ON THE COMPLETENESS OF TOPOLOGICAL VECTOR LATTICES

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Nakano [6, Theorem 4.2] has given a theorem which asserts that a topological vector lattice, under suitable relations between order and topology, is topologically complete, that is, complete for the unique translation invariant uniformity associated with its topology. (For basic definitions, see [2] or [3]. Definitions specifically used in this paper are explained in Section 1.) For the metrizable case, weaker sufficient conditions have been given, for instance, in [1] and [10]; but the most interesting feature of Nakano's theorem (see below, Theorem 2) is that it does not require the hypothesis of metrizability. The surprising fact is that the theorem derives, roughly speaking, the convergence of a class of (in general unbounded) filters (namely, the Cauchy filters) from the order-convergence of a certain family of bounded filters (namely, the section filters of directed, topologically bounded sets). However, Nakano's proof of his theorem contains a gap; it is one of the aims of this paper to fill that gap (Theorem 1). Also, it is of interest to find general classes of topological vector lattices to which Theorems 1 and 2 apply. We exhibit one such class (see Theorem 3 and the Corollary) that includes all reflexive locally convex lattices. Goffman [4] has applied Nakano's theorem to Köthe spaces. Further, for a certain class of locally convex lattices, a characterization of the decisive completeness condition of Theorem 2 is given (Theorem 4), and a stronger form of Theorem 2 is obtained.

1. DEFINITIONS

Let L be a vector lattice. A subset A is *solid* [7] if $x \in A$ and $|y| \leq |x|$ imply that $y \in A$. If L is also a topological vector space (over the real field), it is said to be *locally solid* if the family of all solid 0-neighborhoods forms a neighborhood base of 0. A *topological vector lattice* is a vector lattice and a real topological vector space which is locally solid. An equivalent property is as follows: $K = \{x: x \geq 0\}$ is a normal cone, and the lattice operations are continuous. (Here, K is *normal* if $\lim \phi = 0$ implies $\lim[\phi] = 0$ for every filter ϕ on L [10, (1.a)], where $[\phi] = \{[F]: F \in \phi\}$ with $[F] = (F + K) \cap (F - K)$. The notion of normality plays a fundamental role in the theory of partially ordered topological vector spaces; see [7] to [10].) A *locally convex vector lattice* is a locally convex Hausdorff space and a topological vector lattice.

A subset $A \subset L$ is *order-complete* if for each directed subset $X \subset A$, majorized in L, sup X exists and is a member of A. (Throughout this paper, "directed" will mean "directed for $\leq$." The term "bounded" will be used exclusively in its topological sense, and "majorized" will be substituted for the common "order-bounded above.") A topological vector lattice is *locally order-complete* if there exists a 0-neighborhood base consisting of solid order-complete sets. Finally, we shall say

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that a topological vector lattice is *boundedly order-complete* if for every bounded, directed subset X of L, sup X exists.

Let L be a vector lattice that is order-complete (as a subset of itself). A filter ϕ on L is *order-convergent* if ϕ contains a set X_0 such that both X_0 and $-X_0$ are majorized, and if

$$\sup_{X \in \phi} (\inf X \cap X_0) = \inf_{X \in \phi} (\sup X \cap X_0).$$

The common value of these expressions is called the *order-limit* of ϕ [3, p. 28, Ex. 9]. We say that an order-convergent filter *converges* in a topological vector lattice L if ϕ converges to its order-limit for the topology of L, that is, if ϕ is finer than the neighborhood filter of its order-limit.

An *order-interval* in L is a nonempty subset $[u, v] = \{z: u \leq x \leq v\}$.

The term "complete" (without specification) or "topologically complete" is employed in its usual meaning for uniformities. What uniformity is meant will generally be clear from the context.

2. TOPOLOGICAL VECTOR LATTICES

Nakano based the proof of his theorem [6, 4.2] essentially on his Theorem 3.3, whose proof, however, is insufficient in the nonmetrizable case [6, p. 94]. By proving the following theorem, which is equivalent to [6, 3.3], we shall fill this gap. The proof for the metrizable case is included.

