LUSIN'S THEOREM ON AREAS OF CONFORMAL MAPS

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1. INTRODUCTION

If the function $f(z)$ is of class H_2, then there exists a convex domain D whose boundary is tangent from the interior to the unit circle C at $z = 1$ and which has the following property: the area $A(f, D, \theta)$ of the Riemann surface onto which the function $w = f(e^{i\theta}z)$ maps the domain D is an integrable function of θ. We shall refer to this proposition as Lusin's theorem. Lusin actually claimed a little less; he did not assert that $A(f, D, \theta)$ is an integrable function, merely that it is finite for almost all θ; but his proof [2, pp. 139-149] clearly establishes the stronger proposition.

In Section 2 we give a brief proof of Lusin's theorem. Conceptually, our proof is identical with that of Lusin; technically it is somewhat simpler, because of a profitable reversal of order in an iterated integral. In addition, our slight modification yields a converse of Lusin's theorem.

In a second proof of the theorem, we construct the domain D in terms of the Taylor coefficients of the function f. We also show that even the weak form of Lusin's theorem becomes false if the hypothesis of bounded mean square modulus is replaced by the hypothesis of slow growth of the maximum modulus; also that the domain D in Lusin's theorem can not be chosen independently of the function f.

Lusin [2, p. 151] conjectured that the property of the function f in his theorem is essentially a local rather than a global property. This is indeed the case: Let f be meromorphic in $|z| < 1$; then, for almost all points $e^{i\theta}$ at which the cluster set of f for nontangential approach is not identical with the entire plane, there exists a convex domain $D(\theta)$, touching C at $e^{i\theta}$, such that the image of $D(\theta)$ under f has finite area.

In Section 3, we consider the exceptional set relative to Lusin's theorem, that is, the set of points $e^{i\theta}$ for which $A(f, D, \theta) = \infty$ for every convex domain D in $|z| < 1$ which touches C at $z = 1$. Lusin stated [2, p. 142] that even if f is continuous in $|z| \leq 1$, the exceptional set need not be empty; we illustrate this statement with a simple example.

On the other hand, if the Taylor series of f converges absolutely on C, then the exceptional set is empty; in fact, here the quantity $A(f, D, \theta)$ is a continuous function of θ, for some convex domain D touching the unit circle. This result heightens the remarkable character of the theorems which assert that certain Taylor series converging absolutely on C map C onto a Peano curve (see Salem and Zygmund [4] and Schaeffer [5]).

In Section 4, we waive the requirement that D be convex and that its boundary have a tangent; of the function f we require only that it be holomorphic in $|z| < 1$. From the fact that, for $f(z) = \Sigma a_n z^n$ and $0 \leq r < 1$,

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$$|f'(re^{i\theta})| \leq \sum n|a_n|r^{n-1} = G(r) < \infty,$$

it follows that if f is holomorphic in $|z| < 1$, there exists a domain D with a boundary point at $z = 1$ such that $A(f, D, \theta) < 1$ for all θ. On the other hand, we show that no matter with how narrow a tongue the domain D approaches the unit circle, there exists a function f, holomorphic in $|z| < 1$, such that $A(f, D, \theta) = \infty$ for every θ.

Definitions. A simply connected domain D is a *boundary domain* if it is contained in $|z| < 1$ and has the point $z = 1$ as its only boundary point on $|z| = 1$. A convex boundary domain is a *tangential domain* if the line $x = 1$ is the only straight line through $z = 1$ which does not intersect it. If D is a boundary domain, D_θ denotes the set of points $z = \zeta e^{i\theta}$ $(\zeta \in D)$.

