AMBIGUOUS POINTS OF A FUNCTION CONTINUOUS INSIDE A SPHERE

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Let the function f be defined at all points P in the sphere S: $x^2 + y^2 + z^2 < 1$. A point Q on the surface T of S is called an *ambiguous point* of f provided there exist two Jordan arcs J_0 and J_1 which lie in S, except for their common end point Q, and on which the respective limits

$$\lim_{P \to Q, P \in J_0} f(P) \quad \text{and} \quad \lim_{P \to Q, P \in J_1} f(P)$$

exist and are different.

Bagemihl [1] has constructed an example of a real-valued function in S for which every point on T is an ambiguous point; and he has asked whether it is possible to construct such a function which has the additional property of being continuous at every point of S.

THEOREM. *There exists a real-valued, continuous function in* S *for which every point of* T *is an ambiguous point.*

Proof. Let the surface T be divided into two hemispheres t_{11} and t_{12}; let these hemispheres be divided further into quadrants, octants, ⋯, 2^n-ants, and so forth; and let the diameters of the regions t_{nj} $(j = 1, 2, \cdots, 2^n)$ on T tend to 0 uniformly as $n \to \infty$. Let $\{r_n\}$ be an increasing sequence $(r_0 > 0, r_n \to 1)$; and let S_n and T_n denote the sphere $x^2 + y^2 + z^2 < r_n^2$ and its surface, respectively. We construct two trees G_0 and G_1 in S, subject to the following two restrictions. For each n, the distance between the sets $S_n \cap G_0$ and $S_n \cap G_1$ is positive; and each point Q on T can be approached along two Jordan arcs $J_0(Q)$ and $J_1(Q)$ lying on G_0 and G_1, respectively. In order that the second restriction be satisfied, it is sufficient to observe the following precaution: for each n and each j $(n \geq 1;\ j = 1, 2, \cdots, 2^n)$, let the central projection of t_{nj} on T_n meet a branch of G_0 and a branch of G_1; let all subsidiary branches of the relevant branches lie in the cone determined by t_{nj}; and let S contain no end point of G_0 and G_1.

For each point P in S, let $r_0 = r_0(P)$ denote the distance between P and G_0; similarly, let r_1 denote the distance between P and G_1; and let $f(P) = r_0/(r_0 + r_1)$. The function f is continuous in S; and since f has the values 0 and 1, everywhere on G_0 and G_1, respectively, the theorem is proved.

Remark 1. In Bagemihl's construction, the "pair of paths of ambiguity" is a pair of rectilinear segments, for each point Q on T. *It remains an open question whether there exists a continuous function in* S *which, at every point of* T, *has a pair of rectilinear paths of ambiguity.* On the other hand, it is easily seen that the trees G_0 and G_1 can be constructed in such a way that each point Q on T has a pair of rectifiable paths of ambiguity for which the radius of Q is a half-tangent at Q.

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Remark 2. Bagemihl points out that his example can be modified so that, at each point on T, every real number is an asymptotic value for f. The same is true of the example in the present note. The technicalities can be handled as follows. We replace the trees G_0 and G_1 by a single tree G. This tree is rooted at the origin, where it produces a large (but finite) number of rectilinear branches which extend as far as S_0. In general, the tree G branches into rectilinear segments, wherever it meets a surface S_n; and these segments reach the surface S_{n+1}, where they branch in turn. The function f is defined on G in such a way that it is linear on each segment of G; and at each point where G meets the surface S_n, the function f takes one of the $1 + 2^{2n+1}$ values $k/2^n$ $(-4^n \leq k \leq 4^n)$. It is easily seen that, with appropriate precautions taken, every real number (as well as each of the values $+\infty$ and $-\infty$) is an asymptotic value for f, at each point on T. The continuous extension of f to the remainder of S presents no special difficulties.

Remark 3. For further generalizations of the theorem described in this note, the reader is referred to papers by Bagemihl [2] and Church [3].

REFERENCES

1. F. Bagemihl, *Rectilinear limits of a function defined inside a sphere*, Michigan Math. J. 4 (1957), 147-150.

2. ———, *Ambiguous points of a function harmonic inside a sphere*, Michigan Math. J. 4 (1957), 153-154.

3. P. Church, *Ambiguous points of a function homeomorphic inside a sphere*, Michigan Math. J. 4 (1957), 155-156.

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