A UNIQUENESS THEOREM ON TWO-DIMENSIONAL RIEMANNIAN MANIFOLDS WITH BOUNDARY

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1. INTRODUCTION

The purpose of this paper is to establish the following

THEOREM. *Let* M_2 *and* M_2^* *be two oriented two-dimensional Riemannian manifolds of class* C^2 *imbedded in a Euclidean space* E_{N+2} *of dimension* $N+2$ $(N > 0)$, *with boundaries* C *and* C*, *respectively, and with positive Gaussian curvatures in every normal direction. Suppose that there exists an orientation-preserving differentiable homeomorphism* H *of the manifold* M_2 *onto the manifold* M_2^* *such that at corresponding points the manifolds* M_2 *and* M_2^* *have parallel tangent planes and equal sums of the principal radii of curvature associated with every common normal direction. If the homeomorphism* H *restricted to the boundary* C *is a translation (strictly speaking: is induced by a translation in the space* E_{N+2}*) carrying the boundary* C *onto the boundary* C*, *then the homeomorphism* H *is a translation carrying the whole manifold* M_2 *onto the whole manifold* M_2^*.

For the case where $N = 1$ and the boundaries C and C* of the two manifolds are empty, this theorem was proved by Christoffel [3]; for the general case where $N = 1$, it is due to the author [4]. The method used in this paper is an extension of that used by Chern [2] in proving the uniqueness theorem for Minkowski's problem for closed convex surfaces imbedded in a three-dimensional Euclidean space. (We recall that, by a result of Nash [6] on C^3 isometric imbeddings, every two-dimensional Riemannian manifold with a C^3 positive metric can be imbedded in some Euclidean space.)

2. PRELIMINARIES

Let M_2 be an orientable two-dimensional Riemannian manifold of class C^3 imbedded in a Euclidean space E_{N+2} of dimension $N + 2$ $(N > 0)$. To avoid confusion, we shall use the following ranges of indices throughout this paper:

(2.1) $$\alpha, \beta = 1, 2; \quad 3 \leq r \leq N+2; \quad 1 \leq i, j, k \leq N+2.$$

Associated with a point P in the space E_{N+2} we introduce a right-handed rectangular frame $Pe_1 \cdots e_{N+2}$ such that $e_1, \cdots, e_{N+2}$ form an ordered set of mutually perpendicular unit vectors with the determinant $(e_1, \cdots, e_{N+2})$ equal to +1. Let X denote the position vector of the point P with respect to a fixed point O in the space E_{N+2}; then we can write

(2.2) $$dX = \sum_i \omega_i e_i, \qquad de_i = \sum_j \omega_{ij} e_j,$$

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where ω_i and ω_{ij} are Pfaffian forms in the manifold of frames and satisfy

$$\omega_{ij} + \omega_{ji} = 0. \tag{2.3}$$

Since the exterior derivative of an exact differential is zero, $d(dX) = 0$ and $d(de_i) = 0$, and therefore we have, from equations (2.2),

$$\begin{aligned} d\omega_i &= \sum_j \omega_j \wedge \omega_{ji}, \\ d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj}, \end{aligned} \tag{2.4}$$

where $\wedge$ denotes the exterior product.

To study the manifold M_2, we consider the submanifold of the frames $Pe_1 \cdots e_{N+2}$ such that $P \in M_2$ and e_1, e_2 are two tangent vectors of the manifold M_2 at the point P. Denoting by the same symbols the forms on this submanifold of frames induced by the identity mapping, we have

$$\omega_r = 0, \tag{2.5}$$

and therefore, from the first of equations (2.4),

$$d\omega_r = \sum_\alpha \omega_\alpha \wedge \omega_{\alpha r} = 0. \tag{2.6}$$

