SEPARATION AND UNION THEOREMS FOR GENERALIZED MANIFOLDS WITH BOUNDARY

Frank Raymond

INTRODUCTION

Generalized manifolds are a class of spaces which reflect many of the local and global homology properties of locally Euclidean manifolds. Moreover, they form a class of spaces in which certain operations are closed with respect to this class, whereas this may not be true for classical manifolds; for example, if a generalized manifold is the Cartesian product of two spaces, then both factors are generalized manifolds. (See Theorem 6 for a proof of this fact.)

It is well-known that every 2-sphere imbedded in a 3-sphere separates the 3-sphere into two open connected sets both of whose frontiers coincide with the 2-sphere. However, neither open set need be an open 3-cell, and even if the sets are 3-cells, the 2-sphere may not fit onto a complementary domain to form a closed 3-cell. Wilder [10] has shown that these complementary domains, nevertheless, are generalized cells. This follows as an application of the Jordan-Brouwer separation theorem and its converse [10; Chap. 10]. Wilder's theorems are cast in the framework of orientable, sphere-like, compact generalized n-manifolds over a field. P. A. White [9], has extended the results by relaxing some of the sphere-like conditions. The purpose of this paper is to extend the results to noncompact, locally orientable generalized n-manifolds with boundary defined over an arbitrary principal ideal domain (see Section 3).

The separation of a generalized n-manifold with boundary by a generalized (n - 1)-manifold with boundary is converse to the problem of showing that the union, along the boundary, of two generalized n-manifolds with boundary is again a generalized n-manifold with boundary. Consequently, we shall treat both problems.

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1. NOTATION AND DEFINITIONS

By a *space* we shall mean a locally compact Hausdorff space. By a *neighborhood of a point* in a space we shall mean any open set containing the particular point. If A is a subset of the space X, then by A^- we shall mean the closure of the subset A in X.

The p-*dimensional Čech cohomology group of* X *with compact supports and coefficients in the principal ideal domain* L will be denoted by $H_c^p(X; L)$. However, in general, we shall omit the ring L from the notation, since no confusion can arise. If U is an open subset of the space X, then the sequence

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$$\to H_c^{p-1}(X - U) \overset{d^*}{\to} H_c^p(U) \overset{j^*}{\to} H_c^p(X) \overset{i^*}{\to} H_c^p(X - U) \overset{d^*}{\to}$$

is exact.

If V and U are open sets of X, and $j: V \subset U$ is the inclusion map, then $H_c^r(V \subset U)$ will denote the image of $H_c^r(V)$ in the group $H_c^r(U)$ under the map j^*, induced by the inclusion j.

By $\dim_L X$ we shall mean the *cohomology dimension* of X with respect to the coefficients L, as defined by H. Cohen in [4].

DEFINITION 1. We say that the space X has *local* r-*co-Betti number* 0 *at* $x \in X$, if whenever we are given a neighborhood U, we can find a neighborhood V $(x \in V \subset U)$ such that $H_c^r(V \subset U) = 0$. We write this in the form $p^r(x; X) = 0$.

We shall say that X has *local* r-*co-Betti number equal to* k *at* $x \in X$ $(p^r(x; X) = k)$ if, given a neighborhood U, we can find neighborhoods V and W such that if W' is a neighborhood of x $(x \in W' \subset W \subset V \subset U)$, then $H_c^r(W' \subset V) = H_c^r(W \subset V)$ and $H_c^r(W \subset V)$ is a free L-module of rank k.

If X is a space and A a closed subset, then X has *local* r-*co-Betti number at* $x \in X$, mod A, equal to k, if given U we can find V and W as before such that if W' is as before, then

$$H_c^r((W' - A) \subset (V - A)) = H_c^r((W - A) \subset (V - A))$$

and $H_c^r((W - A) \subset (V - A))$ is a free L-module of rank k. We write this in the form $p^r(x; X - A) = k$.

These definitions are given in [10] for a field L; some have been generalized to a principal ideal domain, for example in [2].

PROPOSITION 1.1. *If* $\dim_L X < \infty$, *then a necessary and sufficient condition that* $\dim_L X \leq n$ *is that* $p^r(x; X) = 0$, *for all* $r > n$ *and for all* $x \in X$.

This proposition is merely a cohomological version for a space of a theorem of Alexandroff [1] concerning Euclidean space.

DEFINITION 2. By a *generalized* n-*manifold* (n-gm) with respect to the ring L (usually suppressed) we shall mean a space X such that

(i) $\dim_L X < \infty$,

(ii) $p^r(x; X) = 0 \quad (r \neq n,\ x \in X)$,

(iii) $p^n(x; X) = 1$.

By an *orientable* n-gm we shall mean an n-gm X such that if O is any component of X, and U is a connected open subset with compact closure $(U \subset O)$, then $j^*: H_c^n(U) \to H_c^n(0)$ is an isomorphism.

By a *locally orientable* n-gm (l. o. n-gm) we mean an n-gm X such that there exists a covering of X by open sets, each element of which is an orientable n-gm.

REMARKS. The definitions given above are equivalent to the plethora of definitions that have recently been given to generalize the definition adopted by Wilder.

It is easy to show that any open subset of an orientable n-gm is orientable. With the help of a minimality argument, it follows that $H_c^n(A) = 0$, for all closed proper subsets A of any component of a l. o. n-gm. A *connected* l. o. n-gm X *is orientable if and only if* $H_c^n(X) \approx L$. An n-gm is locally connected and in fact is clc [2], or in Wilder's terminology, is lc^{∞}.

Yang in [11] and I in [7] show that a locally compact subset A of a l. o. n-gm X has $\dim_L \leq n$ if and only if A contains interior points.

In fact, if A is closed in X, then the interior points of A are identical with the points x of A for which $p^n(x; A) \neq 0$.