Preliminary computations. It follows at once from the definitions that, for any function f meromorphic in $|z| < 1$ and any boundary domain D,

$$A(f, D, \theta) \equiv \iint_{D_\theta} |f'|^2 \, d\sigma .$$

Now let the boundary domain D be given by the relations

$$z = re^{i\phi}, \quad r_0 < r < 1, \quad \lambda_1(r) < \phi < \lambda_2(r), \tag{1}$$

and let

$$\lambda(r) = \lambda_2(r) - \lambda_1(r)$$

(we shall assume, throughout this paper, that $\lambda(r) < 2\pi$). Then

$$\int_0^{2\pi} A(f, D, \theta)\, d\theta = \int_0^{2\pi} \int_{r_0}^{1} \int_{\lambda_1}^{\lambda_2} |f'(re^{i(\theta+\phi)})|^2 \, r \, d\phi \, dr \, d\theta .$$

Since the integrand is nonnegative, the order of integration can be changed, and it follows that if $f(z) = \sum a_n z^n$, then

$$\begin{aligned} \int_0^{2\pi} A(f, D, \theta)\, d\theta &= \int_{r_0}^{1} \int_{\lambda_1}^{\lambda_2} \int_0^{2\pi} |f'(re^{i(\theta+\phi)})|^2 \, r \, d\theta \, d\phi \, dr \\ &= 2\pi \sum n^2 |a_n|^2 \int_{r_0}^{1} \lambda(r) r^{2n-1} \, dr . \end{aligned} \tag{2}$$

In particular, let the boundary domain D be a triangle. For our purposes, there will be no loss of generality in supposing that no circle $|z| = r$ meets any of the sides of D more than once. Then D can be represented in the form (1), and the relation

$$(1 - r)\, C_1 < \lambda(r) < (1 - r)\, C_2$$

holds for all r sufficiently near to 1 (the positive constants C_1 and C_2 depend on D). It follows that, in the case of a triangular boundary domain,

$$C_3/n^2 < \int_{r_0}^{1} \lambda(r)\, r^{2n-1}\, dr < C_4/n^2,$$

that is,

(3) $$C_3 \sum |a_n|^2 \leq \int_0^{2\pi} A(f, D, \theta)\, d\theta \leq C_4 \sum |a_n|^2.$$

2. LUSIN'S THEOREM

THEOREM 1. *Let* f *be holomorphic in* $|z| < 1$. *Then the relation*

$$\int_0^{2\pi} A(f, D, \theta)\, d\theta < \infty$$

holds for every triangular boundary domain D *if* f *is of class* H_2, *and for no such domain if* f *is not of class* H_2.

This theorem follows immediately from the estimate (3).

THEOREM 2. *If* $f \in H_2$, *there exists a tangential domain* D *such that*

$$\int_0^{2\pi} A(f, D, \theta)\, d\theta < \infty .$$

This result can be deduced from Theorem 1 by a standard method (see [1, p. 149]). We give an independent, more constructive proof.

Let $f(z) = \Sigma a_n z^n$; since $\Sigma |a_n|^2 < \infty$, there exists a sequence $\{\omega_n\}$ $(\omega_1 > 1, \omega_n \nearrow \infty)$ such that

(4) $$\sum |a_n|^2 \omega_n < \infty .$$

For $1 \leq t < \infty$ we can define a function $\alpha(t)$ such that $\alpha(t)/t < \pi$ and

(5) $$\alpha(1) = 1, \quad \alpha(n) < \omega_n,$$

and such that $\alpha(t) \nearrow \infty$ and $\alpha(t)/t \searrow 0$ as $t \to \infty$. Now let

$$\psi(r) = \alpha\left(\frac{1}{1-r}\right) \quad (0 \leq r < 1) .$$

Then

(6) $$\psi(r) \nearrow \infty \quad (r \to 1)$$

and

(7) $$(1 - r)\psi(r) \searrow 0 \quad (r \to 1).$$

By (6), the set D composed of all points $z = re^{i\phi}$ with

$$0 < r < 1, \qquad |\phi| < (1 - r)\psi(r)$$

contains a tangential domain D*. It will therefore be sufficient to prove that

$$\int_0^{2\pi} A(f, D, \theta)\, d\theta < \infty.$$

Now

$$\int_0^{2\pi} A(f, D, \theta)\, d\theta = \iint_D \left(\int_0^{2\pi} |f'(re^{i(\theta + \phi)})|^2 \, d\theta\right) d\sigma$$

$$= 2\pi \iint_D \sum_1^\infty n^2 |a_n|^2 r^{2n-2}\, d\sigma$$

$$= 4\pi \sum n^2 |a_n|^2 \left(\int_0^{1-n^{-1}} + \int_{1-n^{-1}}^1\right) r^{2n-1} (1 - r)\psi(r)\, dr.$$

