Close-to-Convex Schlicht Functions

by

Wilfred Kaplan

1. Principal results. Known theorems yield the following: if $\phi(z)$ is a convex schlicht function for $|z| < R$ and f(z) is a function analytic for $|z| < R$ such that

$$\operatorname{Re}\left[\frac{f'(z)}{\phi'(z)}\right] > 0, \quad |z| < R,$$

then f(z) is also schlicht for $|z| < R$. Since the vectors f', ϕ' never differ in direction by more than 90°, it is natural to call f close-to-convex:

Definition. Let f(z) be analytic for $|z| < R$. Then f(z) is close-to-convex for $|z| < R$ if there exists a function $\phi(z)$, convex and schlicht for $|z| < R$, such that $f'(z)/\phi'(z)$ has positive real part for $|z| < R$.

When $R = 1$, it will be convenient to omit reference to the circular domain of definition. Therefore, a close-to-convex function will mean a function which is close-to-convex for $|z| < 1$.

We verify that the close-to-convex functions include several familiar classes of schlicht functions: e.g., the star functions, as well as some less familiar ones: e.g., the functions f(z) having a Poisson integral representation in terms of a function $P(e^{i\theta})$ which is monotone in θ within each of two complementary arcs of $|z| = 1$.

It is of interest to characterize the close-to-convex functions intrinsically, without reference to a convex function ϕ. Such a characterization is obtained as follows: f(z) is close-to-convex if and only if

$$\int_{\theta_1}^{\theta_2} \text{Re}\left[1 + z\frac{f''(z)}{f'(z)}\right] d\theta > -\pi$$

when $\theta_1 < \theta_2$, $z = re^{i\theta}$ and $r < 1$.

2. The class of close-to-convex functions. It is known ([6] p. 582, V.) that if g(z) is analytic in a convex domain D and

(1) $\quad \text{Re}\,[g'(z)] > 0$ in D ,

then g(z) is schlicht in D. If $\phi(z)$ is a schlicht map of $|z| < 1$ onto D, then $f(z) = g[\phi(z)]$ is also schlicht. Since $f'(z) = g'(\phi)\,\phi'(z)$, f(z) satisfies the condition

(2) $\quad \text{Re}\left[\frac{f'(z)}{\phi'(z)}\right] > 0$ for $|z| < 1$.

Conversely, if f(z) satisfies (2), then $g(z) = f[\phi^{-1}(z)]$ satisfies ( ) and $f(z) = g[\phi(z)]$ is schlicht for $|z| < 1$.

Theorem 1. Every close-to-convex function is schlicht.

The class of close-to-convex functions clearly includes the convex functions themselves, as well as the functions f(z) whose derivative has positive real part in the unit circle. The normalized schlicht functions f(z) which map the unit circle onto a domain star-shaped with respect to the origin are characterized by the inequality ([2] pp. 92-94):

(3) $\quad \text{Re}\left[z\frac{f'(z)}{f(z)}\right] > 0$;

since ([2] p. 93)

(4) $$\phi(z) = \int_0^z \frac{f(z)}{z}\, dz$$

is known to be convex, it follows that the star mappings are included in the close-to-convex functions. By specializing the choice of $\phi(z)$, one obtains other subclasses:

(5) $$\mathrm{Re}\,[(z-1)^2 f'(z)] > 0\,,$$

(6) $$\mathrm{Re}\,[(z-e^{i\alpha})(z-e^{i\beta}) f'(z)] > 0 \quad (\alpha, \beta \text{ real})$$

(7) $$\mathrm{Re}\,\Big[\prod_{j=1}^{n} (z-e^{i\alpha_j})^{k_j} f'(z)\Big] > 0\,,$$

$$0 \le k_j \le 1, \quad \sum_{j=1}^{n} k_j \le 2,$$

$$0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_n \le 2\pi.$$

For the class (7) ϕ is a Schwarz - Christoffel mapping; for $\alpha \neq \beta$, (6) is a special case of (7), and (5) is a special case of (6).