Similar methods yield

PROPOSITION 1.2. *If* A *is a closed subset of a* l. o. n-gm X, *then the points* x *of* A *such that* $p^{n-1}(x; A) \neq 0$ *are identical with those points* x *of* A *for which* A *separates* X *locally at* x.

DEFINITION 3. By *an* n-gm *with boundary* B we shall mean a space X and a closed subset B such that

(i) B is an (n - 1)-gm,

(ii) X - B is an n-gm,

(iii) $p^r(x; X) = 0$, for all $x \in B$, all r.

The definition was first given by White [8] for compact X, and then by Brahana [3] for locally compact X. In both instances L was a field. Our definition, when L is a field, is not the same as theirs, but is equivalent to it, as the next lemma shows.

2. THE UNION OF TWO n-gm's WITH BOUNDARY

LEMMA 2.1. *Let* X *be an* n-gm *with boundary* B. *Then* $(X - B)^- = X$ *and* $p^r(x; X - B) = 0$ $(r \neq n,\ x \in B)$. *Moreover, if* L *is a field, or if* B *is* l. o., *then* $p^n(x; X - B) = 1$.

Proof. If B contains an interior point y, then $p^{n-1}(y; X) \neq 0$, contradicting (iii) of Definition 3. Hence $(X - B)^- = X$.

Let $x \in B$, and let U be any neighborhood of x. Choose neighborhoods V and W of x such that $H_c^r((V \cap B) \subset (U \cap B)) = 0$ $(r \neq n - 1)$, and $H_c^r(W \subset V) = 0$ (all r). This is clearly possible, by the definitions. For $r \neq n - 1$, consider the commutative diagram

$$\begin{array}{ccccc}
 & & H_c^{r+1}(W - B) & \to & H_c^{r+1}(W) \\
 & & \downarrow & & \downarrow \\
H_c^r(V \cap B) & \to & H_c^{r+1}(V - B) & \to & H_c^{r+1}(V) \\
\downarrow & & \downarrow & & \\
H_c^r(U \cap B) & \to & H_c^{r+1}(U - B) & &
\end{array}$$

That $p^r(x; X - B) = 0$ $(r \neq n)$, follows immediately from Lemma 6.3 (Appendix).

Let us assume now that B is l. o. and that $x \in B$. Choose a neighborhood U of x such that $U \cap B$ is connected and orientable. Choose neighborhoods W and V such that $W \cap B$ and $V \cap B$ are connected, $W \subset V \subset U$, and $H_c^{n-1}(V \subset U) = H_c^n(W \subset V) = 0$.

Consider the commutative diagram

$$\begin{array}{ccccccccc} & & H_c^{n-1}(W \cap B) & \overset{d_1}{\to} & H_c^n(W - B) & \to & H_c^n(W) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ H_c^{n-1}(V) & \to & H_c^{n-1}(V \cap B) & \overset{d_2}{\to} & H_c^n(V - B) & \to & H_c^n(V) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & & & \\ H_c^{n-1}(U) & \to & H_c^{n-1}(U \cap B) & \overset{d_3}{\to} & H_c^n(U - B). & & & & \end{array}$$

The vertical maps of the second column are all isomorphisms, and

$$H_c^{n-1}(W \cap B) \approx L.$$

Since $H_c^{n-1}(V \subset U) = 0$, d_2 is an injection. Let

$$K = H_c^n((W - B) \subset (V - B)).$$

Because $H_c^n(W \subset V) = 0$, $d_2(H_c^{n-1}(V \cap B)) \supset K$, and obviously $d_2(H_c^{n-1}(V \cap B)) \subset K$. If $W' \subset W$,

$$H_c^{n-1}(W' \subset V) = H_c^{n-1}(W \subset V).$$

Consequently,

$$H_c^n((W' - B) \subset (V - B)) \supset H_c^n((W - B) \subset (V - B)).$$

From these facts, it follows that $p^n(x; X - B) = 1$.

When L is a field, one applies [10; 1.4, p 291], and this completes the proof.

LEMMA 2.2. *Let* X *be an* n-gm *with boundary* B. *Let* X - B *and* B *be* l. o. *Then, given* $x \in B$, *there exists a connected neighborhood* U *of* x *such that* $U \cap B$ *and* U - B *are connected and orientable.*

Proof. Choose connected neighborhoods U and V of $x \in B$ such that $U \cap B$ is connected and orientable, $V \cap B$ is connected, $V \subset U$, $H_c^{n-1}(V \subset U) = H_c^n(V \subset U) = 0$, and such that d maps $H_c^{n-1}(U \cap B)$ isomorphically onto $H_c^n((V - B) \subset ((U - B)) \approx L$. This last condition follows from the argument used in Lemma 2.1. Therefore, there exist orientable components of V - B which are mapped into an orientable component of U - B. (Each component $(V - B)_\alpha$ of (V - B) is open and orientable if and only if $H_c^n(V - B)_\alpha \approx L$. Furthermore,

$$H_c^n(V - B) = \Sigma_\alpha H_c^n(V - B)_\alpha,$$

and $(dH_c^{n-1}(V \cap B)) \subset H_c^n(V - B)$ is mapped isomorphically into $H_c^n(U - B)$.) Let $(U - B)'$ denote the orientable component whose n-th cohomology group contains $dH_c^{n-1}(U \cap B)$. Clearly, $U \cap B$ must meet $((U - B)')^-$, for otherwise d would be trivial. Let $D = (U \cap B) \cap ((U - B)')^-$. We shall show that $D = B \cap U$, and hence the lemma will be proved. Clearly, D is the frontier of $(U - B)'$ in U, and therefore we may consider the commutative diagram

$$\begin{array}{ccc} H_c^{n-1}(U \cap B) & \xrightarrow{d} & H_c^n((U - B)') \,. \\ \downarrow i^* & \nearrow d' & \\ H_c^{n-1}(D) & & \end{array}$$

Since $d\colon H_c^{n-1}(U \cap B) \to H_c^n((U - B)')$ is nontrivial, $H_c^{n-1}(D) \neq 0$. However, by the remarks following Definition 2, this is only possible if $D = B \cap U$.