By a Lemma of E. Cartan (see, for instance, [1], p. 117) on exterior algebra, equation (2.6) implies that for each value of r

$$\omega_{\alpha r} = \sum_\beta A_{r\alpha\beta}\omega_\beta \tag{2.7}$$

with the conditions

$$A_{r\alpha\beta} = A_{r\beta\alpha}. \tag{2.8}$$

Let $Oi_1 \cdots i_{N+2}$ be a frame in the space E_{N+2} such that $i_1, \cdots, i_{N+2}$ form an ordered set of mutually perpendicular unit vectors at a point O. With respect to this frame we define the vector product of $N + 1$ vectors $A_1, \cdots, A_{N+1}$ through the point O in the space E_{N+2} to be the vector A_{N+2} (denoted by $A_1 \times \cdots \times A_{N+1}$) satisfying the following conditions:

(a) the vector A_{N+2} is perpendicular to the $(N + 1)$-dimensional subspace of the space E_{N+2} spanned by the vectors $A_1, \cdots, A_{N+1}$,

(b) the magnitude of the vector A_{N+2} is equal to the volume of the parallelepiped whose edges are the vectors $A_1, \cdots, A_{N+1}$,

(c) the two frames $OA_1 \cdots A_{N+1} A_{N+2}$ and $Oi_1 \cdots i_{N+2}$ have the same orientation.

Let σ be a permutation on the $N + 1$ numbers $1, \cdots, N + 1$; then

$$A_{\sigma(1)} \times \cdots \times A_{\sigma(N+1)} = (\operatorname{sgn} \sigma) A_1 \times \cdots \times A_{N+1},$$

where sgn σ is +1 or -1 according as the permutation σ is even or odd. Furthermore, the scalar product of the vector $A_1 \times \cdots \times A_{N+1}$ and a vector I through the point O in the space E_{N+2} is given by

$$I \cdot (A_1 \times \cdots \times A_{N+1}) = (-1)^{N+1} (I, A_1, \cdots, A_{N+1}). \tag{2.9}$$

From equation (2.9) it follows that

$$e_1 \times \cdots \times e_{r-1} \times e_{r+1} \times \cdots \times e_{N+2} = (-1)^{N+r} e_r. \tag{2.10}$$

Let dA be the area element of the manifold M_2 at a point P, and let K_r and H_r, respectively, be the Gaussian curvature and the mean curvature of the manifold M_2 associated with the normal vector e_r at the point P. Then, by means of the combined operation $\otimes$ of the vector product $\times$ and the exterior product $\wedge$ (for this operation $\otimes$ see, for instance, [5]), we obtain

$$dX \otimes dX \otimes e_3 \otimes \cdots \otimes e_{r-1} \otimes e_{r+1} \otimes \cdots \times e_{N+2} = (-1)^{N+r} 2 e_r dA, \tag{2.11}$$

$$de_r \otimes de_r \otimes e_3 \otimes \cdots \otimes e_{r-1} \otimes e_{r+1} \otimes \cdots \otimes e_{N+2} = (-1)^{N+r} 2K_r e_r dA, \tag{2.12}$$

$$dX \otimes de_r \otimes e_3 \otimes \cdots \otimes e_{r-1} \otimes e_{r+1} \otimes \cdots \otimes e_{N+2} = (-1)^{N+r+1} 2H_r e_r dA. \tag{2.13}$$

From equations (2.2, (2.5), (2.10), (2.11), (2.12), (2.13) it follows that

$$dA = \omega_1 \wedge \omega_2, \tag{2.14}$$

$$K_r dA = \omega_{r1} \wedge \omega_{r2}, \tag{2.15}$$

$$2H_r dA = \omega_{r2} \wedge \omega_1 - \omega_{r1} \wedge \omega_2. \tag{2.16}$$

It is known that the vector $\mathfrak{H} = \sum_r H_r e_r$ is independent of the choice of the mutually perpendicular unit vectors $e_3, \cdots, e_{N+2}$ in the normal space of the manifold M_2 at the point P; the vector $\mathfrak{H}$ and its magnitude are respectively called the mean curvature vector and the mean curvature of the manifold M_2 at the point P.

