(\bar{k}_1)$. The lemma follows immediately from Lemmas 12 and 13, if it can be shown that $k_1 g = gk_1$ for all g in G. To see this, let $M_0 = \pi^{-1}([\bar{k}_1, \infty))$. Then M_0 is a subsemigroup of M with minimal ideal $\pi^{-1}(\bar{k}_1)$, as can easily be proved. Thus, by Koch, $k_1 x = xk_1$ for each x in M_0, since the minimal ideal is a group. Clearly, G is contained in M_0, so that $k_1 g = gk_1$ for each g in G, and the proof of Lemma 15 and Case 2.2.1 is complete.

Case 2.2.2. $\bar{K}$, the minimal ideal of S, is empty.

LEMMA 16. *An element θ may be adjoined to S as a minimal element so that $S_0 = S \cup \theta$ is an M-thread. Also, an element ϕ may be adjoined to M so that $M_0 = M \cup \phi$ is a semigroup with zero, ϕ, and M is a subsemigroup of M_0. Finally, a function π_0 can be defined from M_0 onto S_0 in such a way that $\pi_0 | M = \pi$, and π_0 is an open continuous homomorphism.*

Proof. Since the minimal ideal of S is empty and $\bar{e}$ is, by assumption, less than $\pi(1)$, by Storey [9], a zero θ may be adjoined to S so that $S_0 = S \cup \theta$ is an M-thread.

Now let ϕ be an element with $\phi \notin M$, and let $M_0 = M \cup \phi$. In order to make M_0 into a topological space, define the neighborhood system $\mathfrak{N}$ at ϕ by the condition that $P \in \mathfrak{N}$ if and only if $P = \pi^{-1}(V) \cup \phi$, where $V = (\theta, a)$ for some $a \in S$. The topology on M_0 is to be that generated by the open sets of M and the collection $\mathfrak{N}$. Clearly, M_0 is a locally compact Hausdorff space.

Define $\pi_0 \colon M_0 \to S_0$ by

$$\pi_0(x) = \begin{cases} \pi(x) & \text{if } x \in M, \\ \theta & \text{if } x = \phi. \end{cases}$$

The claim is now made that π_0 is an open continuous homomorphism of M_0 onto S_0, where we define $\phi x = \phi = x\phi$ for all x in M_0.

Clearly, π_0 is continuous and open at each point x in M, so that it suffices to show that π_0 is open and continuous at ϕ. This, however, follows immediately from the definition of π_0, $\mathfrak{N}$, and the neighborhoods of θ. It also is obvious that π_0 is a homomorphism onto, and that π_0 restricted to M is π.

In order to prove that M_0 is a semigroup under the definition of multiplication given above, it must be shown that multiplication is continuous and associative. It is trivial to show associativity. For the continuity of multiplication, let us consider the diagram

$$\begin{array}{ccc} S_0 \times S_0 & \xrightarrow{\mu_0} & S_0 \\ {\scriptstyle \pi_0 \times \pi_0}\uparrow & & \uparrow{\scriptstyle \pi_0} \\ M_0 \times M_0 & \xrightarrow{m_0} & M_0 \end{array}$$

where m_0 is the multiplication function in M_0. Since the diagram is analytic, π_0 is open, and μ_0 is continuous, it follows that m_0 is continuous. Clearly M is a subsemigroup of M_0, so that the proof of Lemma 16 is complete.

Lemma 17. *If there exists a sequence $\{\bar{e}_n\}$ of idempotents in S such that $\cdots < \bar{e}_{n+1} < \bar{e}_n < \cdots < \bar{e}_1 < \pi(1)$ and $\lim_{n\to\infty} \bar{e}_n = \theta$, then $T = \cup \{T(e_n) \mid n = 1, 2, \cdots\}$, where e_n is the idempotent in $\pi^{-1}(\bar{e}_n)$, is a closed subsemigroup of M meeting each orbit of G in exactly one point. Also, $xg = gx$ for $x \in T$ and $g \in G$, and π restricted to T is an isomorphism of T onto S.*

Proof. For each n, let $M_n = \pi^{-1}([\bar{e}_n, \infty))$. Then, clearly, M_n is a closed subsemigroup of M, and $T(e_n) \cap M_n$ is a closed subsemigroup of M_n. Since, furthermore, $\pi^{-1}(\bar{e}_n) = e_n G$ is the minimal ideal of M_n, it follows as before that $e_n g = ge_n$ for each g in G, since $G \subset M_n$ for each n. Since $\bar{e}_n$ acts as an identity for $[\theta, \bar{e}_n)$, it follows quite easily that $T(e_n) \subset M_n$ for each n.