For convenience, we shall say that X is a l.o. n-gm with l.o. boundary B, when we mean that X is an n-gm with boundary B such that X - B and B are l.o. The following theorem is a generalization of results of White [8] and Brahana [3].

THEOREM 1. *Let* X_1 *and* X_2 *be closed subsets of a space* $X = X_1 \cup X_2$, *and suppose that they are* l.o. n-gm's *with* l.o. *boundaries* B_1 *and* B_2. *Suppose that* $X_1 \cap X_2 = B \subset B_1 \cap B_2$ *is a* l.o. (n - 1)-gm. *Then* X *is a* l.o. n-gm *with* l.o. *boundary* $(B_1 \cup B_2) - (X_1 \cap X_2)$.

Proof. The points of $B = X_1 \cap X_2$ form an open set relative to both B_1 and B_2. (See the remarks following Definition 2.) Therefore, we need only show that the points of B have the correct local co-Betti numbers as points of X, and that they lie within an orientable part of X. Choose $x \in B$ and neighborhoods of x, $U_1 \subset U_2 \subset U_3$, so that

$$H_c^r((U_j \cap B) \subset (U_{j+1} \cap B)) = 0 \qquad (j = 1, 2;\ r \neq n - 1)\,.$$

Let $U_j^i = U_j \cap X_i$ (i = 1, 2; j = 1, 2, 3). Since U_j is locally connected and $X_1 \cap X_2 = B$ is closed (relative to U_j) and separates U_j,

$$H_c^r(U_j - B) = H_c^r(U_j^1 - B) \oplus H_c^r(U_j^2 - B) \qquad (r = 1, 2, \cdots;\ j = 1, 2, 3)\,.$$

By Lemma 2.1, U_j^i can be chosen so that

$$H_c^r((U_j^i - B) \subset (U_{j+1}^i - B)) = 0 \qquad (j = 1, 2;\ i = 1, 2;\ r \neq n)\,.$$

Let us therefore consider the commutative diagram

$$\begin{array}{ccccc} H_c^r(U_1 - B) & \to & H_c^r(U_1) & \to & H_c^r(U_1 \cap B) \\ \downarrow & & \downarrow & & \downarrow \\ H_c^r(U_2 - B) & \to & H_c^r(U_2) & \to & H_c^r(U_2 \cap B) \\ \downarrow & & \downarrow & & \downarrow \\ H_c^r(U_3 - B) & \to & H_c^r(U_3) & \to & H_c^r(U_3 \cap B)\,. \end{array}$$

From the information above and Lemma 6.3, it follows that $H_c^r(U_1 \subset U_3) = 0$ $(r \neq n, n - 1)$.

If $r = n - 1$, then the proposition that the first column is mapped onto the second column would imply that $H_c^{n-1}(U_1 \subset U_3) = 0$. That this is the actual case will follow from the discussion of orientability.

For $x \in B = X_1 \cap X_2$, choose a connected neighborhood U such that $U \cap B$, $U - X_i$, and $U \cap X_i$ are connected, and $U - X_i$, $U \cap B$ are orientable (i = 1, 2). This is possible, by Lemma 2.2. Let $U^i = U \cap X_i$ (i = 1, 2). Consider the exact sequence:

$$H_c^{n-1}(U \cap B) \xrightarrow{d} H_c^n(U^1 - B) \oplus H_c^n(U^2 - B) \to H_c^n(U) \to 0\,.$$

By Lemma 2.2, d is an injection, and consequently $H^n_c(U) \approx L$. Clearly, $p^n(x; X) = 1$, and therefore X is an n-gm with boundary. If A is any proper subset of U, closed relative to U, then the exact sequence

$$H^{n-1}_c(A \cap B) \xrightarrow{d} H^n_c((U^1 - B) \cap A) \oplus H^n_c((U^2 - B) \cap A) \to H^n_c(A) \to 0,$$

together with the fact that each proper closed subset of a l. o. n-gm has trivial n-dimensional cohomology, implies that $H^n_c(A) = 0$. Because U is a connected n-gm, $H^n_c(U) \approx L$, and $H^n_c(A) = 0$ for all proper closed subsets A of U, it easily follows that U is an orientable n-gm. This completes the proof.

3. SEPARATION THEOREMS

THEOREM 2. *Let* X *be a connected* l. o. n-gm, *and* X' *a connected* l. o. (n - 1)-gm *imbedded as a closed subset of* X. *If* X - X' *is separated, then* X - X' *is the union of exactly two disjoint connected open sets each of whose frontiers is* X', *and onto each of which* X' *fits as a manifold with boundary.*

Proof. Let us first assume that both X and X' are orientable. Consider the exact sequence

$$H^{n-1}_c(X') \xrightarrow{d} H^n_c(X - X') \to H^n_c(X) \to 0.$$

Since X - X' is separated and $H^{n-1}_c(X') \approx H^n_c(X) \approx L$, exactness implies that $H^n_c(X - X') \approx L \oplus L$. Thus, X - X' is the union of two disjoint connected open sets O_1 and O_2. Let A be any closed proper subset of X'. The fact that $H^{n-1}_c(A) = 0$, together with exactness, implies that $H^n_c(X - A) \approx L$. Consequently, X - A is connected. Therefore A cannot be the entire frontier of either of the domains O_1 and O_2, which implies that X' is the frontier of both O_1 and O_2.