By (6),

$$\int_0^{1-n^{-1}} r^{2n-1} (1 - r)\psi(r)\, dr < \psi(1 - n^{-1}) \int_0^1 (r^{2n-1} - r^{2n})\, dr$$

$$= \alpha(n)[1/2n - 1/(2n + 1)]$$

$$< \alpha(n)/n^2;$$

and by (7),

$$\int_{1-n^{-1}}^1 r^{2n-1} (1 - r)\, \psi(r)\, dr < n^{-1}\psi(1 - n^{-1}) \int_{1-n^{-1}}^1 dr$$

$$= \alpha(n)/n^2.$$

Therefore $\int_0^{2\pi} A(f, D, \theta)\, d\theta \leq 8\pi \Sigma |a_n|^2 \alpha(n)$, and the result follows from (4) and (5).

THEOREM 3. *If* $\mu(r) > 0$ *and* $\mu(r) \to \infty$ *as* $r \to 1$, *there exists a function* f, *holomorphic in* $|z| < 1$, *such that*

$$|f(re^{i\phi})| < \mu(r) \qquad (0 \leq r < 1,\ 0 \leq \phi < 2\pi)$$

and $A(f, D, \theta) = \infty$ *for all* θ *and all triangular boundary domains* D.

To prove the theorem, we construct a function of the form

$$f(z) = \sum z^{n_k} \qquad (n_{k+1} > 2n_k).$$

It is clear that if the exponents n_k tend to infinity fast enough as $k \to \infty$, then $|f(re^{i\phi})| < \mu(r)$, for all ϕ and all r in $0 \leq r < 1$. Also, if the n_k grow fast enough as $k \to \infty$, then

$$|f'(re^{i\phi})| > n_k/2e$$

in the annulus $1 - 1/n_k < r < 1 - 1/2n_k$. Since the area of the intersection of this annulus and an arbitrary but fixed triangular boundary domain is of the order of magnitude $1/n_k^2$, the theorem is proved.

The following theorem generalizes a result of Lohwater and Piranian [1, Theorem 1].

THEOREM 4. *If* D *is a tangential domain, there exists a function* $f(z) = \Sigma a_n z^n$ $(\Sigma |a_n| < \infty)$ *such that* $A(f, D, \theta) = \infty$ *for all* θ.

If D is a tangential domain, it contains a region of the form (1), with $\lambda(r)$ decreasing for $r > r_0$, and such that $\lambda(r)/(1 - r) \to \infty$ as $r \to 1$. Let $\{a_k\}_1^\infty$ be a sequence of positive constants, with $\Sigma a_k < \infty$, and let $f(z) = \Sigma a_k z^{n_k}$, where $\{n_k\}$ is an increasing sequence of positive integers $(n_1 > (1 - r_0)^{-1})$. If $n_k \to \infty$ fast enough, then

$$|f'(re^{i\phi})| > n_k a_k/2e$$

throughout the annulus $1 - 1/n_k < r < 1 - 1/2n_k$. It follows that

$$A(f, D, \theta) \geq \sum_{k=1}^{\infty} \frac{n_k^2 a_k^2}{4e^2} \int_{1-1/n_k}^{1-1/2n_k} \int_0^{\lambda(r)} r \, d\phi \, dr$$

$$\geq \sum_{k=1}^{\infty} \frac{n_k^2 a_k^2}{4e^2} \lambda\left(1 - \frac{1}{2n_k}\right)/3n_k$$

$$= \Sigma \; a_k^2 n_k \lambda \left(1 - \frac{1}{2n_k}\right)/12e^2 .$$

If $n_k \to \infty$ rapidly enough, the last series diverges; this completes the proof.