If $h(z) = \log \phi(z)$ is chosen to be analytic, then the condition that ϕ be convex is expressed by the inequality

(8) $$\mathrm{Re}\,[1 + z\,h'(z)] > 0\,.$$

Accordingly, if $f(z)$ is analytic for $|z| < 1$, then $f(z)$ is close-to-convex if and only if there exists a function $h(z)$, analytic for $|z| < 1$, such that

(9) $$\mathrm{Re}\,[f'(z)\,e^{-h(z)}] > 0, \quad \mathrm{Re}\,[1 + z\,h'(z)] > 0\,.$$

From the familiar integral representation of a function with positive real part ([7] p. 185) we obtain the expressions

$$(10)\qquad f'(z) = e^{h(z)}\left[\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\Psi(\theta) + i\alpha\right],$$

$$(11)\qquad h'(z) = \frac{1}{z}\left[\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\chi(\theta) + i\beta - 1\right],$$

from which an integral representation for $f(z)$ in terms of two monotone non-decreasing functions $\Psi(\theta)$ and $\chi(\theta)$ can be obtained.

3. Intrinsic characterization. Let $f(z)$ and $\phi(z)$ be given as in Section 1; let $p(z) = \arg f'(z)$ and $q(z) = \arg \phi(z)$ be chosen to be continuous for $|z| < 1$. Since $f'(z)$ and $\phi'(z)$ have no roots for $|z| < 1$, such a choice is possible. Because of (2), at each z

$$|p(z) - q(z) + 2k\pi| < \frac{1}{2}\pi$$

for some $k = 0, \pm 1, \ldots$. Because of the continuity of $p(z)$ and $q(z)$, k must be independent of z. If $p(z)$ is properly chosen, $k \equiv 0$, and it will be assumed that such a choice has been made, so that

$$(12)\qquad |p(z) - q(z)| < \frac{1}{2}\pi \text{ for } |z| < 1 .$$

We now introduce the functions

$$(13)\qquad P(r, \theta) = p(re^{i\theta}) + \theta, \quad Q(r, \theta) = q(re^{i\theta}) + \theta ,$$

which are defined for $0 \leq r < 1$ and all real θ. Condition (12) becomes

$$(14)\qquad |P(r, \theta) - Q(r, \theta)| < \frac{1}{2}\pi.$$

The condition that $\phi(z)$ be a convex mapping is described by (8) or equivalently by the condition

(15) $$\frac{\partial Q}{\partial \theta} > 0 ;$$

Thus $Q(r, \theta)$ is monotone increasing in θ for each fixed r. Now, if $\theta_1 < \theta_2$,

$$\begin{aligned} P(r, \theta_1) - P(r, \theta_2) &= [P(r, \theta_1) - Q(r, \theta_1)] \\ &\quad - [P(r, \theta_2) - Q(r, \theta_2)] \\ &\quad + [Q(r, \theta_1) - Q(r, \theta_2)] \\ &< [P(r, \theta_1) - Q(r, \theta_1)] - [P(r, \theta_2) - Q(r, \theta_2)] . \end{aligned}$$

Accordingly, by (14),

(16) $$P(r, \theta_1) - P(r, \theta_2) < \pi \quad \text{for } \theta_1 < \theta_2 .$$

Condition (16) is thus a necessary condition that f(z) be close-to-convex; it can be expressed in other equivalent forms:

(16') $$\arg f'(re^{i\theta_1}) - \arg f'(re^{i\theta_2}) < \pi + (\theta_2 - \theta_1)$$

for $\theta_1 < \theta_2$, provided $\arg f'(z) = p(z)$ is chosen as above to be continuous for $|z| < 1$;

(16'') $$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left[1 + re^{i\theta} \frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right] d\theta > -\pi,$$

for $\theta_1 < \theta_2$. The condition is also sufficient:

Theorem 2. A necessary and sufficient condition that a function f(z), analytic and with non-van-

ishing derivative for $|z| < 1$, *be close-to-convex is that* (16'') *hold for* $\theta_1 < \theta_2$ *and* $r < 1$.

The necessity being established above, it remains to prove the sufficiency. Given $f(z)$, we choose $p(z) = \arg f'(z)$ to be continuous and then define $P(r, \theta)$ by (13). The condition (16'') is then replaced by (16). In addition,

$$(17) \qquad P(r, \theta + 2\pi) - P(r, \theta) = 2\pi,$$

since $p(z)$ has period 2π with respect to θ.