We now show that X' fits onto O_1 and O_2 as a manifold with boundary. Let $x \in X'$, and choose connected neighborhoods U_1, U_2, U_3 of x $(U_1 \subset U_2 \subset U_3)$, so that each $U_j \cap X'$ (j = 1, 2, 3) is connected and

$$H^r_c((U_j \cap X') \subset (U_{j+1} \cap X')) = 0 \qquad (r \neq n - 1),$$

$$H^r_c(U_j \subset U_{j+1}) = 0 \qquad (r \neq n;\ j = 1, 2).$$

Let

$$U^i_j = U_j \cap O^-_i \qquad (j = 1, 2, 3;\ i = 1, 2).$$

Clearly, $U^1_j \cup U^2_j = U_j$, $U^1_j \cap U^2_j = U_j \cap X'$, and the U^i_j are closed in U_j. Consider the commutative diagram

$$\begin{array}{ccccc}
H^r_c(U_1) & \to & H^r_c(U^1_1) \oplus H^r_c(U^2_1) & \to & H^r_c(U_1 \cap X') \\
\downarrow & & \downarrow & & \downarrow \\
H^r_c(U_2) & \to & H^r_c(U^1_2) \oplus H^r_c(U^2_2) & \to & H^r_c(U_2 \cap X') \\
\downarrow & & \downarrow & & \downarrow \\
H^r_c(U_3) & \to & H^r_c(U^1_3) \oplus H^r_c(U^2_3) & \to & H^r_c(U_3 \cap X').
\end{array}$$

From the conditions on the U_j, Corollary 6.2 and Lemma 6.3 imply that

$$H_c^r(U_1^1) \oplus H_c^r(U_1^2) \to H_c^r(U_3^1) \oplus H_c^r(U_3^2)$$

is trivial. Consequently $H_c^r(U_1^i \subset U_3^i) = 0$ $(r \neq n, n-1)$.

Since the U_j^i are proper closed subsets of U_j, $H_c^n(U_j^i) = 0$. Consequently, the maps

$$\Delta_j\colon H_c^{n-1}(U_j \cap X') \to H_c^n(U_j)$$

are onto. Since

$$H_c^{n-1}(U_j \cap X') \approx H_c^n(U_j) \approx L,$$

Δ_j is onto, and L is a principal ideal domain, it follows that Δ_j is an isomorphism. Hence, $H_c^{n-1}(U_j)$ is mapped onto $H_c^{n-1}(U_j) \oplus H_c^{n-1}(U_j^2)$, which implies that $H_c^{n-1}(U_1^i \subset U_3^i) = 0$. Thus, $\bar{O_i}$ is the union of an orientable n-gm O_i with an orientable (n - 1)-gm X' such that X' fits onto O_i as a manifold with boundary.

Next, we shall localize the argument above to obtain

LEMMA 3.1. *Let* X', *a* l.o. (n - 1)-gm, *be a closed subset of a* l.o. n-gm X. *Then* X' *separates* X *locally, as in the conclusion of the Theorem* 2.

Proof. Choose $x \in X'$, and choose connected, orientable neighborhoods V and U $(V \subset U)$ such that $V \cap X'$ and $U \cap X'$ are connected and orientable. If we choose V so that $H_c^{n-1}(V \subset U) = 0$, then by a simple argument, $H_c^n(V - X') \supset L \oplus L$, and, therefore V - X' is not connected. Now apply the argument above to V and $V \cap X'$.

REMARK. Lemma 3.1 also follows from the Proposition 1.2.

We now complete the proof of the theorem. Let $x \in X'$. By Lemma 3.1, we can choose a neighborhood U_x of x such that U_x and $U_x \cap X'$ are connected and orientable and $U_x - X'$ is the union of two connected open disjoint sets U_x^1 and U_x^2, both of which have frontier $X' \cap U_x$ fitting on as a manifold with boundary. Let

$$U = \bigcup_{x \in X'} U_x,$$

and let $X - X' = O_1 \cup O_2$ be a partition into nonempty disjoint open sets. Suppose that U - X' were connected. Then U - X' would be either in O_1 or O_2, say in O_1. Let $y \in X'$; then there exists a connected neighborhood U_y of y such that $U_y - X'$ lies completely in $U - X' \subset O_1$. Thus O_2 must be closed in X. But this is impossible, since X is assumed to be connected. Consequently, U - X' cannot be connected.

Let $U_x \cap U_y \cap X' \neq \emptyset$, and let $z \in U_x \cap U_y \cap X'$. Then z is a limit point of both components of $U_x - X'$, as well as of both components of $U_y - X'$. Hence U_x^1, a component of $U_x - X'$, must meet a component of $U_y - X'$, say U_y^1. Likewise, U_x^2, the other component of $U_x - X'$, must meet U_y^2, the other component of $U_y - X'$. From these facts and a standard simple chain argument, it follows that U - X' has at most two components, and that for each U_x the components of $U_x - X'$ lie in distinct components of U - X'. Now standard connectedness arguments show that X - X' is the union of two components O_1 and O_2, each containing a component of U - X', and this completes the proof of the theorem.

COROLLARY 3.2. *Let* X *be a connected* l.o. n-gm *with* l.o. *boundary* B. *Then* X - B *is connected.*

Proof. Let us assume for the moment that B is connected. Consider the *double* of X. This is formed by attaching two distinct copies of X along B. Clearly, this union is a connected l.o. n-gm without boundary, by Theorem 1. The double of X is separated into two disjoint open sets by B, and therefore, by Theorem 2, into exactly two disjoint components having B as common boundary. By the uniqueness of components, X - B is one of these. Now, if B is not connected, let B_i denote a component of B, and U_i a connected open set of X containing B_i and such that its closure does not meet $B - B_i$. By an argument similar to that used above, $U_i - B_i$ is connected. That X - B is connected now follows from a standard type of simple chain argument.

