# Intrinsic Relations Satisfied by the Vorticity and Velocity Vectors in Fluid Flow Theory

by

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1. Introduction. In plane fluid flows, it is well known[1) that the following relation exists between the magnitude of the vorticity vector, $\omega$, and that of the velocity vector, q,

$$(1.1) \qquad \omega = -\frac{\partial q}{\partial n} + \kappa q .$$

Here, $\kappa$ is the curvature of the stream line at the point under consideration, and $\partial q/\partial n$ represents the rate of change of q with respect to arc length along the direction normal to the stream line.

Our first problem is to generalize the above relation to three dimensional fluid flow theory. To do so, we shall decom pose the vorticity vector into components along the tangent, principal normal, and binormal to the stream lines. From this decomposition, the desired generalization of formula (1.1) is easily obtained. It will be shown that the right hand side of (1.1) is nothing but the component of the vorticity vector along the binormal direction.

Secondly, we shall determine an intrinsic relation satisfied by the Bernoulli function for the case of the steady flow of a non-viscous fluid. From this relation, one can easily obtain a necessary and suf-

1) Theoretical Hydrodynamics, L. M. Milne - Thomson, Macmillan Co, London, 1938, p. 99.

ficient condition for the Bernoulli equipotentials to be a family of parallel surfaces.

Finally, by eliminating the density between the equations of motion and the equation of continuity, we obtain a relation which determines the rate of change of the magnitude of the velocity vector with respect to displacement along the stream lines in terms of the two principal values of a tensor which lies in the local subspace orthogonal to the stream lines. In the case when there exist $\infty^1$ surfaces orthogonal to the stream lines, the principal values of this tensor reduce to the sum of the two principal normal curvatures (the mean curvature) of any one of these $\infty^1$ surfaces.

2. The Basic Decomposition. Let $x^j (j = 1, 2, 3)$ denote the variables of a system of Cartesian orthogonal coordinates. In order to use the summation convention, we shall write indices in covariant and contravariant positions. The velocity vector will be denoted by $v^j$, and if $t^j$ is a unit vector in the direction of $v^j$, then we may write

(2.1) $$v^j = q\, t^j .$$

If $e^{ijk}$ denotes the permutation tensor with components 1, 0, -1 according to whether (ijk) is an even permutation, contains a repeated integer, or is an odd permutation and if

(2.2) $$\partial_j v_k = \frac{\partial v_k}{\partial x^j}$$

then the vorticity vector, $\omega^k$, is defined by

(2.3) $$\omega^k = e^{kij}\, \partial_i v_j .$$

For a congruence of curves determined by the unit vector, $t^j$, it is well known that[2)]

(2.4) $\partial_j t_k = L_{jk} + t_j u_k$

where $u_k$ is the curvature vector of the congruence, $t_k$,

(2.5) $t^j \partial_j t_k = u_k$

and $L_{jk}$ is a tensor which lies in the local space orthogonal to $t_k$. By differentiation of the velocity vector in (2.1) and use of (2.4) we obtain the basic relation

(2.6) $\partial_j v_k = v_k \partial_j (\ln q) + v_j u_k + q L_{jk}$ .

If we multiply this last relation by $e^{ijk}$ and use the definition of $\omega^k$ in (2.3), we find that

(2.7) $\omega^i = e^{ijk} v_k \partial_j (\ln q) + e^{ijk} v_j u_k + q e^{ijk} L_{jk}$ .

First, we evaluate the last term in the right hand side of (2.7). Let us decompose the tensor, $L_{jk}$, into a symmetric tensor, $S_{jk}$, and a skew symmetric tensor, $w_{jk}$, so that

(2.8) $L_{jk} = S_{jk} + w_{jk}$ .