THEOREM 1. *In every locally order-complete topological vector lattice, each order-interval is a complete uniform space (for the induced uniformity).*

Proof. We shall prove the theorem first in the case where the topology of the vector lattice L in question is metrizable. Let $\{V_n\}$ be a 0-neighborhood base consisting of symmetric, solid, and order-complete sets such that $V_{n+1} + V_{n+1} \subset V_n$ for all n. If p_n is the gauge of V_n, we shall set $\pi = \{p_n\}$ and refer to the topology with 0-base $\{V_n\}$ as the π-topology. Let [u, v] be an arbitrary, fixed order-interval in L; to show that [u, v] is complete, it is obviously sufficient to show that each Cauchy sequence in [u, v] contains a convergent subsequence. Given a Cauchy sequence in [u, v], there exists a subsequence $\{x_k\}$ such that $x_{k+1} - x_k \in V_{k+1}$ for every k. Let m be an arbitrary integer $(m > k)$; then

$$\sup_{k \leq \ell \leq m} x_\ell - x_k = \sup_{k \leq \ell \leq m} (x_\ell - x_k) \leq \sup_{k \leq \ell \leq m} |x_\ell - x_k|$$

$$\leq |x_{k+1} - x_k| + \cdots + |x_m - x_{m-1}|.$$

Since (V_k being solid) the last right-hand term represents an element of V_k, it follows that $\sup\{x_\ell: k \leq \ell \leq m\} \in x_k + V_k$. On the other hand, the set of all these suprema $(m > k)$ is directed and majorized by v, hence $y_k = \sup\{x_\ell: \ell > k\}$ is in $x_k + V_k$, because V_k is order-complete. Dually, it follows that $z_k = \inf\{x_\ell: \ell > k\}$ is in $x_k + V_k$, and consequently $y_k - z_k \in V_{k-1}$. $\{y_k\}$ and $\{z_k\}$ are directed sets (the former for $\geq$); letting $y = \inf y_k$ and $z = \sup z_k$, we obtain $y - z \in V_k$ for every k. It follows that $y = z$ and that $\lim_k x_k = y$ for the π-topology. Since [u, v] is closed in L (this is a consequence of the continuity of the lattice operations), $y \in [u, v]$.

Before taking up the proof for the general, nonmetrizable case, we shall point out a number of consequences of our assumptions. Recall that a band in an order-complete vector lattice is a vector subspace which is solid and order-complete as a subset of L. If N is a band in L, the set of all $y \in L$ for which $x \in N$ implies $\inf(x, y) = 0$ is a band N' in L; $L = N + N'$ is a direct sum decomposition such that, if P denotes the projection of L onto N vanishing on N', one has $0 \leq P \leq I$ (for the order induced on the space of linear maps of L into itself). Under the present assumptions, every such P is continuous (a proof of this statement and further relevant material can be found in [5, Section 2]). Further, let X be an arbitrary directed, majorized subset of L, and P a projection in L such that $0 \leq P \leq I$; then $P(\sup X) = \sup P(X)$.

Let $\mathscr{V}$ denote a neighborhood base of 0 in L, consisting of symmetric, solid, order-complete sets; let $\mathscr{P}$ be the family of all real-valued functions on L, each of which is the gauge of some $V \in \mathscr{V}$. By π we shall denote an arbitrary countable subfamily of $\mathscr{P}$ such that the subfamily of $\mathscr{V}$ corresponding to π is a neighborhood basis of 0 for a topology (not necessarily a Hausdorff topology) which is compatible with the vector space structure of L. Let Π be the set of all these π; it is easy to establish that Π is nonempty and directed under inclusion. Since the intersection of any number of bands in L is a band, the set $N_\pi' = \bigcap \{p^{-1}(0) : p \in \pi\}$ is a band. Denote by N_π the band in L complementary to N_π', and by P_π that projection of L onto N_π which vanishes on N_π'. It is not difficult to establish that $\pi \subset \rho$ implies $P_\pi \leq P_\rho$ (in the above sense). If $N_0' = \bigcap \{p^{-1}(0) : p \in \mathscr{P}\}$, N_0 is the band complementary to N_0', and P_0 is the associated projection onto N_0, then $P_0 = \sup\{P_\pi : \pi \in \Pi\}$, and $P_0 = I$ if and only if the topology of L is a Hausdorff topology.