COROLLARY 3.3. *If* X *is a* l.o. n-gm *with* l.o. *boundary* B, *and* X - B *is orientable, then* B *is orientable.*

Proof. Consider a component B_i of B, and a connected open set U of X $(U \supset B_i)$ such that $U^- \cap B = B_i$. The double of U is a connected l.o. n-gm which can be separated by B_i into exactly two components. Each component, homeomorphic to $U - B_i$, is an orientable n-gm. Consideration of the exact sequence

$$H_c^{n-1}(B_i) \rightarrow H_c^n(U - B_i) \oplus H_c^n(U - B_i) \rightarrow H_c^n(U) \rightarrow 0$$

shows that B_i must be orientable.

THEOREM 3. *Let* X', *a closed subset of a connected* l.o. n-gm X *with* l.o. *boundary* B, *be a connected* l.o. (n - 1)-gm *with* l.o. *boundary* B'. *Suppose that* $(X' - B') \subset (X - B)$ *and that* X - X' *is separated. Then* X *is the union of exactly two connected* l.o. n-gm's D_i *with* l.o. *boundaries* $(D_i \cap B) \cup X'$ (i = 1, 2) *such that* $D_1 \cap D_2$ *is* X' *and constitutes the frontier of both* D_1 *and* D_2.

Proof. Let $x \in B'$, and suppose that $x \in X - B$. Choose U, a connected and orientable neighborhood of x, so that $U \cap X'$ is connected. By doubling $U \cap X'$ and using the remarks following Definition 2, one can show that $H_c^{n-1}(U \cap X') = 0$. Thus U - X' cannot be disconnected. Now, since X' - B' must separate X locally (Lemma 3.1), a standard simple chain argument shows that X' could not have separated X. Thus $B \cap X' = B'$.

Let Y = X - B and Y' = X' - B'. Obviously, Y - Y' is separated, for if not, then in any partition $X - X' = C_1 \cup C_2$ into disjoint open sets, Y - Y' would have to be in one of them, say C_1. Hence $C_2 \subset B$ has interior points in X, which contradicts Lemma 2.1. Since Y and Y' are connected, by Corollary 3.2, $Y - Y' = O_1 \cup O_2$, where the O_i are disjoint open connected sets. The set Y' is the frontier (in Y) of each of the sets O_1 and O_2, and forms with each of them a manifold with boundary.

Consider the double, DX, of X. Now DX is separated by DX', and DC_1 and DC_2 are the two disjoint components whose union is DX - DX'. From the fact that the O_i are connected and disjoint, it follows that the C_i are connected and disjoint. Furthermore, both C_1 and C_2 contain points of the boundary B, for otherwise DC_1 and DC_2 could not be connected. Consequently B' disconnects the boundary B, and therefore B' fits locally onto B - B' as a manifold with boundary.

The proof will be complete if it is shown that B' is the frontier of both $C_1 \cap B$ and $C_2 \cap B$, and that at points $x \in B'$, $p^r(x; D_i) = 0$ for all r, where $D_i = C_i^-$ (i = 1, 2).

Let B_j' be a component of B'. Let U be a connected neighborhood of B_j' such that $U \cap X'$ and $U \cap B$ are connected and $U \cap B' = B_j'$. It is easily seen that $U \cap X'$

separates U. Let $U - X' = G_1 \cup G_2$ be a partition by open sets. By Corollary 3.2, $Y' \cap U$ and $U - B$ are connected. Therefore, $(U - B) - (Y' \cap U)$ can be written as $(O_1 \cap U) \cup (O_2 \cap U)$, where $O_1 \cap U$ and $O_2 \cap U$ are disjoint, open nonempty connected sets. Since

$$(O_1 \cap U) \cup (O_2 \cap U) \subset G_1 \cup G_2,$$

and since G_1 and G_2 contain interior points of X, the sets G_1 and G_2 must be connected. Moreover, since DU is separated by $D(X' \cap U)$ into two disjoint parts DG_1 and DG_2, both $G_1 = C_1 \cap U$ and $G_2 = C_2 \cap U$ contain points of B. Thus, $B \cap U$ is separated by $B' \cap U$ into exactly two parts such that $B' \cap U$ fits onto each part as a manifold with boundary. Obviously, $B \cap G_1$ and $B \cap G_2$ are the two components of $(B \cap U) - (B' \cap U)$. The set $(B \cap G) \cup (X' \cap U)$ is the union of two l.o. (n - 1)-gm's with common l.o. (n - 2)-boundary $(B' \cap U)$, and hence it is a l.o. (n - 1)-gm. The set B' is the common frontier of $C_1 \cap B$ and $C_2 \cap B$, and it fits onto each disjoint part as a manifold with boundary.

We have seen, so far, that $D_1 \cap D_2$ is X', and that it coincides with the frontier in X of both D_1 and D_2. Moreover, D_i contains a closed subset $(C_i \cap B) \cup X'$ which is a l.o. (n - 1)-gm fitting onto D_i as a manifold with boundary at all points except perhaps at B'. We must therefore prove that $p^r(x; D_i) = 0$ for all r and all $x \in B_j'$ (i = 1, 2). Choose neighborhoods V_k (k = 1, 2, 3) of x, sufficiently small so that

$$H_c^r(V_2 \subset V_3) = H_c^r((V_1 \cap X') \subset (V_2 \cap X')) = 0$$

for all r. Consider the commutative diagram

$$\begin{array}{ccccc}
 & & H_c^r(V_1 \cap D_1) \oplus H_c^r(V_1 \cap D_2) & \to & H_c^r(V_1 \cap X') \\
 & & \downarrow & & \downarrow \\
H_c^r(V_2) & \to & H_c^r(V_2 \cap D_1) \oplus H_c^r(V_2 \cap D_2) & \to & H_c^r(V_2 \cap X') \\
\downarrow & & \downarrow & & \\
H_c^r(V_3) & \to & H_c^r(V_3 \cap D_1) \oplus H_c^r(V_3 \cap D_2). & &
\end{array}$$

Applying Corollary 6.2 and Lemma 6.3, we obtain

$$H_c^r((V_1 \cap D_1) \subset (V_3 \cap D_1)) = H_c^r((V_1 \cap D_2) \subset (V_3 \cap D_2)) = 0,$$

and this completes the proof.