Further, let $n^j$, $b^j$ denote unit vectors along the principal normal and binormal, respectively, of the stream lines (with unit tangent vector, $t^j$). Since $L_{jk}$ lies in the subspace orthogonal to $t^j$, the tensors $S_{jk}$, $W_{jk}$ lie in this subspace. Hence, we may write

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2) Einführung in die Neueren Methoden der Differentialgeometrie, J. A. Schouten and D. J. Struik, P. Noordhoff, Groningen, Batavia, 1938, p. 28

$$(2.9) \qquad w_{jk} = w(n_j b_k - n_k b_j) ,$$

where w is a scalar. Further, by forming the scalar product of (2. 6) with the skew symmetric tensor

$$n^j b^k - n^k b^j$$

we obtain

$$(2.10) \qquad b^k \frac{\partial v_k}{\partial n} - n^k \frac{\partial v_k}{\partial b} = q \, L_{jk} (n^j b^k - n^k b^j) ,$$

where $\partial v_k/\partial n$, $\partial v_k/\partial b$, denote the rate of change of $v_k$ with respect to arc length along the $n^j$, $b^j$ directions, respectively. Replacing the tensor $L_{jk}$, by (2. 8), (2. 9), we find that the unknown scalar, w, is determined by

$$(2.11) \qquad b^k \frac{\partial v_k}{\partial n} - n^k \frac{\partial v_k}{\partial b} = 2\, qw .$$

If we introduce the relation (2. 1), $v_k = qt_k$, into (2. 11) and note that $t_k$, $n_k$, $b_k$ are mutually orthogonal, we find that

$$(2.12) \qquad b^k \frac{\partial t_k}{\partial n} - n^k \frac{\partial t_k}{\partial b} = 2w .$$

Now, through use of (2. 8), (2. 9), the last term on the right hand side of (2. 7) reduces to

$$(2.13) \qquad qe^{ijk} L_{jk} = qe^{ijk} w_{jk} = qwe^{ijk}(n_j b_k - n_k b_j) .$$

If we recall the cross-product relations

$$(2.14) \qquad e^{ijk} n_j b_k = t^i , \quad e^{ijk} t_j n_k = b^i , \quad e^{ijk} b_j t_k = n^i ,$$

evaluate (2. 13), and replace w by its value as given by (2. 12), we see that

$$(2.15)\quad q e^{ijk} \mathcal{L}_{jk} = q t^i \left( b^k \frac{\partial t_k}{\partial n} - n^k \frac{\partial t_k}{\partial b} \right).$$

The terms in the parenthesis on the right hand side are known as rotation coefficients.[3]

Secondly, we evaluate the first and second terms in the right hand side of (2.7). By decomposing $\partial_j q$, we obtain

$$(2.16)\quad \partial_j q = \frac{\partial q}{\partial s} t_j + \frac{\partial q}{\partial n} n_j + \frac{\partial q}{\partial b} b_j$$

where $\partial q/\partial s$ indicates the rate of change of q with respect to arc length along the $t_j$ direction. Forming the scalar product of (2.16) with $e^{ijk} v_k$, recalling that $v_k = q\, t_k$, and (2.14), we find that

$$(2.17)\quad e^{ijk} v_k \partial_j (\ln q) = - \frac{\partial q}{\partial n} b^i + \frac{\partial q}{\partial b} n^i .$$

Finally, since the curvature vector, $u_k$, is given by

$$u_k = \kappa n_k$$

where $\kappa$ is the curvature of the stream lines, the second term in (2.7) reduces to

$$(2.18)\quad e^{ijk} v_j u_k = q \kappa b^i .$$

In order to obtain the desired generalization of formula (1.1), we substitute (2.15), (2.17), (2.18) into the right hand side of (2.7) and find that

3) Einführung ...., J. A. Schouten and D. J. Struik, loc. cit., p. 33

$$(2.19)\quad \omega^j = q\left(b^k \frac{\partial t_k}{\partial n} - n^k \frac{\partial t_k}{\partial b}\right) t^j + \frac{\partial q}{\partial b} n^j + \left(-\frac{\partial q}{\partial n} + q\kappa\right) b^j .$$