3. AN INTEGRAL FORMULA

Suppose that M_2 and M_2^* are two orientable two-dimensional Riemannian manifolds of class C^3 imbedded in a Euclidean space E_{N+2} of dimension $N + 2$ $(N > 0)$ with boundaries C and C*, respectively, and with positive Gaussian curvatures in every normal direction. Furthermore, suppose that there exists an orientation-preserving differentiable homeomorphism H of the manifold M_2 onto the manifold M_2^* such that at corresponding points the manifolds M_2 and M_2^* have parallel tangent planes. Then the definitions in Section 2 can be applied to the manifold M_2; and for the corresponding quantities and equations for the manifold M_2^* we shall use the same symbols and numbers with a star, respectively.

By using equations (2.2), (2.3), (2.5), (2.9), (2.10), (2.16), (2.5)* and the first of equations (2.2)*, and applying the ordinary rules for differentiation of determinants, we can obtain the differential form

$$d(X^*, dX, e_3, \cdots, e_{N+2}) = (-1)^{N+1} e_3 \cdot (dX^* \otimes dX \otimes e_4 \otimes \cdots \otimes e_{N+2})$$

$$+ (-1)^{N+1} X^* \cdot \sum_r dX \otimes e_3 \otimes \cdots \otimes e_{r-1} \otimes de_r \otimes e_{r+1} \otimes \cdots \otimes e_{N+2} \tag{3.1}$$

$$= \omega_1^* \wedge \omega_2 - \omega_2^* \wedge \omega_1 - 2 \sum_r p_r^* H_r \, dA,$$

where

$$p_r^* = X^* \cdot e_r. \tag{3.2}$$

Integrating both sides of equation (3.1) over the manifold M_2 and applying Stokes' Theorem to the left side, we then arrive at the integral formula

$$\int_C (X^*, dX, e_3, \cdots, e_{N+2}) = \iint_{M_2} (\omega_1^* \wedge \omega_2 - \omega_2^* \wedge \omega_1) - 2 \iint_{M_2} \sum_r p_r^* H_r \, dA. \tag{3.3}$$

4. PROOF OF THE THEOREM

First we should notice that the given differentiable homeomorphism H between the manifolds M_2 and M_2^* induces a homeomorphism between the two frames $Pe_1 \cdots e_{N+2}$ and $P^*e_1 \cdots e_{N+2}$ at two corresponding points P and P^* of the manifolds M_2 and M_2^*, whence

$$\omega_{r\alpha}^* = \omega_{r\alpha}. \tag{4.1}$$

From equations (2.7), (2.14), and (2.15) it follows that

$$K_r = A_{r11} A_{r22} - A_{r12} A_{r21}; \tag{4.2}$$

this and the assumption that $K_r > 0$ for each value of r imply that the matrix $(A_{r\alpha\beta})$ is nonsingular for each value of r. Therefore by equations (2.7) and (2.8) we can write, for any value of r,

$$\omega_\alpha = \sum_\beta \lambda_{r\alpha\beta} \omega_{\beta r}, \tag{4.3}$$

where $(\lambda_{r\alpha\beta})$ is the inverse matrix of $(A_{r\alpha\beta})$ and where

$$\lambda_{r\alpha\beta} = \lambda_{r\beta\alpha}. \tag{4.4}$$

Using equations (2.3), (2.15), (2.16), (4.1), (4.3), (4.4), (4.3)*, (4.4)*, we find immediately that

$$2H_r = (\lambda_{r11} + \lambda_{r22}) K_r,$$

$$\omega_1^* \wedge \omega_2 - \omega_2^* \wedge \omega_1 = (\lambda_{r11}^* \lambda_{r22} - 2\lambda_{r12}^* \lambda_{r12} + \lambda_{r22}^* \lambda_{r11}) \omega_{r1} \wedge \omega_{r2}. \tag{4.5}$$

The first of equations (4.5) implies that the sum of the principal radii of curvature at the point P of the manifold M_2, which are associated with the normal vector e_r, is equal to

