A NOTE ON TOPOLOGIES ON $2^{\mathfrak{R}}$

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In this paper we consider topologies on $2^{\mathfrak{R}}$, the set of functions on $\mathfrak{R}$ into the two-element set $\{0, 1\}$. We make no distinction between $2^{\mathfrak{R}}$ and the collection of subsets of $\mathfrak{R}$; that is, we identify each subset of $\mathfrak{R}$ with its characteristic function.

We define the *sequential order topology* τ_1 as follows: A subset B of $2^{\mathfrak{R}}$ is closed if it contains the limit of each convergent sequence contained in B; the sequence $\{x_n\}$ is said to converge to x whenever

$$\bigcup_m \bigcap_{n \geq m} x_n = x = \bigcap_m \bigcup_{n \geq m} x_n$$

(see [1]). The *transfinite sequential order topology* τ_2 and the *order topology* τ_3 are defined analogously, with "sequence" replaced by "transfinite sequence" for τ_2 and by "net" for τ_3. By "transfinite sequence" we mean a function on a well ordered set into a set, and by "net" we mean a function on a directed set into a set. The topology of pointwise convergence is the usual strong product topology where the discrete topology is taken as the topology for $\{0, 1\}$. We denote this topology by τ_4.

Notation. For convergence under τ_i (i = 1, 2, 3, 4), we write $x_\alpha \to x$ under τ_i. If x is a point of accumulation under τ_i of a set A, we write $x \in A'$ (τ_i). As usual, the statement that $x \in A'$ (τ_i) means that every set open under τ_i and containing x contains points of A - (x). We order topologies as follows: τ is stronger than or equal to τ' $(\tau \geq \tau')$ if each set open (closed) under τ is also open (closed) under τ'. For the characteristic function f_x of a subset x of $\mathfrak{R}$, and for j in $\mathfrak{R}$, we mean by the statements $f_{x_n}(j) \to f_x(j)$ and $f_{x_\alpha}(j) \to f_x(j)$ that the sequence $\{f_{x_n}(j)\}$ in $\{0, 1\}$ converges to $f_x(j)$ and that the net $\{f_{x_\alpha}(j)\}$ converges, as a Moore-Smith limit, to $f_x(j)$, respectively. By $2^{\mathfrak{R}}$ (τ) we mean the space determined by the set $2^{\mathfrak{R}}$ with topology τ.

THEOREM 1. *Every projection* π_j $(j \in \mathfrak{R})$ *of* $2^{\mathfrak{R}}$ *is continuous under* τ_i (i = 1, 2, 3, 4). *Also,* $\tau_1 \leq \tau_2 \leq \tau_3 \leq \tau_4$.

Proof. For τ_4 the result is well known and is an immediate consequence of the definition of τ_4. For the other cases we note that $\pi_j^{-1}(1)$ is the collection of subsets of $\mathfrak{R}$ which contain j. It is both open and closed under τ_1, τ_2 and τ_3; and $\pi_j^{-1}(0)$ is the complement of $\pi_j^{-1}(1)$. This shows that τ_i (i = 1, 2, 3) is weaker than or equal to τ_4, since τ_4 is the strongest topology under which every projection is continuous. Since every sequence is a transfinite sequence and every transfinite sequence is a net, we have $\tau_1 \leq \tau_2 \leq \tau_3$. This implies the ordering $\tau_1 \leq \tau_2 \leq \tau_3 \leq \tau_4$.

THEOREM 2. *If* $\mathfrak{R}$ *is countable, then* $\tau_1 = \tau_2 = \tau_3 = \tau_4$.

Proof. Suppose $x \in A'$ (τ_4). Then, since τ_4 is metric, $\mathfrak{R}$ being countable, there exists a sequence $\{x_n\}$ in A - (x) such that $x_n \to x$ under τ_4. But this means that $f_{x_n}(j) \to f_x(j)$ for each j in $\mathfrak{R}$, where f_{x_n} and f_x are the characteristic functions

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of x_n and x, respectively. Clearly, then, $x_n \to x$ under τ_1 and $x \in A'$ (τ_1). Thus $\tau_1 \geq \tau_4$. Combining this inequality with the inequalities in Theorem 1, we obtain the desired result.