THEOREM 5. *If the function* f *is meromorphic in* $|z| < 1$, *then there exists a set* E *on the unit circle, of measure* 2π, *such that for each point* $e^{i\theta}$ *in* E *one of the following two statements holds:*

i) *if* A *is an angle in* $|z| < 1$ *with its vertex at* $e^{i\theta}$, *the set of cluster values of* f *at* $e^{i\theta}$ *(for approach in* A*) constitutes the entire plane;*

ii) *there exists a convex domain* $D(\theta)$ *whose boundary is tangent to* $|z| = 1$ *at* $e^{i\theta}$ *and whose image under* f *is a Riemann surface of finite area.*

Before proving this theorem, we point out that if f is of bounded characteristic in $|z| < 1$, then condition (ii) is satisfied for almost all $e^{i\theta}$. The present result therefore goes considerably beyond Lusin's theorem. We also call attention to an unsolved problem. Let $w = P(t)$ $(0 \leq t < \infty)$ be a curve which is everywhere dense in the finite plane and whose arcs $0 \leq t \leq T < \infty$ are all rectifiable, and let R be a ribbon of finite area which contains the curve $w = P(t)$. If f maps the unit disc conformally onto the Riemann surface R, there exists a point $e^{i\theta}$ at which both (i) and (ii) hold. It is easy to modify the construction so that the set of those points at which both (i) and (ii) hold has the power of the continuum on every arc of C. The question remains whether there exists an analytic function f such that the set where both (i) and (ii) hold has positive measure.

To prove the theorem, we use a result of Plessner [3, Theorem 1]: if f is meromorphic in $|z| < 1$ then, at almost all points for which (i) does not hold, the non-tangential limit of f exists and has a finite value $f(e^{i\theta})$. We denote by E^* the set of points where this second condition is satisfied; generally, we use the symbol $|E|$ to represent the Lebesgue measure of a set E.

For $n = 1, 2, \cdots$, let $E_1(n)$ denote the subset of E^* where $|f(e^{i\theta})| < n$. Then $\lim |E_1(n)| = |E^*|$. Let T be a triangular boundary domain, and for $0 < r < 1$ let $E(n, r)$ denote the subset of points $e^{i\theta}$ in $E_1(n)$ for which $|f(z)| < n$ throughout the intersection of T_θ with the annulus $r < |z| < 1$. Since

$$\lim_{r\to 1} |E(n, r)| = |E(n)| ,$$

there exist constants r_n $(n = 1, 2, \cdots)$ such that

$$\lim_{n\to\infty} |E(n, r_n)| = |E^*| .$$

Moreover, there exist closed subsets $E_2(n)$ $(n = 1, 2, \cdots)$ of $E(n, r_n)$ such that

$$\lim_{n\to\infty} |E_2(n)| = |E^*| . \tag{8}$$

Now let R_n denote the intersection of the annulus $r_n < |z| < 1$ with the union of all the sets T_θ $(\theta \in E_2(n))$. Then R_n it the union of finitely many simply connected domains R_{nk} $(k = 1, 2, \cdots, k_n)$ in which $|f(z)| < n$ and which are bounded by rectifiable Jordan curves C_{nk} (see Figure 1). Let $w_{nk} = w = g(z)$ map the fixed component R_{nk} conformally onto the disc $|w| < 1$, and let $F(w) = f(g^{-1}(w))$ $(|w| < 1)$. Since $|F(w)| < n$ there exists, by Theorem 2, a tangential domain D in $|w| < 1$ such that $A(F, D, \phi) < \infty$ for almost all ϕ. Since C_{nk} is rectifiable, the function $z = g^{-1}(w)$ maps sets of measure zero on $|w| = 1$ into sets of measure zero on C_{nk}. It follows that, for almost all points p on C_{nk}, the function f(z) maps the domain $g^{-1}(D_{g(p)})$ onto a Riemann surface of finite area. Moreover, the curve C_{nk} has a tangent almost everywhere; and (see Schlesinger [6], §47, especially page 161) if C_{nk} has a tangent at p, then the boundary of $g^{-1}(D_{g(p)})$ has a tangent at p. Therefore, for almost all points p of $E_2(n) \cap C_{nk}$, the domain $g^{-1}(D_{g(p)})$ meets the requirement stated in (ii). The theorem now follows from (8).