Lemma. *Let* $t(\theta)$ *be a real function of* θ *for* $-\infty < \theta < \infty$ *such that*

$$(18) \qquad t(\theta + 2\;\;) - t(\theta) = 2\pi,$$

$$(19) \qquad t(\theta_1) - t(\theta_2) < \pi \quad \text{for } \theta_1 < \theta_2 .$$

Then there exists a real function $s(\theta)$ *which is monotonic non-decreasing and satisfies the conditions*

$$(20) \qquad s(\theta + 2\pi) - s(\theta) = 2\pi,$$

$$(21) \qquad |s(\theta) - t(\theta)| \leq \frac{1}{2}\pi .$$

Proof. Let

$$s(\theta) = \underset{\theta' \leq \theta}{\text{l. u. b.}}\; t(\theta') - \frac{1}{2}\pi .$$

Then $s(\theta)$ is non-decreasing. By (19), $t(\theta')$ is bounded above, for $\theta' < \theta$, by $t(\theta) + \pi$. Hence the l. u. b. is finite and

$$s(\theta) \leq t(\theta) + \frac{1}{2}\pi .$$

Furthermore, since $t(\theta) \leq \text{l.u.b.}\ t(\theta')$ for $\theta' \leq \theta$,

$$s(\theta) \geq t(\theta) - \frac{1}{2}\pi .$$

Hence (21) is proved; (20) follows from (18). The lemma is thus established.

We now set $P(\rho, \theta) = t(\theta)$, for a fixed $\rho < 1$, and apply the lemma, denoting the corresponding function $s(\theta)$ by $s(\rho, \theta)$. For $r < \rho$ we define

$$(22) \quad q_\rho(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(\rho^2 - r^2)(s(\rho, \alpha) - \alpha)}{\rho^2 + r^2 - 2\rho r \cos(\alpha - \theta)} d\alpha,$$

so that $q_\rho(r, \theta)$ is harmonic for $r < \rho$. Moreover, the function

$$(23) \qquad Q_\rho(r, \theta) = q_\rho(r, \theta) + \theta$$

is monotone increasing in θ for each fixed $r < \rho$. For, if $\theta_1 < \theta_2$,

$$Q_\rho(r, \theta_2) - Q_\rho(r, \theta_1) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(\rho^2 - r^2)(s(\rho, \alpha + \theta_2) - s(\rho, \alpha + \theta_1))}{\rho^2 + r^2 - 2\rho r \cos \alpha} d\alpha$$

and, since $s(\rho, \alpha)$ is non-decreasing in α, the right hand side is positive or 0. Hence $Q_\rho(r, \theta)$ is non-decreasing in θ, so that $\partial Q_\rho / \partial \theta \geq 0$. Since this derivative is harmonic, the equality is ruled out, so that $Q_\rho(r, \theta)$ is strictly increasing.

We now choose an analytic function $h_\rho(z)$ whose imaginary part is $q_\rho(r, \theta)$ and such that $\text{Re}[h_\rho(0)] = 0$. Then set

(24) $\phi_\rho(z) = \int_0^z e^{h_\rho(z)}\, dz$,

so that

(25) $\phi_\rho(0) = 0, \ |\phi'_\rho(0)| = 1.$

The function $\phi_\rho(z)$ is then analytic for $|z| < \rho$. Moreover,

(26) $\operatorname{Re}[1 + z \frac{\phi''_\rho(z)}{\phi'_\rho(z)}] = \frac{\partial Q_\rho}{\partial \theta} > 0, \ |z| < \rho.$

Hence $\phi_\rho(z)$ is a convex function for $|z| < \rho$. Furthermore, since

$$|P(\rho, \theta) - s(\rho, \theta)| \leq \frac{1}{2}\pi ,$$

we conclude from (22) and the Poisson integral for $p(r, \theta)$ in terms of $p(\rho, \theta)$ that

$$|P(r, \theta) - Q_\rho(r, \theta)| < \frac{1}{2}\pi \quad \text{for} \quad r < \rho.$$

Accordingly,

(27) $\operatorname{Re}[\frac{f'(z)}{\phi'_\rho(z)}] > 0$ for $|z| < \rho$,

so that $f(z)$ is close-to-convex for $|z| < \rho$. It remains to show that we can pass to the limit, $\rho \to 1$, and get a unique function $\phi(z)$ for $|z| < 1$.

If we choose the sequence $\rho_n = 1 - \frac{1}{n}$, then the corresponding functions $\phi_{\rho_n}(z)$ are defined for an increasing sequence of domains. For each fixed n, the functions $\phi_{\rho_m}(z)$ for $m \geq n$ form a normal family for $|z| < \rho_n$; this follows from the normality of the family of normalized schlicht functions and condition (25).