We consider the following property of convergence. Let $\{x_\alpha^\beta\}$ be a double sequence or net such that $x_\alpha^\beta \to x^\beta$ for each β and such that $x^\beta \to x$. Then there exists a function f on $\{\alpha\}$ into $\{\beta\}$ such that $x_\alpha^{f(\alpha)} \to x$. Here $\{\alpha\}$ and $\{\beta\}$ are elements of some class of directed sets—the class consisting of the set of positive integers in the case of τ_1, the class of well ordered sets in the case of τ_2, and the class of arbitrary directed sets in the cases of τ_3 and τ_4. This concept is due to G. Birkhoff [2], and the property will here be referred to as property P.

It is easy to show that convergence defined in terms of a given topology has this property (example: the topology of pointwise convergence [2, 4]).

It has been shown that for any set $\mathfrak{R}$ the order topology and the topology of pointwise convergence coincide on $2^{\mathfrak{R}}$ [3, 5]. Moreover, it is easy to see that a net order converges if and only if it converges under the topology of pointwise convergence. Thus convergence under τ_3 has property P, and this implies that convergence under τ_1 and under τ_2 has property P. This shows that sequences adequately describe the topology τ_1. By this we mean that, for $A \subset \mathfrak{R}$, $x \in A'$ (τ_1) if and only if there exists a sequence in A - (x) which converges to x. The corollary of Theorem 3 (see below) shows that transfinite sequences are not adequate for the order topology and for the topology of pointwise convergence when the cardinality of $\mathfrak{R}$ is at least c.

THEOREM 3. *If the cardinality of* $\mathfrak{R}$ *is at least* c, *then* τ_2 *is not compact.*

Proof. It is sufficient to prove that $2^{\mathfrak{R}}$ (τ_2) is not compact when $\mathfrak{R}$ has cardinality c. For suppose that $\mathfrak{R}'$ has cardinality c and that $2^{\mathfrak{R}'}$ (τ_2) is not compact. Now, if $\mathfrak{R}$ has cardinality greater than or equal to c, then $2^{\mathfrak{R}'}$ is a closed subset of $2^{\mathfrak{R}}$, and compactness of $2^{\mathfrak{R}}$ (τ_2) would imply compactness of $2^{\mathfrak{R}'}$ (τ_2). We let $\mathfrak{R}'$ be the collection of subsets of positive integers, and we denote by x_n in $2^{\mathfrak{R}'}$ the collection consisting of those sets of integers which contain n. Let $A = \{x_n\} (n = 1, 2, 3, \cdots)$. We show that A has no points of accumulation under τ_2.

To this end, it is sufficient to show that if x is any point in $2^{\mathfrak{R}}$, then A - (x) fails to contain a transfinite sequence converging to x. Suppose that $\{x_\alpha\}$ is a transfinite sequence on $D = \{\alpha\}$ in A - (x) which converges to x. For each positive integer n, there exists a β_n in D such that $\beta > \beta_n$ implies that $x_\beta \neq x_n$. To show this, suppose it is false. There can be no last element in D, for otherwise $\{x_\alpha\}$ would converge to x_{α_0}, where α_0 is this last element. Thus for some integer n_0 and for each β in D there exists a $\beta' > \beta$ such that $x_{\beta'} = x_{n_0}$. This implies that $\{x_\alpha\}$ converges to x_{n_0}. Hence $x = x_{n_0}$, which contradicts the assumption that $\{x_\alpha\}$ is contained in A - (x).

