THE POINT SPECTRUM OF WEAKLY ALMOST PERIODIC FUNCTIONS

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1. INTRODUCTION

We adopt the terminology and notation of our first paper [1] on the family $\mathfrak{W}$ of weakly almost periodic functions on a locally compact Abelian group G. With every w.a.p. function x(t) we associate a formal Fourier series

$$\sum_{\lambda \in G^*} a(\lambda)(t, \lambda),$$

where $a(\lambda) = M_s[x(s)(-s, \lambda)]$. We show that the Fourier series is the Fourier series of an almost periodic (a.p.) function $x_1(t)$; that is, every w.a.p. function x(t) admits a unique decomposition $x = x_1 + x_2$, where x_1 is a.p. and $M(|x_2|^2) = 0$. The set $[\lambda : a(\lambda) \neq 0]$ becomes the discrete or discontinuous part of the spectrum $\sigma(x)$ (see [2]).

The basic ergodic theorem which underlies the mean value theory of w.a.p. functions in [1] now reappears in the guise of a summability theorem.

2. ABSTRACT SUMMABILITY THEORY

In the present context, summability theory rests on the almost periodic properties of the kernel:

LEMMA 1. *Let* x *be w.a.p., and let* y *be a.p. with the properties* $y(t) > 0$, $y(-t) = y(t)$, $M(y) = 1$. *Then* $x*y$ *lies in* $\bar{O}(x)$, *the closed convex hull of the translates of* x.

Proof. For every t,

$$(x*y)(t) = M_s[x(s)y(t - s)] = M_s[x(s)y(s - t)] = \lim_{\alpha} T_\alpha(xy_t),$$

where the T_α run through the semi-group of finite convex combinations of translation operators $x(s) \to x_u(s) = x(s - u)$ ordered by multiplication (see [1], p. 225). Since the T_α are equi-uniformly continuous and the set $\{y_t\}$ and (hence) the set $\{x \cdot y_t\}$ $(t \in G)$ are conditionally compact in the norm topology of C(G), the convergence is uniform in t. It follows that for every $\varepsilon > 0$ there exists a finite set $\{s_n\}$ in G and a set $\{a_n\}$ of positive real numbers with $\Sigma a_n = 1$ such that simultaneously

$$(1) \qquad |(x*y)(t) - \Sigma a_n x(s - s_n)y(s - s_n - t)| < \varepsilon/2 \quad (s, t \in G),$$

$$(2) \qquad \Sigma a_n y(-s_n) = b, \quad \text{where } |1 - 1/b| < \frac{\varepsilon}{2\|x\| \cdot \|y\|}.$$

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We set $s = t$ in (1); then

$$|(x*y)(t) - \Sigma a_n x(t - s_n) y(-s_n)| < \varepsilon/2 \qquad (t \in G).$$

Since $\Sigma(a_n/b)y(-s_n) = 1$, $\Sigma(a_n/b)y(-s_n)x_{s_n}$ lies in $\overline{O}(x)$ and

$$\|x*y - \Sigma(a_n/b)y(-s_n)x_{s_n}\| < \varepsilon .$$

Recall that the algebra $\mathfrak{U}$ of almost periodic functions on G may be identified with (all) the continuous functions on a compact Abelian group $\overline{G} \supset G$. Moreover, for $x \in \mathfrak{U}$, $M(x) = \int_{\overline{G}} x(s)\,ds$, where ds is normalized Haar measure on $\overline{G}$. One can then introduce an approximate identity in $\mathfrak{U} = C(\overline{G})$; that is, there exists a net $\{x_\alpha\}$ of a.p. functions such that $\lim x_\alpha * x = x$ uniformly for every $x \in \mathfrak{U}$, while $x_\alpha \geq 0$ and $M(x_\alpha) = 1$ for all α (for these and other standard results in harmonic analysis see, for example, Loomis [4]). It is convenient to require also that $x_\alpha(-s) = x_\alpha(s)$.