Hence a subsequence converges uniformly in this domain. By applying the diagonal process in the familiar fashion, we obtain a subsequence of $\phi_{\rho_n}(z)$ which converges uniformly in each circle $|z| < \rho < 1$ and hence has as limit a unique function $\phi(z)$, analytic for $|z| < 1$. Since the $\phi_\rho(z)$ are schlicht and convex, $\phi(z)$ must also be so. Since (27) holds for $\rho = \rho_n$, we conclude that

$$\text{(28)} \quad \mathrm{Re}\left[\frac{f'(z)}{\phi'(z)}\right] > 0 \quad \text{for} \quad |z| < 1 ;$$

i.e., f(z) is close-to-convex.

4. Geometric interpretation. The condition (16'') or its equivalent, condition (16), has the following geometric meaning: $w = f(z)$ maps each circle $z = re^{i\theta}$ (r fixed and $r < 1$) onto a simple closed curve whose unit tangent vector $T = i \exp[iP(r, \theta)]$ either rotates in a counterclock-wise direction, as θ increases, or else rotates clockwise in such a manner that $\arg T = P + \frac{1}{2}\pi$ never drops to a value π radians below a previous value; i.e., $\Delta \arg T$ exceeds $-\pi$, as θ increases. This is illustrated in Fig. 1. Here $\arg T_2 - \arg T_1$ is only slightly

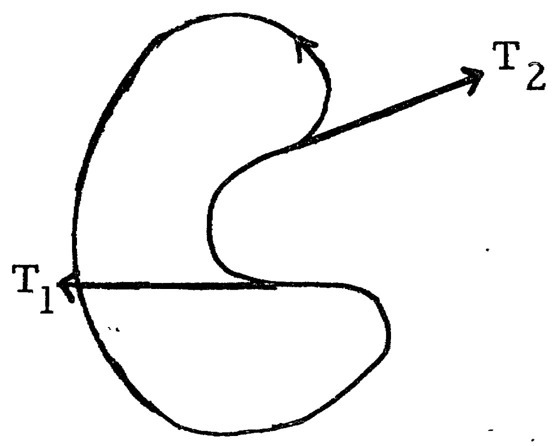

Fig. 1.

greater than $-\pi$. Thus such a "hairpin turn" is permitted, provided one does not make a complete reversal of direction.

5. Extremal aspects. For each function f(z), analytic and with non-vanishing derivative for $|z| < 1$, we make the definition:

$$(29) \quad c[f] = \underset{\phi}{\text{g.l.b.}} \left[\underset{|z|<1}{\text{l.u.b.}} \left| \arg f'(z) - \arg \phi'(z) \right| \right],$$

where ϕ ranges over the class of all convex schlicht functions for $|z| < 1$; for each z, the arguments of f' and ϕ' are to be chosen to give the absolute value of the difference its smallest value. In general, $c[f] \leq \pi$. If $c[f] < \pi$, then as in the preceding section one can compute c[f] by restricting ϕ by the conditions (25); the restricted family is normal and accordingly there exists a convex ϕ such that

$$(30) \quad |\arg f'(z) - \arg \phi'(z)| \leq c[f], \quad |z| < 1 ;$$

c[f] is the smallest constant for which such a ϕ can be found. The function ϕ in (30) can be termed a "best convex approximation to f(z)".

If $c[f] = 0$, then f must itself be convex; if $c[f] \leq \frac{1}{2}\pi$, then f is close-to-convex.

The constant c[f] and a corresponding extremal ϕ satisfying (30) can be found directly by the procedure of Section 3. We introduce the function $P(r, \theta) = \arg f'(z) + \theta$. Then

$$(31) \quad c[f] = \min \left(\frac{1}{2} \text{ l.u.b. } [P(r, \theta_1) - P(r, \theta_2)], \pi \right),$$

where the l.u.b. is taken over all r, θ_1, θ_2 for which

$\theta_1 < \theta_2$ and $r < 1$. For, if $c[f] < \pi$, then we choose ϕ satisfying (30) and let $Q(r,\theta) = \arg \phi'(z) + \theta$; as in Section 3, arg f' and arg ϕ' can be chosen so that

(32) $\quad |P(r,\theta) - Q(r,\theta)| \leq c[f]$.

The method of derivation of (16) then yields the relation

(33) $\quad P(r,\theta_1) - P(r,\theta_2) \leq 2c[f]$ for $\theta_1 < \theta_2$.

On the other hand, if

$$\frac{1}{2} \text{ l.u.b. } [P(r,\theta_1) - P(r,\theta_2)] = \alpha < \pi ,$$

then the proof of Theorem 2 can be repeated to yield a convex function ϕ such that

(34) $\quad |\arg f'(z) - \arg \phi'| \leq \alpha$.