(4.6) $$2H_r/K_r = \lambda_{r11} + \lambda_{r22}.$$

From the assumption and from equations (4.6) and (4.6)* we thus obtain

(4.7) $$\lambda^*_{r11} + \lambda^*_{r22} = \lambda_{r11} + \lambda_{r22}.$$

Substituting equations (2.15), (4.4), (4.5) in equation (3.3), we have

(4.8)
$$\int_C (X^*, dX, e_3, \cdots, e_{N+2})$$
$$= \iint_{M_2} (\lambda^*_{r11}\lambda_{r22} - 2\lambda^*_{r12}\lambda_{r12} + \lambda^*_{r22}\lambda_{r11})\,\omega_{r1}\wedge\omega_{r2}$$
$$- \iint_{M_2} \sum_r p^*_r(\lambda_{r11} + \lambda_{r22})\,\omega_{r1}\wedge\omega_{r2}.$$

The replacement of the manifold M_2 by the manifold M_2^* in equation (4.8), together with the use of equation (4.1), gives

(4.9)
$$\int_C (X^*, dX^*, e_3, \cdots, e_{N+2})$$
$$= 2\iint_{M_2} (\lambda^*_{r11}\lambda^*_{r22} - \lambda^{*2}_{r12})\,\omega_{r1}\wedge\omega_{r2} - \iint_{M_2} \sum_r p^*_r(\lambda^*_{r11} + \lambda^*_{r22})\,\omega_{r1}\wedge\omega_{r2}.$$

Subtracting equation (4.8) from equation (4.9), and using equation (4.7) and the assumption that $dX^* = dX$ along the boundary C, we obtain

(4.10) $$\iint_{M_2}[2(\lambda^*_{r11}\lambda^*_{r22} - \lambda^{*2}_{r12}) - (\lambda^*_{r11}\lambda_{r22} - 2\lambda^*_{r12}\lambda_{r12} + \lambda^*_{r22}\lambda_{r11})]\,\omega_{r1}\wedge\omega_{r2} = 0.$$

Subtracting equation (4.10) from the one obtained by interchanging the roles of the two manifolds M_2 and M_2^* in equation (4.10), we get

(4.11) $$\iint_{M_2} (\lambda^*_{r11}\lambda^*_{r22} - \lambda^{*2}_{r12})\,\omega_{r1}\wedge\omega_{r2} = \iint_{M_2} (\lambda_{r11}\lambda_{r22} - \lambda_{r12}^2)\,\omega_{r1}\wedge\omega_{r2}.$$

Thus the substitution of equation (4.11) in equation (4.10) yields

(4.12) $$\iint_{M_2}[(\lambda^*_{r11} - \lambda_{r11})(\lambda^*_{r22} - \lambda_{r22}) - (\lambda^*_{r12} - \lambda_{r12})^2]\,\omega_{r1}\wedge\omega_{r2} = 0.$$

But

$$(\lambda^*_{r11} - \lambda_{r11})(\lambda^*_{r22} - \lambda_{r22})$$

$$= \frac{1}{2}[(\lambda^*_{r11} - \lambda_{r11}) + (\lambda^*_{r22} - \lambda_{r22})]^2 - \frac{1}{2}[(\lambda^*_{r11} - \lambda_{r11})^2 + (\lambda^*_{r22} - \lambda_{r22})^2],$$

which is reduced, by means of equation (4.7), to

$$(4.13) \qquad (\lambda^*_{r11} - \lambda_{r11})(\lambda^*_{r22} - \lambda_{r22}) = -\frac{1}{2}[(\lambda^*_{r11} - \lambda_{r11})^2 + (\lambda^*_{r22} - \lambda_{r22})^2] .$$

By the assumption and equation (2.15), $\omega_{r1} \wedge \omega_{r2} > 0$. Thus the integrand in equation (4.12) is nonpositive, and therefore equation (4.12) holds when and only when

$$(4.14) \qquad \lambda^*_{r\alpha\beta} = \lambda_{r\alpha\beta} \qquad (\alpha, \beta = 1, 2).$$

Using equations (4.14), (4.3), (4.3)*, we obtain

$$(4.15) \qquad \omega^*_\alpha = \omega_\alpha \qquad (\alpha = 1, 2).$$

From equations (4.15), (2.2), (2.5), (2.2)*, (2.5)* it follows that dX* = dX over the whole manifold M_2, and hence the proof of the theorem is complete.

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