Theorem 2 is a converse of Theorem 1. To establish a converse of Theorem 3, we generalize Theorem 1 as follows:

THEOREM 4. *Let* X_1 *and* X_2 *be* l.o. n-gm's *with* l.o. *boundaries* B_1 *and* B_2. *Let* $X_1 \cap X_2 = B \subset (B_1 \cap B_2)$ *be a* l.o. (n - 1)-gm *with* l.o. *boundary* B', *and let* X_1 *and* X_2 *be closed subsets of* $X_1 \cup X_2$. *Then* $X_1 \cup X_2$ *is a* l.o. n-gm *with* l.o. *boundary* $((B_1 \cup B_2) - B) \cup B'$.

The following lemma will be useful in the proof of this theorem. The lemma follows from Theorem 2.

LEMMA 3.4. *Let* D, *a closed subset of a* l.o. n-gm X, *be a* l.o. n-gm *with* l.o. *boundary* D'. *Then* D' *is the common frontier of* X - D *and the interior of* D, *and* D' *fits onto* X - D *as a manifold with boundary.*

This lemma, together with Theorem 1, enables one to construct a proof. The only difficulty will be to show that $p^r(x; X) = 0$ for all $x \in B'$ and all r. This is overcome by considering a diagram similar to that employed in the proof of Theorem 2. The vertical map of the second and third columns can always be chosen to be trivial. (A slight argument is needed for the cases $r = n - 1, n$.) Thus, the composition of the maps of the first column is trivial, and this is precisely what we need to prove.

4. A PROBLEM

Attempts to weaken the local orientability assumptions of Theorem 2 seem to lead to quite difficult problems. In this section, after restricting ourselves to the case where L is a field or the integers, we shall show that the following two assertions are equivalent.

ASSERTION A. *If* X', *a closed subset of a* l.o. n-gm X, *is an* $(n - 1)$-gm, *then* X' *is locally orientable.*

ASSERTION B. *If* X', *a connected* $(n - 1)$-gm, *is a closed subset of a connected* l.o. n-gm X *which is separated by* X', *then* $X - X'$ *is the union of two connected sets both of which have* X' *as frontier.*

For the case where X is compact and L is a field, a proposed proof of Assertion B has been published [9; Theorem 2]. But an essential omission occurs in the argument, and consequently the validity of Assertion B without the assumption that X' is locally orientable remains uncertain.

If we assume the validity of the Assertion A, then the separation theorem (Theorem 2) implies the validity of Assertion B.

Conversely, let us assume the truth of Assertion B, and let X' and X be as in the hypothesis of Assertion A. Let $x \in X'$, and let U be a connected orientable neighborhood of x such that $U \cap X'$ is connected. Since $p^{n-1}(x; X') \neq 0$ $(x \in X')$, we can choose U so that $U - X'$ is not connected, (see Proposition 1.2). Thus, by Assertion B, $U - X'$ is the union of exactly two connected sets both having $U \cap X'$ as their frontiers in U.

Let A be a subset of $U \cap X'$, closed in U. Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & H_c^{n-1}((U \cap X') - A) & & & & \\
 & \swarrow j & & d \searrow & & & \\
H_c^{n-1}(U \cap X') & & \xrightarrow{d_1} & H_c^{n}(U - X') & \xrightarrow{j_1} & H_c^{n}(U) & \rightarrow 0 \\
\downarrow i' & & & \downarrow i & & \downarrow & \\
H_c^{n-1}(A) & & \xrightarrow{d_2} & H_c^{n}(U - A) & \xrightarrow{j_2} & H_c^{n}(U) & \rightarrow 0
\end{array}$$

The horizontal rows are exact, and the first two vertical columns are exact. Let us restrict ourselves momentarily to the case where L is a field.

Since $U \cap X'$ is the common frontier of the two components of $U - X'$, $U - A$ must be connected. Hence

$$j_2 \colon H_c^n(U - A) \rightarrow H_c^n(U) \approx L$$

is an isomorphism, and $H_c^n(U - X') \approx L \oplus L$. The image $d_1 H_c^{n-1}(U \cap X')$ is the kernel of j_1 and is isomorphic to L. Thus $H_c^{n-1}(U \cap X')$ may be written as

$G \oplus$ kernel d_1, where G is isomorphic to L. Since d_2 is trivial, $id_1\, G = 0$. Therefore there exists a $K \subset H_c^{n-1}((U \cap X') - A)$ such that $j(K) \supset G$. Consequently, i' maps G trivially into $H_c^{n-1}(A)$, for every closed proper subset A of $U \cap X'$. It can easily be seen that the following lemma holds:

LEMMA 4.1. *Let* L *be a field. Then a connected* n-gm X *is orientable if and only if there exists a nontrivial subgroup of* $H_c^n(X)$ *whose image is always trivial under the homomorphism* $i: H_c^n(X) \to H_c^n(A)$, *for each closed proper subset* $A \subset X$. (If L is not assumed to be a field, then one must assume that the nontrivial subgroup of $H_c^n(X)$ has no elements of finite order.)

The lemma implies that $U \cap X'$ is an orientable (n - 1)-gm over the field. Therefore, if L is Z, the group of integers, then U and $U \cap X'$ are orientable over every field, because U is orientable over every field and $U' \cap X$ can be shown to be an (n - 1)-gm over every field. Furthermore, since $U \cap X'$ is an (n - 1)-gm over Z, $\dim_Z (U \cap X') < \infty$, and $U \cap X'$ is clc over Z. This implies that $U \cap X'$ is orientable over Z (see for example [6]). Hence Assertions A and B are equivalent.