We show now that every order-interval is complete in L. Since

$$[u, v] = u + [0, v - u]$$

and $x \to u + x$ is a uniform automorphism of L, it is sufficient to show that every order-interval $[0, v]$ is complete ($v \geq 0$). Let ϕ be an arbitrary Cauchy filter in $[0, v]$. Since P_π ($\pi \in \Pi$) is uniformly continuous, the image ϕ_π of ϕ under P_π is a Cauchy filter base for the restriction of the π-topology to N_π. Since, by the definition of N_π, that topology is a Hausdorff topology (hence metrizable), it follows from the first part of the proof that there exists a unique $z_\pi \in N_\pi$ such that $\pi\text{-}\lim \phi_\pi = z_\pi$. If $\rho \supset \pi$, then also $\rho\text{-}\lim \phi_\pi = z_\pi$, since ϕ_π has a unique ρ-limit in N_π that must be identical with z_π. Since the positive cone in N_π is closed for the ρ-topology whenever $\rho \supset \pi$, it follows that $z_\pi \in [0, v]$. Moreover, $\pi \subset \rho$ implies $P_\pi \leq P_\rho$ and hence, by a similar argument, $z_\pi \leq z_\rho$. Thus, $\{z_\pi : \pi \in \Pi\}$ is directed and majorized in L (namely, by v); let $u = \sup\{z_\pi : \pi \in \Pi\}$. It follows from the remarks in the preceding paragraph that for each fixed $\pi \in \Pi$, $P_\pi u = \sup\{P_\pi z_\rho : \rho \in \Pi\}$. But since $P_\pi P_\rho = P_\pi$ for all $\rho \supset \pi$ (see [5, Section 2]), it follows, whenever $\rho \supset \pi$, that

$$P_\pi z_\rho = P_\pi(\rho\text{-}\lim \phi_\rho) = \rho\text{-}\lim P_\pi \phi_\rho = \rho\text{-}\lim \phi_\pi = \pi\text{-}\lim \phi_\pi = z_\pi .$$

Hence $\sup\{P_\pi z_\rho : \rho \in \Pi\} = \sup\{P_\pi z_\rho : \rho \supset \pi\} = z_\pi$ for π fixed, and therefore $P_\pi u = z_\pi$. Our final assertion is that u is a limit point of ϕ for the given topology on L; this is equivalent to saying that u is a limit point of ϕ for every π-topology ($\pi \in \Pi$). Since for each π, $L = N_\pi + N_\pi'$ is a topological direct sum, and since every $p \in \pi$ vanishes on N_π', u is a π-limit point of ϕ if and only if $P_\pi u$ is a π-limit point of $P_\pi \phi = \phi_\pi$; but we have even $\pi\text{-}\lim \phi_\pi = z_\pi = P_\pi u$, as shown. Clearly, u is the unique limit in N_0 of the projection $\phi_0 = P_0 \phi$; the set of all limit points of ϕ in L is $u + N_0'$. The proof of the theorem is complete.

THEOREM 2 (Nakano). *Let* L *be a topological vector lattice which is both locally order-complete and boundedly order-complete. Then* L *is topologically complete.*

We shall not reproduce the details of Nakano's proof; they can be found in [6, Section 4]. Roughly, the proof proceeds as follows: If ϕ is a Cauchy filter, then ϕ^+, the image of ϕ under $x \to x^+$, is a Cauchy filter base; it is enough to show that ϕ^+ converges. Let ϕ_x^+ denote the Cauchy filter base which is the image of ϕ^+ under $y \to \inf(x, y)$, for arbitrary fixed $x \geq 0$; by Theorem 1, ϕ_x^+ has a unique limit a_x, if L is assumed to be a Hausdorff space. The crux of the proof then consists in showing that the directed set $\{a_x : x \geq 0\}$ is (topologically) bounded. Then, by assumption, $\sup\{a_x : x \geq 0\}$ exists, and it is shown to be the limit of ϕ^+. The case where L is not a Hausdorff space is settled without difficulty.