Hence

$$c[f] \leq \alpha \leq c[f]$$

and (31) is proved.

If $f(z)$ is analytic with non-vanishing derivative for $|z| < R$, we can define $c_R[f]$, for example by (31) with r restricted to be less than R. Then $c_R[f]$ will be a monotone non-decreasing function of R. For example, if $f(z) = e^z$, we find

$$c_r[e^z] = 0, \quad r \leq 1$$

$$c_r[e^z] = \sqrt{r^2 - 1} - \arccos \frac{1}{r}, \quad r \geq 1 .$$

The equation $c_r = \frac{1}{2}\pi$ is satisfied for r slightly less than 3; this gives the largest circle $|z| = a$ within which e^z is close-to-convex; e^z remains schlicht for $|z| < \pi$, is convex only for $|z| < 1$.

For $w = (z - 1)^2$, we find

$$c_r[(z-1)^2] = 0, \ r \leq \frac{1}{2},$$

$$c_r[(z-1)^2] = \cos^{-1}\frac{2r^2+1}{3r} + \tan^{-1} 3r\sqrt{\frac{4r^2-1}{1-r^2}}, \ \frac{1}{2} \leq r < 1.$$

As $r \to 1$, $c_r \to \frac{1}{2}\pi$, so that $c_1[(z-1)^2] = \frac{1}{2}\pi$. Accordingly, this function is convex for $|z| < 1/2$, is close-to-convex for $r < 1$; the latter domain is the largest circular domain, with center at 0, in which the function is schlicht.

It is natural to ask whether $\frac{1}{2}\pi$ is the best possible value for $c[f]$ in the sense that it is the smallest value which guarantees schlichtness. For each α, $\frac{1}{2}\pi < \alpha \leq \pi$, we can indeed construct a function f such that $c_r[f] = \alpha$, but f is not schlicht for $|z| < r$. To this end we let

$$f_\epsilon(z) = (1-z)^{2+\epsilon}, \ f_\epsilon(0) = 1, \ |z| < 1, \ 0 \leq \epsilon \leq 1.$$

We then find that $c_1[f_\epsilon(z)]$ is continuous in ϵ and equals $\frac{1}{2}\pi$ for $\epsilon = 0$, equals π for $\epsilon = 1$. Accordingly, $c_1[f_\epsilon]$ takes on every value α between $\frac{1}{2}\pi$ and π as ϵ goes from 0 to 1, but $f_\epsilon(z)$ is not schlicht for $|z|<1$ and $\epsilon > 0$.

As remarked above, the largest r for which

$c_r[f] = 0$ gives the largest circle $|z| = r$ within which $f(z)$ is convex; it is known that for normalized schlicht functions this r is $\geq 2 - \sqrt{3}$ ([2] p. 92) and can equal this value. It would be of interest to obtain a similar lower bound for the largest circle within which f is close-to-convex.

6. *A sub-class of the close-to-convex functions.* In a previous paper ([5]), the author demonstrated that the functions $f(z)$ representable for $|z| < 1$ by a Poisson integral

$$(35) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} h(\theta)\, d\theta ,$$

in which $h(\theta)$ is monotone non-decreasing, are schlicht for $|z| < 1$. It was also shown that each such $f(z)$ satisfies

$$(36) \quad \operatorname{Im}[(z-1)^2 f'(z)] > 0 ,$$

from which we conclude that $f(z)$ is close-to-convex.

We now show that these conclusions remain valid if we assume that $h(\theta)$ is monotone non-decreasing in one interval of θ and monotone non-increasing for the remaining values:

Theorem 3. *Let $h(\theta)$ be defined and non-constant for $\pi < \theta \leq 2\pi$; let $h(\theta)$ be monotone non-decreasing for $0 \leq \theta \leq \pi$ and monotone non-increasing for $\leq \theta \leq 2\pi$. Then (35) defines a function $f(z)$ which is schlicht and close-to-convex for $|z| < 1$.*

Proof: We shall verify that

$$\operatorname{Re}\left[\frac{f'(z)}{\phi'(z)}\right] > 0, \quad \phi(z) = \log \frac{z-1}{z+1} ,$$

i.e., that (6) holds with $\alpha = 0$ and $\beta = \pi$. As in [5], we find that

$$\mathrm{Re}\,[(z^2 - 1) f'(z)] = \mathrm{Re}\,[\frac{i}{\pi} \int_0^{2\pi} \frac{(z+1)(1-e^{i\theta})}{e^{i\theta} - z}\, dh(\theta)]$$

$$= \frac{1}{\pi} \int_0^{2\pi} \frac{\sin\theta\,(1+r)}{|e^{i\theta} - z|^2}\, dh(\theta) .$$

Since $h(\theta)$ is non-decreasing for $0 \leq \theta \leq \pi$, where $\sin\theta \geq 0$, and non-increasing for $\pi \leq \theta \leq 2\pi$, where $\sin\theta \leq 0$, the integral is positive as asserted and $f(z)$ is close-to-convex.