For plane flow , we know that

$$\frac{\partial q}{\partial b} = 0 , \qquad \omega^j t_j = 0 .$$

Hence, (2.19) reduces to (1.1). Finally, we note that formula (2.19) may be generalized by replacing the principal normal and binormal vectors, $n^j$, $b^j$, by two arbitrary unit vectors, $r^j$, $m^j$ which lie in the local subspace orthogonal to $t^j$ and which are mutually orthogonal. Then, if the principal normal is expressed in terms of these vector fields by the relation

$$(2.20)\quad n^j = a\, m^j + b\, r^j , \quad a^2 + b^2 = 1$$

where a, b are scalars, a routine computation shows that (2.19) must be replaced by

$$(2.21)\quad \omega^j = q\left(m^k \frac{\partial t_k}{\partial r} - r^k \frac{\partial t_k}{\partial m}\right) t^j + \left(\frac{\partial q}{\partial m} + \kappa q\, b\right) r^j + \left(-\frac{\partial q}{\partial r} + \kappa q\, a\right) m^j$$

where $\partial q/\partial m$ and $\partial q/\partial r$ denote the rate of change of q with respect to arc length in the $m^j$, $r^j$ directions, respectively.

3. <u>An Intrinsic Expression for the Bernoulli Function in Steady State Non-Viscous Fluid Flow</u>. It is well known that for the steady state flow of a non-viscous

incompressible fluid, the Bernoulli function is defined by

$$B = \int \frac{dp}{\rho} + \frac{q^2}{2} + F ,$$

where p is the pressure, $\rho$ the density, F the potential of the body forces. Further, it is known[4] that the surfaces

$$B = \text{constant}$$

consist of stream lines and vortex lines, and

$$(3.1) \quad \frac{\partial B}{\partial p} = q \omega \sin \beta \quad ,$$

where $\partial B / \partial p$ denotes the rate of change of B with respect to a displacement normal to surfaces, B = constant, and $\beta$ is the angle between the velocity and vorticity vectors.

We shall express the right hand side of (3.1) in terms of intrinsic expressions involving q, the magnitude of the velocity vector. Evidently, the right hand side is the magnitude of the cross-product of $\omega^j$ and $v^j$. By use of (2.19), we may express this cross product by

$$(3.2) \quad e^{ijk} \omega_j v_k = - q \frac{\partial q}{\partial b} b^i + q \left( - \frac{\partial q}{\partial n} + q \kappa \right) n^i .$$

Thus, (3.1) may be written in the form

4) Hydrodynamics, H. Lamb, Dover Publications, New York, 1945, p. 244.

$$(3.3) \qquad \frac{\partial B}{\partial p} = \left[ \left(q \frac{\partial q}{\partial b}\right)^2 + q^2 \left( - \frac{\partial q}{\partial n} + qK \right)^2 \right]^{1/2}.$$

For parallel surfaces, the p is constant between two surfaces of the family, B = constant. Hence, a necessary and sufficient condition that the surfaces, B = constant, consist of parallel surfaces is that

$$q^2 \left[ \left(\frac{\partial q}{\partial b}\right)^2 + \left(qK - \frac{\partial q}{\partial n}\right)^2 \right] = \text{constant}$$

along each surface of the family.

In the case of the compressible non-viscous fluid, the above result remains valid providing we consider only isentropic fluids (that is, fluids of constant entropy). This result follows by modifying Lamb's argument in a trivial fashion.

4. Intrinsic Significance of the Equations of Motion, Continuity, and Energy for Compressible Non-Viscous Fluids. In this section, we consider the steady flow of a non-viscous, compressible fluid when the body forces vanish. For this case as well as for the incompressible case, the equations of motion are

$$(4.1) \qquad v^j \partial_j v_k = - \frac{1}{\rho} \partial_k p,$$

where p is the pressure and $\rho$ is the density. However, the continuity relation for the compressible and incompressible cases will be discussed separately. In the compressible case, the continuity relation is

$$(4.2) \qquad v^j \partial_j \rho + \rho \partial_j v^j = 0 ;$$

and in the incompressible case, this relation reduces to

$$(4.3) \quad \partial_j v^j = 0 .$$

We shall consider the energy relation for the case of polytropic gases. In this case, the condition that entropy is constant along any stream line may be expressed by the Bernoulli relation[5)]