Now for each n we can pick β_n in D such that $\beta_{n+1} > \beta_n$, and such that $\beta > \beta_n$ implies that $x_\beta \neq x_n$. Clearly, $\{\beta_n\}$ is cofinal in D, since each x_β is an x_n and D is simply ordered. Now for each β_n an integer i_n may be chosen such that (1) $x_{i_n} = x_{\beta_n'}$ for some $\beta_n' \geq \beta_n$, and (2) i_n is greater than i_{n-1}. Let $j = \{i_1, i_3, \cdots, i_{2n+1}, \cdots\}$. Clearly, j is in

$$\bigcap_{\gamma \in D} \bigcup_{\beta \geq \gamma} x_\beta - \bigcup_{\gamma \in D} \bigcap_{\beta \geq \gamma} x_\beta,$$

which contradicts the assumption that $x_\alpha \to x$ under τ_2.

COROLLARY. *If the cardinality of* $\mathfrak{N}$ *is at least* c, *then* $\tau_2 < \tau_3$.

The following theorem contradicts the second part of 75.5 and 75.6, page 285 in [6].

THEOREM 4. *The collection of finite subsets of* $\mathfrak{N}$ *is dense in* $2^{\mathfrak{N}}$ *under the transfinite sequential order topology. Also,* $\tau_1 < \tau_2$ *if* $\mathfrak{N}$ *is uncountable.*

Proof. Let x be the set of ordinal numbers less than the first uncountable ordinal. For each ω in x, define $x_\omega = \{\alpha | \alpha \leq \omega\}$. Each x_ω is countable, and

$$\bigcup_\omega \bigcap_{\alpha \geq \omega} x_\alpha = x = \bigcap_\omega \bigcup_{\alpha \geq \omega} x_\alpha .$$

Thus $x_\omega \to x$ under τ_2. Clearly, any countable set is the limit of a sequence of finite sets. This proves the first part of the theorem for the first uncountable cardinal, since convergence in this case has property P. It is clear that the proof can be extended by transfinite induction. The last part of the theorem follows from the fact that if $\mathfrak{A}$ is a subset of $\mathfrak{N}$, then the topologies τ_i (i = 1, 2, 3, 4) are the same on $2^{\mathfrak{A}}$ as the corresponding relative topologies on $2^{\mathfrak{A}}$ as a subset of $2^{\mathfrak{N}}$.

THEOREM 5. *The set of finite subsets is dense in* $2^{\mathfrak{N}}$ *under* τ_3 *and* τ_4.

Proof. This follows from Theorem 4. We also note that if $x \subset \mathfrak{N}$, the net $\{x_\alpha\}$ of finite subsets of x converges to x under τ_3. We make a directed set out of the index set $\{\alpha\}$ as follows: $\alpha \leq \beta$ whenever $x_\alpha \subset x_\beta$.

If L is a complete lattice, we make a complete lattice out of $L^{\mathfrak{N}}$ by ordering the elements coordinatewise; that is, we write $x \leq y$ if each coordinate of x is less than or equal to the corresponding coordinate of y.

THEOREM 6. *Let* L *be a complete lattice which is compact in the order topology, and such that* $L^{\mathfrak{N}}$ *contains* $2^{\mathfrak{N}}$ *as a sublattice. Then the topology of pointwise convergence on* $L^{\mathfrak{N}}$ *is properly stronger than the transfinite sequential order topology on* $2^{\mathfrak{N}}$, *when the cardinality of* $\mathfrak{N}$ *is at least* c. *Also, transfinite sequences are not adequate for the topology of pointwise convergence.*

Proof. Frink [3] has shown that the order topology and the topology of pointwise convergence on $L^{\mathfrak{N}}$ are equivalent. It is easy to see by the proof of Theorem 3 that the transfinite sequential order topology on $L^{\mathfrak{N}}$ is not compact if the cardinality of $\mathfrak{N}$ is at least c.

COROLLARY. *If the cardinality of* $\mathfrak{N}$ *is at least* c, *then transfinite sequences are not adequate for the topology of the tychonoff cube* $J^{\mathfrak{N}}$.

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