3. LOCALLY CONVEX VECTOR LATTICES

In this section, we assume that the reader is familiar with the basic theory of locally convex vector spaces. Recall that a locally convex space E is a disk space (espace tonnelé) if every disk in E, that is, every convex, closed, absorbing subset of E, is a neighborhood of 0. The dual E' of E is the vector space of all continuous linear forms on E; the strong topology on E' is the topology of uniform convergence on all bounded subsets of E (the bounded-open topology). Finally, if E is a locally convex vector lattice, E' is a vector lattice whose positive cone is

$$K' = \{f \in E' : f(x) \geq 0 \quad \text{for} \quad x \geq 0\};$$

we call this the natural order on E'.

THEOREM 3. *Let* E *be a locally convex vector lattice and a disk space. Equipped with its natural order and the strong topology,* E' *is a locally convex lattice which is locally order-complete and topologically complete.*

Proof. Without using the assumption that E is a disk space, we show first that E' is a locally order-complete vector lattice for the strong topology $\beta(E', E)$. (It has been shown [5, 1.18] that E' is a topological vector lattice for this topology.) From the hypothesis it follows that the family $\mathscr{B}$ of all solid bounded subsets $B \subset E$ is a fundamental system of bounded sets; that is, $\beta(E', E)$ is the topology of uniform convergence on $\mathscr{B}$. We shall show that each polar set $U_B = B^o$ $(B \in \mathscr{B})$ is a solid, order-complete subset of E'. It is clear that $-g \leq f \leq g \in U_B$ implies $f \in U_B$. Further, let $f \in U_B$; in order to see that $|f| \in U_B$, it is sufficient to show that $|f|(x) \leq 1$ for $x \in B \cap K$. By definition of $|f|$ [3, p. 36], we have

$$|f|(x) = \sup\{f(u) - f(v) : u \geq 0, v \geq 0, u + v = x\}.$$

But for each such pair (u, v), one has $-x \leq u - v \leq x$, hence $u - v \in B$ if $x \in B$; thus $|f|(x) \leq 1$ for $x \in B$. Now let F be a directed subset of E', majorized by h, say; it is no restriction of generality to assume $F \subset K'$. The linear form f_0 on E, defined on K as $f_0(x) = \sup\{f(x) : f \in F\}$, is in E'; for if $x \to 0$ in E, then $|x| \to 0$, since the lattice operations in E are continuous, and hence

$$\limsup_{x \to 0} |f_0(x)| \leq \limsup_{x \to 0} f_0(|x|) \leq \lim_{x \to 0} h(|x|) = 0,$$

because h is continuous. Assume further that F is directed and majorized in E', and that $F \subset U_B$ (but not necessarily $F \subset K'$). If $f_0 = \sup F$, then

$$f_0(x) = \sup \{ f(x) \colon f \in F \} \quad \text{for} \quad x \in K.$$

Since $E = K - K$, f_0 is in the weak*-closure of F. Hence U_B, being weak*-closed, is order-complete.

The proof will be complete if we can show that E' is boundedly order-complete; here we shall have to use the assumption that E is a disk space. Let $F \subset E'$ be directed and bounded. Then, since every bounded set is relatively weak*-compact, F has a limit point f_0 in E'. It is clear that $f_0(x) = \sup \{ f(x) \colon f \in F \}$, for every $x \in K$. Hence $f_0 = \sup F$ exists. Now an application of Theorem 2 completes the proof.

Let E denote a locally convex space, E'' its strong bidual, ψ the evaluation map of E into E''. Then E is reflexive if ψ is a homeomorphism of E onto E''. If, in addition, E is a topological vector lattice, then ϕ is also a lattice isomorphism. Since every reflexive space is a disk space, and since the strong dual of a reflexive space is reflexive, we obtain from Theorem 3 the following result.

COROLLARY. *Every reflexive locally convex vector lattice is locally order-complete and topologically complete.*

It is not known whether every reflexive locally convex space is complete.