Define T as in the statement of the lemma. In order to prove that T is a closed subsemigroup of M, meeting each orbit of G in exactly one point, let us first prove two statements concerning the idempotents of M. The first statement is that if e is an idempotent of M, and $x \in T(e)$, then $xe = e$. Since $\pi(e) = \pi(e)\pi(x)$, and S is an M-thread, it follows that $\pi(e) \leq \pi(x)$, so that it is also true that $\pi(e) = \pi(x)\pi(e)$. Now, since $x \in T(e)$, $ex = e$, so that $exe = ee = e$ and $xexe = xe$. But now, $xe \in eG$, a group, and xe is an idempotent. Since e is the unique idempotent in eG, the claim that $e = xe$ is true.

The second claim concerning idempotents of M is this: if e and f are idempotents and $e \prec f$, then T(f) is contained in T(e). To see this, let $x \in T(f)$. By assumption, $e \prec f$, so that $\pi(e) < \pi(f)$, and therefore $\pi(e)\pi(f) = \pi(e)$, hence $e = ef$. From this it follows that $ex = (ef)x = e(fx) = ef = e$, so that x does belong to T(e).

With these two claims established, it is easily seen that T is a subsemigroup of M, since it is the union of a tower of subsemigroups. T clearly meets each orbit of G in M; for if $x \in M$, then by hypothesis there exists an idempotent e_n with $e_n \prec x$. Hence $T(e_n) \cap xG \neq \Box$, and therefore $T \cap xG \neq \Box$. Also, T cannot meet xG in more than one point, since the same is true for each set in the tower of which T is the union.

To see that T is closed in M, suppose $\{x_n\}$ is a sequence of elements of T such that $x_n \to x$, $x \in M$. Since π is continuous and since $x_n \to x$, it follows that $\pi(x_n)$ converges to $\pi(x) \neq \theta$. Now the sequence $\pi(e_n)$ converges monotonically to θ, and $\pi(x) > \theta$, so that there exists an integer N such that $\pi(e_N) < \pi(x)$. But $\pi(x_n)$ converges to $\pi(x)$, hence there exists an integer N_0 such that $\pi(e_N) < \pi(x_n)$ for all $n > N_0$. Thus $x_n \in T(e_N)$ for all $n > N_0$. Since $T(e_N)$ is closed and $x_n \to x$, we have $x \in T(e_N)$, hence $x \in T$, which proves that T is closed. Clearly, $tg = gt$ for $t \in T$ and $g \in G$. The application of Lemma 13 to T now establishes Lemma 17.

LEMMA 18. *Suppose that there exists an idempotent* $\bar{f}$ *in* S *such that, if* $\bar{g}^2 = \bar{g}$ *for* $\bar{g} \in S$, *then* $\bar{f} \leq \bar{g}$, *in other words, such that* $\bar{f}$ *is a minimal idempotent for* S. *Then the interval* $[\theta, \bar{f}]$ *is a subsemigroup of* S_0 *isomorphic to the unit interval under real multiplication, and* $[\bar{f}, \infty)$ *is an* M*-thread.*

Proof. Since θ is a zero for $I_0 = [\theta, \bar{f}]$, and $\bar{f}$ is a unit for I_0, and since these two elements are exactly the idempotents of I_0, it follows from [7] that I_0 is either isomorphic to the unit interval under real multiplication, or else isomorphic to the real interval [1/2, 1] under the multiplication $x \circ y = \max(1/2, xy)$, where xy denotes the real product of x and y. In the latter case, however, each element is nilpotent, which is clearly not the case here, for if $\bar{x} \in (\theta, \bar{f}]$, then $\bar{x}^n \in S$ for each n, and therefore $\bar{x}^n \neq \theta$ for any n. From this the lemma follows.

LEMMA 19. *Let* $\bar{f}$ *be as in Lemma* 18, *and let* $D = \pi_0^{-1}([\theta, \bar{f}])$. *Then* D *is a compact connected subsemigroup of* M_0. *If* f *denotes the idempotent of* $\pi^{-1}(\bar{f})$, *then* fG *is a Lie group with identity* f, *and if* x *and* y *are in* D, *with* $xy = f = yx$, *then* x *and* y *both belong to* fG.