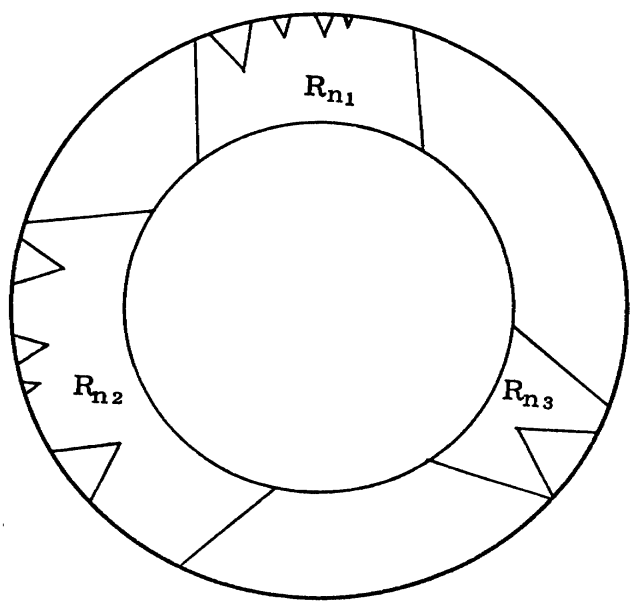

Fig. 1.

3. THE EXCEPTIONAL SET

THEOREM 6. *There exists a function* f, *holomorphic in* $|z| < 1$ *and continuous in* $|z| \leq 1$, *such that* $A(f, D, 0) = \infty$ *for every triangular boundary domain* D.

Let

$$f(z) = [\log(z-1)]^{-1/2} \exp\{i \log(z-1)\} ;$$

then

$$f'(z) = \frac{\exp\{i \log(z-1)\}}{(z-1)[\log(z-1)]^{1/2}} \left\{ i - \frac{1}{2\log(z-1)} \right\}.$$

In $|z| < 1$, the real part of $i \log(z-1)$ is bounded, and therefore there exists a positive constant C_5 such that

$$|f'(z)|^2 > \frac{C_5}{|z-1|^2 \, |\log(z-1)|}.$$

A slight computation now gives the desired result.

THEOREM 7. *If* $f(z) = \Sigma a_n z^n$ *and* $\Sigma |a_n| < \infty$, *there exists a tangential domain* D *such that the quantity* $A(f, D, \theta)$ *is a continuous function of* θ.

We will construct a tangential domain D of the form (1). By Minkowski's inequality,

$$[A(f, D, \theta)]^{1/2} = \left\{ \iint_{D_\theta} |\, \Sigma\, n a_n z^{n-1}|^2 d\sigma \right\}^{1/2}$$

$$\leq \Sigma \left\{ \iint_{D_\theta} |n a_n z^{n-1}|^2 d\sigma \right\}^{1/2}$$

$$= \Sigma\, n|a_n| \left(\int_{r_0}^{1} \lambda(r) r^{2n-1}\, dr \right)^{1/2}.$$

Since $\Sigma|a_n| < \infty$, there exists a sequence $\{\omega_n\}$ $(\omega_1 > 1,\ \omega_n \nearrow \infty)$ such that $\Sigma|a_n|\omega_n < \infty$. Also, the function $\lambda(r)$ can be chosen (analogously to the function $(1-r)\psi(r)$ in the proof of Theorem 2) so that $\lambda(r)/(1-r) \to \infty$ as $r \to 1$ and so that

$$n \left(\int_{r_0}^{1} \lambda(r) r^{2n-1}\, dr \right)^{1/2} < \omega_n .$$

For such a function $\lambda(r)$, the quantity $A(f, D, \theta)$ is a bounded function of θ.