The equivalence of Assertion A and Assertion B enables us to connect the union theorem (Theorem 1) with what Wilder has called the Converse of the Jordan-Brouwer Separation Theorem. The assumption that an open set of a l. o. n-gm has r-uniform local connectedness (r-ulc) (see [10] for the definition) is equivalent to the assumption that the (n - r)-local co-Betti numbers modulo the complement of the given open set vanish. This is readily seen by employing Poincaré duality to the definition of r-ulc. Thus, [10; Chap. 10, Theorem 3.3] of Wilder and [9; Corollary 6.11] of White may be extended:

THEOREM 5. *If* X' *is a closed subset of a connected* l. o. n-gm X *(over a field), and if* X - X' *is the union of two disjoint connected open* r-ulc *sets* (r = 0, 1, ⋯, n - 2) *of which* X' *is the common frontier, then* X' *is a* l. o. (n - 1)-gm *which fits onto both domains as a manifold with boundary.*

5. AN APPLICATION

THEOREM 6. *Let* C *be a* l. o. n-gm *with* l. o. *boundary* C' *over a field or over the integers. Let* $C = A \times B$. *Then* A *is a* l. o. r_0-gm *with possible* l. o. *boundary* A', *and* B *is a* l. o. s_0-gm *with possible* l. o. *boundary* B'; *moreover*, $r_0 + s_0 = n$.

Proof. Let us consider the case where L is a field. Let $(a_0 \times b_0) \in (C - C') = C^o$. Then $p^r(a_0 \times b_0; C) = \delta_n^r$, where δ_j^i is the Kronecker delta. From the Künneth theorem for cohomology with compact supports, it follows that

$$p^r(a_0; A) = \delta_{r_0}^r \quad \text{and} \quad p^r(b_0; B) = \delta_{s_0}^r \quad (r_0 + s_0 = n).$$

Let B^o be the set of points of B such that $p^n(a_0 \times b; C) = 1$. Let $p^n(a' \times b'; C) = 1$ for some $(a' \times b') \in C^o$. Then

$$p^r(a'; A) = \delta_{r_1}^r \quad \text{and} \quad p^r(b; B) = \delta_{s_1}^r \quad (r_1 + s_1 = n).$$

If $r_1 = r_0$, then $b' \in B^o$. If $r_1 > r_0$, then

$$p^{r_1 + s_0}(a' \times b_0; C) \neq 0,$$

and if $r_1 < r_0$, then

$$p^{r_0+s_1}(a_0 \times b'; C) \neq 0.$$

Since this is impossible, $r_1 = r_0$ and $s_1 = s_0$. Thus, $C^o = A^o \times B^o$, where A^o is defined similarly to B^o. Clearly, A^o is an r_0-gm and B^o is an s_0-gm. Since C^o is a l.o. n-gm, every point $(a_0 \times b_0)$ in C^o has an orientable connected product neighborhood $U_0 \times V_0$ with $U_0 \subset A^o$ and $V_0 \subset B^o$. From the Künneth theorem and the facts that $H^n_c(U_0 \times V_0) \cong L$ and that the cohomology dimensions of U_0 and V_0 are r_0 and s_0, respectively, it easily follows that U_0 and V_0 are orientable neighborhoods for a_0 and b_0, respectively.

Let $A' = A - A^o$ and $B' = B - B^o$. Then

$$C' = (A^o \times B') \cup (A' \times B^o) \cup (A' \times B'),$$

where all three sets are disjoint. Since A^o and B^o are open in A and B, respectively, $A' \times B^o$ and $A^o \times B'$ are open in C'. Therefore A' and B' are an l.o. $(r_0 - 1)$-gm and an l.o. $(s_0 - 1)$-gm, respectively, if they exist. Locally, $A' \times B'$ separates C' into two parts such that $A' \times B'$ is the common frontier. Therefore $A' \times B'$ fits onto $A' \times B^o$ and $A^o \times B'$ as a manifold with boundary. Thus $A' \times B$ is a l.o. (n - 1)-gm with l.o. boundary $A' \times B'$. It is now easily seen that $p^r(b; B) = 0$ for all r and all b in B'. Similarly, using $A \times B'$, we conclude that $p^r(a; A) = 0$, for all r, and for all a in A'.

We have seen that for a given field, A and B are locally orientable generalized manifolds with locally orientable boundaries. Let us assume now that L is Z, the group of integers. Then C is also a l.o. n-gm with l.o. boundary C' over every field. Therefore, by the argument above, A and B are locally orientable generalized manifolds with locally orientable boundaries over every field.

Let $(a \times b) \in A \times B$, and let O be any neighborhood of $a \times b$. Since $A \times B$ is clc over Z, we may choose *compact* connected neighborhoods U' and U of a $(U \supset U')$ and V' and V of b $(V \supset V')$ such that $U \times V \subset O$ and $i^*: H^*(U \times V) \to H^*(U' \times V')$ is trivial except in dimension 0. The Künneth theorem implies that the diagram

$$\begin{array}{ccccc} 0 \to & H^r(U) \otimes H^0(V) & \to & H^r(U \times V) \\ & \downarrow & & \downarrow i^* \\ 0 \to & H^r(U') \otimes H^0(V') & \to & H^r(U' \times V') \end{array}$$

is commutative and that its rows are exact. Because i^* is trivial, the map $i^*: H^r(U) \to H^r(U')$ is obviously trivial. Therefore A is clc over Z. Similarly, we can show that A', B, and B' are clc over Z.