The following theorem has resulted from the attempt to give a characterization, in terms of topological completeness, of boundedly order-complete locally convex vector lattices. Although the condition given is sufficient without further restriction, it fails to be necessary in the general case. As the proof of Theorem 4 shows, to obtain necessity one has to require at least that every order-convergent filter converges weakly in E; this is the case if and only if $\sup X = x_0$ implies $\sup f(X) = f(x_0)$ for every directed subset X and every continuous positive linear form f on E. If in the latter statement the continuity of f is omitted, one is led to the algebraic condition of minimality [10, Section 14].

THEOREM 4. *Let* E *be a locally convex vector lattice in which every order-convergent filter converges weakly. Then* E *is boundedly order-complete if and only if* E *is topologically complete for the topology of uniform convergence on all order intervals in* E'.

Proof. Since the polars in E of the sets $[-f, f]$ ($f \in K'$) are convex solid sets, it follows that E, equipped with the topology $o(E, E')$ of uniform convergence on all order-intervals in E', is a locally convex vector lattice. (It follows immediately that $o(E, E')$ is a Hausdorff topology.) Next we observe that $o(E, E')$ is a locally convex topology for which the dual of E is E'. Clearly, $o(E, E')$ is finer than the weak topology on E; on the other hand, every order interval in E' is weak*-compact. For this, it is enough to show that every order-interval $[0, f]$ in E' is equicontinuous with respect to the original topology E. (From the following argument it can be seen that $o(E, E')$ is the coarsest locally convex topology on E whose dual is E' and such that the lattice operations are continuous. Under somewhat more general assumptions, this result is due to A. L. Peressini.) Let U be a 0-neighborhood in E such that $x \in U$ implies $f(x) \leq 1$. Since the lattice operations are continuous, there exists a symmetric 0-neighborhood V such that $V \subset V^+ - V^+ \subset U$, where $V^+ = \{ x^+ \colon x \in V \}$. Hence $|g(x)| \leq 1$ for every $x \in V$ and $g \in [0, f]$. It follows now from a well-known theorem of Mackey that the dual of E with respect to $o(E, E')$ is E'.

Assume now that E is boundedly order-complete. Then E is obviously boundedly order-complete for o(E, E'). We show that E is locally order-complete for o(E, E'). Let $U = [-f, f]^0$ be a basic neighborhood of 0, and let $X \subset U$ be directed. Let $x_0 = \sup X$; we must show that $x_0 \in U$. By assumption, the filter of sections of X converges weakly to x_0; since U is weakly closed, we obtain $x_0 \in U$. Hence, by Theorem 2, E is complete for o(E, E').

Conversely, let E be complete for o(E, E'). Let H be a directed bounded subset of E; then the filter $\phi(H)$ of sections of H is a Cauchy filter with respect to o(E, E'). Since the positive cone K is closed for o(E, E'), we have $\lim \phi(H) = x_0$ and $x_0 = \sup H$ by [3, p. 26, Prop. 6]. Hence E is boundedly order-complete. This completes the proof.

COROLLARY. *Every boundedly order-complete, locally convex vector lattice in which every order-convergent filter converges weakly is topologically complete.*

Proof. Theorem 4 implies that E is complete for o(E, E'). On the other hand, we have observed that o(E, E') is compatible with the duality between E and E', and that it is coarser than the given topology on E. From this we conclude by a standard argument that E is complete for its original topology.

We could have proved the Corollary above by a direct application of Theorem 2, observing that every locally convex vector lattice satisfying the above assumption is also locally order-complete. This may be shown directly, as in the proof of Theorem 4, or by using [10, (14.1)]. Theorem 1 implies that in such a vector lattice every order-interval is complete. We remark in conclusion that a complete topological vector lattice need not be boundedly order-complete. An example is furnished by the Banach lattice of all null sequences of real numbers, equipped with its usual order and norm. If X is the set of all vectors of the form $(1, 1, \cdots, 1; 0, 0, \cdots)$, then X is a bounded, directed set that has no upper bound.

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