Proof. Clearly, D is compact and connected, and it is a subsemigroup of M_0. By Lemma 10, fG is isomorphic to G, hence fG is a Lie group with identity f. Now suppose that x and y in D are such that $xy = f = yx$. Then

$$\pi(xy) = \pi(x)\pi(y) = \pi(f) = \bar{f} = \pi(y)\pi(x).$$

But $[\theta, \bar{f}]$ is isomorphic to the unit interval under real multiplication, hence $\pi(x) = \bar{f} = \pi(y)$, so that x and y both belong to fG.

LEMMA 20. *Let* $Z = \{x \in D | xg = gx$ *for* $g \in fG\}$, *where* f *is the idempotent defined in Lemma* 19. *Then* Z *is a compact subsemigroup of* D *meeting each orbit of* fG *in* D.

Proof. This lemma is due to Mostert and Shields, and the proof is to be found in [7].

LEMMA 21. *There exists in* D *an* I-*semigroup* J, *with endpoints* f *and* ϕ, *such that* π *restricted to* J *is an isomorphism onto* $[\theta, \bar{f}]$.

Proof. The first step in the construction of J is the construction of a one-parameter semigroup in Z. The proof of the existence of this one-parameter semigroup proceeds as follows. By Lemma 20, Z is a compact semigroup, and it is an adequate local semigroup whose maximal subgroup $C = Z \cap fG$ is the center of fG (see [7]). Clearly, C is not open in Z, and there exists a neighborhood of f containing only the idempotent f. Hence, by Mostert and Shields [7], there exists a one-parameter semigroup $\delta: [0, 1] \to Z$ such that $\delta(0) = f$ and $\delta(t) \notin C$ for $t > 0$. (Here $[0, 1]$ denotes the real unit interval.)

Now, for a positive real number $r = n + a$, where n is an integer and $0 < a \leq 1$, define $\alpha(r) = \delta(1)^n \delta(a)$. In [3], the author proves that such a function α is a homomorphism from the nonnegative reals under addition to M. Also, as in [3], there exists a real number $t_0 > 0$ such that α is either one-to-one on $[0, t_0)$ or else one-to-one on $[0, \infty)$. In either case, $\alpha([0, t_0)) \cup \phi = (\alpha([0, t_0)))^*$, where t_0 is real or $t_0 = \infty$, so that α is one-to-one on $[0, t_0)$. Thus $J = (\alpha([0, t_0)))^*$ is an I-semigroup.

Clearly, J is connected, $f \in J$, and $\phi \in J$; thus $\pi_0(J)$ is connected and contains θ and $\bar{f}$, and therefore it must equal $[\theta, \bar{f}]$. To see that π_0 restricted to J is an isomorphism, it is necessary only to show that π_0 is one-to-one on J, or that J meets each orbit of fG in at most one point. For the proof of this fact, suppose, for real numbers s and t with $s > 0$ and $t \geq 0$, that $\alpha(s) = \alpha(t)(fg)$ for some $g \in G$. Since f is an identity for J, this implies that $\alpha(s) = \alpha(t)g$. Then, if $t = s + r$ for some real number $r \geq 0$, it follows that

$$\alpha(s) = \alpha(t)g = \alpha(s + r)g = \alpha(s)\alpha(r)g.$$

This implies that $\pi(\alpha(s)) = \pi(\alpha(s))\pi(\alpha(r))$, and by the multiplication in $[\theta, \bar{f}]$, that $\pi(\alpha(r)) = \bar{f}$. Hence $\alpha(r) \in fG$, and therefore $r = 0$ and $s = t$. This completes the proof of the lemma.

LEMMA 22. *There exists a thread* T *in* M_0 *such that* $M_0 = TG$, T *is isomorphic to* S_0, *and* $tg = gt$ *for* $t \in T$ *and* $g \in G$.

Proof. Let $T = T(f) \cup J$, where J is as in Lemma 21. Since $\bar{f}$ is a zero for $[\bar{f}, \infty)$, and $\pi(f) = \bar{f}$, we see that $T(f)$ meets each orbit of G in exactly one point, for each x such that $\pi(f) \leq \pi(x)$; also, π restricted to $T(f)$ is an isomorphism from $T(f)$ onto $[\bar{f}, \infty)$. By Lemma 21, π restricted to J is an isomorphism onto $[\theta, \bar{f}]$, and since $T(f) \cap J = \{f\}$, π restricted to T is an isomorphism onto S_0. Thus T has the desired properties.

With Lemma 22, the proof of the theorem is complete.

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