We shall show that

$$\dim_Z A = \dim_F A, \quad \dim_Z A' = \dim_F A', \quad \dim_Z B = \dim_F B, \quad \dim_Z B' = \dim_F B',$$

where F is any field. Clearly, it suffices to show this equality for F equal to Q, the field of rational numbers, and for F equal to Z_p, the field of integers modulo a prime p (for all primes p). Suppose that

$$\dim_Q A^o = m_0 \quad \text{and} \quad \dim_{Z_p} A^o = m_p \neq m_0,$$

for some prime p. Then there exist integers $n_0 = \dim_Q B^o$ and $n_p = \dim_{Z_p} B^o$ such that $m_0 + n_p > n$, if $m_0 > m_p$, and $m_p + n_0 > n$, if $m_0 < m_p$. Let us assume that $m_0 > m_p$. Choose connected open sets U', U, V', V $((U')^- \subset U \subset A^o,\ (V')^- \subset V \subset B^o)$ such that $(U')^-$ and $(V')^-$ are compact and $U \times V$ is contained within an orientable part of $A^o \times B^o$. Since A^o and B^o are clc over Z, $H_c^*(U' \subset U)$ and $H_c^*(V' \subset V)$ are finitely generated. The universal coefficient theorem implies that $H_c^{m_0}(U' \subset U) = G$, where G is isomorphic to the direct sum of Z and a finite torsion group. The commutative diagram

$$\begin{array}{ccc} H_c^{m_0}(U; Z) \otimes H_c^{n_p}(V; Z) & \to & G \otimes H_c^{n_p}(V; Z) \to 0 \\ & \searrow & \downarrow \\ & & H_c^{m_0}(U'; Z) \otimes H_c^{n_p}(V; Z) \end{array}$$

implies that $H_c^{m_0}(U; Z) \otimes H_c^{n_p}(V; Z) \neq 0$ if $H^{n_p}(V; Z) \neq 0$. From the Künneth theorem we have the exact sequence

$$0 \to H_c^{m_0}(U) \otimes H_c^{n_p}(V) \to H_c^{m_0+n_p}(U \times V).$$

If $H_c^{n_p}(V) \neq 0$, then this would lead to a contradiction, since $m_0 + n_p > n$. Therefore, $m_0 \leq m_p$. That m_0 cannot be less than m_p is the result of a similar argument applied to the inequality $m_p + n_0 > n$. It only remains to show that $H_c^{n_p}(V) \neq 0$. Since $H_c^{n_p}(V; Z_p) \neq 0$, the universal coefficient theorem implies that either $H_c^{n_p}(V; Z) \neq 0$ or $H_c^{n_p+1}(V; Z) \neq 0$. In case $H_c^{n_p}(V; Z) = 0$, we could have used $H_c^{n_p+1}(V; Z)$ in the diagram above. Thus $\dim_Z A^o = \dim_F A^o$, for every field F. Similarly, one can show that

$$\dim_Z A' = \dim_F A', \qquad \dim_Z B^o = \dim_F B^o, \qquad \dim_Z B' = \dim_F B'.$$

Since C and C' are locally orientable over Z, we may choose sufficiently small open sets in A^o, B^o, A', and B' that are orientable over every field. This, together with the fact that A^o, A', B^o, and B' are clc over Z suffices to show that they are locally orientable generalized manifolds over Z (see for example [6]). The universal coefficient theorem now easily yields that $p^r(a; A) = p^r(b; B) = 0$ for $a \in A'$, $b \in B'$, and all r, and with integer coefficients. This completes the proof.

Of course, the converse of the theorem is also true, and the proof is easier. Theorem 6 was first stated by the author in an abstract submitted to the American Mathematical Society (NOTICES, 5 (1958), pp. 298-299), for coefficients in a field and without boundary. The present form of Theorem 6 (but without boundary) was cited in [6], but a proof was not given at that time. The converse, with L a field and without boundary, was first proved by T. R. Brahana.

6. APPENDIX

LEMMA 6.1. *Let* X_1, X_2, *and* F *be closed subsets of a compact space* X, *such that* $X = X_1 \cup X_2$ *and* $X_1 \cap X_2 \supset F$. *Then the modified Mayer-Vietoris cohomology sequence*

$$\to H^{p-1}(X_1 \cap X_2, F) \xrightarrow{\Delta} H^p(X, F) \xrightarrow{\phi} H^p(X_1, F) \oplus H^p(X_2, F) \xrightarrow{\psi} H^p(X_1 \cap X_2, F) \xrightarrow{\Delta}$$

is exact.

COROLLARY 6.2. *Let* X_1 *and* X_2 *be closed subsets of a locally compact space* X *such that* $X_1 \cup X_2 = X$. *Then the sequence*

$$H_c^{p-1}(X_1 \cap X_2) \xrightarrow{\Delta} H_c^p(X) \xrightarrow{\phi} H_c^p(X_1) \oplus H_c^p(X_2) \xrightarrow{\psi} H_c^p(X_1 \cap X_2) \xrightarrow{\Delta}$$

is a modified Mayer-Vietoris sequence and is exact.

Note that the sequence in Lemma 6.1 differs from the relative Mayer-Vietoris sequence [5; 15.6c, p. 44], which yields a Mayer-Vietoris sequence for cohomology with compact supports exactly as that in Corollary 6.2, except that X_1 and X_2 are assumed to be open subsets of X instead of closed subsets (see [2]).

LEMMA 6.3. *Consider the commutative diagram of Abelian groups*

$$\begin{array}{ccccc} & & B_1 & \to & C_1 \\ & & \downarrow{\scriptstyle i_1} & & \downarrow \\ A_2 & \to & B_2 & \to & C_2 \\ \downarrow & & \downarrow{\scriptstyle i_2} & & \\ A_3 & \to & B_3. & & \end{array}$$

Suppose that the vertical maps of the first and third columns are trivial and the second horizontal row is exact. Then the vertical map $i_2 \circ i_1 \colon B_1 \to B_3$ *is trivial.*

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