By applying a suitable linear transformation, we can extend the theorem to the case in which h is non-decreasing from 0 to α and non-increasing from α to 2π, also to the case of functions defined in the half-plane $y > 0$ by a Poisson integral

$$\frac{1}{\pi i} \int_{-\infty}^{\infty} \left\{ \frac{1}{t-z} - \frac{t}{1+t^2} \right\} h(t)\, dt .$$

Each function $u + iv = f(z)$ can be shown to map onto a domain D which is convex in the sense that, if (u, v_1) and (u, v_2) are in D, then so also is the line segment joining these points. When h is a step-function, D is bounded by rays and two lines, all parallel to the v-axis; examples are given on pp. 605-609 of [6].

7. Mapping by non-analytic functions. It would be of interest to generalize the preceding discussion to mappings

(37) $u = F(x, y), \quad v = G(x, y), \quad x^2 + y^2 < 1$,

where F and G are of class C' and the Jacobian

$$(38) \quad J = \frac{\partial(F, G)}{\partial(x, y)}$$

is positive throughout. In conversations with the author, C. J. Titus has conjectured that if the mapping has the property that each circle $x^2 + y^2 = r^2 < 1$ is mapped onto a path C whose tangent never turns back through π, as in Section 4, then it must be one-to-one. One can also ask whether the geometric condition just stated is sufficient to guarantee that a mapping (37) defined only for $x^2 + y^2 = r^2$ is one-to-one; simple counter-examples show that this is not the case. However, C. J. Titus has conjectured that, if one also requires that the image curve has non-negative circulation (in a properly defined sense), then it must indeed be a simple closed curve.

The theorem of the preceding section does have a natural extension to mappings (37):

Theorem 4. Let $F(x,y)$ be continuous for $x^2 + y^2 < 1$ and let $F(\cos\theta, \sin\theta) = h(\theta)$ be non-decreasing for $0 \leq \theta \leq \pi$, non-increasing for $\pi < \theta < 2\pi$. Let $G(x, y)$ be defined for $x^2 + y^2 < 1$ in such a manner that equations (37) define a mapping which is locally a homeomorphism. Then (37) defines a homeomorphism of the set: $x^2 + y^2 < 1$ into the uv-plane.

Proof: Since the mapping is locally a homeomorphism, the level curves of $F(x, y)$ must form a regular curve-family H filling the domain: $x^2 + y^2 < 1$ ([3], p. 155). From the results of [3], it follows that each curve C of the family H can be parametrized by equations $x = x(t)$, $y = y(t)$, $-\infty < t < \infty$, so that $x^2 + y^2 \to 1$ as $t \to \infty$. We denote by $L[C, +]$ (or $L[C, -]$) the set of limit points of sequences $(x(t_n), y(t_n))$ as $t_n \to \infty$ (or $t_n \to -\infty$). Then $L[C, +]$ must be an arc of $x^2 + y^2 = 1$ or a single point; since $F(x, y)$ is continuous, $L[C, +]$ can be an arc only when $F(\cos \theta, \sin \theta) = h(\theta)$ is constant along the arc. From the monotonicity of h it follows that, if C_1, C_2, C_3 are curves of H on which F has the respective values k_1, k_2, k_3, with $k_1 < k_2 < k_3$, then C_2 separates C_1 from C_3 in $x^2 + y^2 < 1$. Hence in general, for every triple C_1, C_2, C_3 in H, one curve separates the other two. By the main theorem (p. 11) of [4], it follows that H must have the structure of a family of parallel lines: i.e., there is a homeomorphism of $x^2 + y^2 < 1$ onto itself transforming H onto the lines $y = $ const.. In the new coordinates, the function F becomes a function $F_o(y)$ which is strictly monotone, while G becomes a function $G_o(x, y)$ which is strictly monotone in x for each y. Accordingly, $u = F_o(y)$, $v = G_o(x, y)$ defines a homeomorphism and therefore a similar conclusion holds for (37).

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University of Michigan
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