LEMMA 2. *Let* S *denote the family of operators* T_y $(y \in \mathfrak{U})$: $T_y x = x - y*x$ $(x \in \mathfrak{W})$. *Then* S *is a convex Abelian semi-group admitting the family of almost invariant integrals* T_α: $T_\alpha x = x - x_\alpha * x$.

Proof. Clearly, S is a convex Abelian semi-group. In fact, $T_y T_z = T_{y \cup z}$, where $y \cup z = y + z - y*z$. Properties (I) and (II) (notation of [1]) of a family of almost invariant integrals are immediate, (III) follows from the crude inequality $\|T_\alpha\| \leq 2$, while (IV) $\lim_\alpha \|T_\alpha(I - T_y)\| \leq \lim_\alpha \|y - x_\alpha * y\| = 0$.

THEOREM 1. *Every* x *in* $\mathfrak{W}$ *admits the unique decomposition* $x = x_1 + x_2$, $x_1 \in \mathfrak{U}$, $M(|x_2|^2) = 0$.

Proof. Note first that the uniqueness follows from the Parseval equation. Since $\overline{O}(x)$ is weakly compact, the net $x_\alpha * x$ has a weak cluster point x_1, whence the net $T_\alpha x = x - x_\alpha * x$ has a weak cluster point $x_2 = x - x_1$. The general ergodic theorem ([1], Theorem 3.1) then implies that $\lim_\alpha T_\alpha x = x_2$ and $T_y x_2 = x_2$ for all $y \in \mathfrak{U}$. But then the function

$$x_1 = x - x_2 = \lim x_\alpha * x$$

is almost periodic, and $y*x_2 = 0$ for all $y \in \mathfrak{U}$. In particular,

$$a(\lambda)\,(t, \lambda) = M_s[(t - s, \lambda)x_2(s)] = (\lambda * x_2)(t) = 0$$

for every character λ, whence all the Fourier coefficients $a(\lambda)$ of x_2 vanish and $M(|x_2|^2) = 0$ by the Parseval equation.

The functions x and x_1 generate the same Fourier series, and explicit algorithms are available (see [5]) for summing the series to x_1 uniformly.

3. THE SPECTRUM OF A W.A.P. FUNCTION

Given an x in $L^\infty(G)$, the spectrum $\sigma(x)$ consists of all characters λ contained in the weak* span of the translates of x. If x is w.a.p., the spectrum $\sigma(x)$ has a richer structure:

THEOREM 2. *If* $x \in \mathfrak{W}$, *let* $\sigma_p(x) \subset \sigma(x)$ *consist of all characters* λ *contained in the uniform span* Γ *of the translates of* x. *Then* $\lambda \in \sigma_p(x)$ *if and only if* $a(\lambda) \neq 0$.

Proof. $a(\lambda) = \lim \Sigma a_n x(s - s_n)(-s + s_n, \lambda)$ uniformly in s. Hence $a(\lambda) \neq 0$ implies that $(s, \lambda) = a(\lambda)^{-1} \lim \Sigma a_n (s_n, \lambda) x(s - s_n)$ uniformly in s, whence λ is in Γ. Conversely, if λ has the form $(s, \lambda) = \lim \Sigma a_n x_{s_n}$ it follows that

$$1 = M_s[(-s, \lambda)(s, \lambda)] = \lim \Sigma a_n M_s[(-s, \lambda) x(s - s_n)]$$

$$= \lim \Sigma a_n (-s_n, \lambda) M_s[(-s, \lambda) x(s)] = a(\lambda) \lim \Sigma a_n (-s_n, \lambda),$$

whence $a(\lambda) \neq 0$.

It is appropriate to call $\sigma_p(x)$ the point spectrum of x. For example, when x is a Fourier-Stieltjes transform $x(t) = \int_{G^*} (t, \lambda) d\mu(\lambda)$, it turns out (see [3]) that $\sigma_p(x) = [\lambda: \mu\{\lambda\} \neq 0]$. The role of the point spectrum in the spectral synthesis of w.a.p. functions is thus settled by Theorem 1, but the role of the remaining (that is, *continuous*) part of the spectrum $\sigma_c(x) = \sigma(x) - \sigma_p(x)$ is more delicate.

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