To establish continuity of the function A, we introduce the notation

$$A_R(f, D, \theta) = \iint_{\substack{z \in D_\theta \\ |z|<R}} |f'|^2 d\sigma \qquad (0 < R < 1) .$$

For each R, the quantity $A_R(f, D, \theta)$ is a continuous function of θ, and it will suffice to show that $A_R(f, D, \theta) \to A(f, D, \theta)$ uniformly. Repeating the computation in the preceding paragraph, we obtain the inequality

$$\{A(f, D, \theta) - A_R(f, D, \theta)\}^{1/2} \leq \sum n|a_n| \left(\int_R^1 \lambda(r) r^{2n-1} dr \right)^{1/2} .$$

Since the right member is independent of θ and tends to zero as $R \to 1$, the theorem is proved.

COROLLARY. *If* $f(z) = \Sigma a_n z^n$ $(\Sigma |a_n| < \infty)$ *and* P *is a polygonal region in* $|z| < 1$, *then* f *maps* P *onto a Riemann surface of finite area.*

4. GENERAL BOUNDARY DOMAINS

THEOREM 8. *If* D *is a boundary domain, there exists a function* f, *holomorphic in* $|z| < 1$, *such that* $A(f, D, \theta) = \infty$ *for every* θ.

Let D be of the form (1); or, if D is not of this form, let $\lambda(r)$ denote the angle subtended, at the origin, by the intersection of the circle $|z| = r$ with D. Without loss of generality we suppose that $\lambda(r)$ is a decreasing function of r, for $r > r_0$. We choose a sequence $\{x_n\} \equiv \{x(n)\}$ such that $x_n \nearrow 1$ and

$$\lambda(x_n) > 1/n \qquad (n > n_0) . \tag{9}$$

By means of the sequence $\{x_n\}$ we construct a sequence $\{n_k\}$ which, together with the definitions $a_k = [x(n_k)]^{-n_k}$ and

$$f(z) = \sum a_k z^{n_k},$$

will lead to a proof of the theorem.

Let n_1 be any integer greater than n_0. Once $n_1, n_2, \cdots, n_{k-1}$ have been chosen, let n_k be an integer large enough so that

$$n_k > 4(a_1 n_1 + \cdots + a_{k-1} n_{k-1}) , \tag{10}$$

$$n_k a_k [x(n_i)]^{n_k} < k^{-2} \qquad (i = 1, 2, \cdots, k-1) , \tag{11}$$

$$y_k \equiv x(n_k)(1 - 1/2n_k) > x(n_{k-1}) , \tag{12}$$

and let R_k denote the annulus $y_k < |z| < x(n_k)$. For z in R_k,

$$|zf'(z)| > n_k a_k |z|^{n_k} - \sum_1^{k-1} n_i a_i - \sum_{k+1}^{\infty} n_i a_i [x(n_k)]^{n_i} .$$

By (10), the first sum on the right is less than $n_k/4$, and by (11) the second sum is less than $1/k$. By (12),

$$a_k y_k^{n_k} = (1 - 1/2n_k)^{n_k} > e^{-1/2} > 1/2,$$

and therefore $|zf'| > n_k/4$ throughout R_k, for k sufficiently large. It follows that

$$\iint_{D_\theta \cap R_k} |f'|^2 d\sigma > \int_{y_k}^{x(n_k)} \int_0^{\lambda(r)} |rf'|^2 d\theta\, dr > x(n_k)\ (2n_k)^{-1} \lambda[x(n_k)]\, n^2/16 .$$

By (9), the last expression is greater than $x(n_k)/32$, and the theorem follows.

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