$$(4.4) \quad v^j \partial_j \left(h + \frac{q^2}{2}\right) = 0$$

where h, the enthalpy, is given by

$$(4.5) \quad h = c_p T = \frac{c^2}{\gamma - 1} .$$

The constant $c_p$ is the specific heat at constant pressure, $\gamma$ is the ratio of the specific heats, T is the absolute temperature, and c, the local sound speed, is defined by

$$(4.6) \quad c^2 = \left.\frac{\partial p}{\partial \rho}\right)_S$$

where S is the entropy. By use of the gas law,

$$p = R \rho T ,$$

where R is the universal gas constant, and of the formula (4.5), we may express the equations of motion and energy in terms of the variables, $v_j$, $T$, $\rho$, or equivalently in terms of $v_j$, $c^2$, $\rho$. A simple computation shows that these relations may be written as

5) Supersonic Flow and Shock Waves, R. Courant and K. O. Friedrichs, Interscience Press, N. Y., see equations (9.06), (14.05).

$$(4.7) \quad v^j \partial_j v_k = - \frac{1}{\gamma \rho} \partial_k (\rho c^2),$$

$$v^j \partial_j \left( \frac{c^2}{\gamma - 1} + \frac{q^2}{2} \right) = 0 .$$

In the special case of isentropic flows ($S$ = constant), we can express the equations of motion in the simplified form (see 4.6)

$$(4.8) \quad v^j \partial_j v_k = - \frac{c^2}{\rho} \partial_k \rho .$$

The energy relation of (4.7) is unaltered.

Now, we express the various terms of equations (4.2) (or (4.3)) and (4.7), (4.8) in terms of intrinsic quantities. The basic relation (2.6) may be written as

$$(4.9) \quad \partial_j v_k = t_k \partial_j q + q t_j u_k + q L_{jk} .$$

Let us choose the directions determined by the unit vectors, $r^j$, $m^j$ so that they coincide with the principal directions of the symmetric tensor $s_{jk}$ ( see (2.8)). Then, if $h_1$, $h_2$ are the principal values of $s_{jk}$, we may write

$$(4.10) \quad s_{jk} = h_1 r_j r_k + h_2 m_j m_k .$$

Similarly to (2.9), we may express the skew symmetric part of $L_{jk}$ by

$$(4.11) \quad w_{jk} = \overline{w} (r_j m_k - r_k m_j)$$

where the scalar, $\overline{w}$, may be expressed in terms of the rotation coefficients by (see 2.12)

(4.12) $$2\overline{w} = \left(m^k \frac{\partial t_k}{\partial r} - r^k \frac{\partial t_k}{\partial m}\right).$$

Further, it should be recalled that the tensor $L_{jk}$ of (4.9) is defined as the sum

(4.13) $$L_{jk} = s_{jk} + w_{jk}.$$

From these last relations and (4.9), we see that the left hand side of the equations of motion (4.7) may be expressed by

(4.14) $$v^j \partial_j v_k = q \frac{\partial q}{\partial s} t_k + q^2 u_k.$$

By contracting (4.9), we find that

(4.15) $$\partial_j v^j = \frac{\partial q}{\partial s} + q g^{jk} L_{jk}$$

where $g^{jk}$ is the metric tensor. With the aid of (4.10), (4.13) we see that

$$g^{jk} L_{jk} = (h_1 + h_2)$$

and (4.15) reduces to

(4.16) $$\partial_j v^j = \frac{\partial q}{\partial s} + q(h_1 + h_2).$$

Let us draw some conclusions from our calculations. First, consider the incompressible case and hence equations (4.1), (4.3). From (4.3), (4.16), we see that for an incompressible fluid, the rate of change of q with respect to arc length is given by

(4.17) $$\frac{\partial q}{\partial s} = -q(h_1 + h_2).$$

Further, by substituting equation (4.14) into the equations of motion (4.1), we find that the rate of change of pressure is determined by

$$(4.18) \qquad q \frac{\partial q}{\partial s} t_k + q^2 \kappa n_k = - \frac{1}{\rho} \partial_k p .$$

Thus, pressure does not vary in the binormal direction and its variations along the stream line and its principal normal are determined by the laws

$$(4.19) \qquad \frac{\partial p}{\partial s} = - \rho q \frac{\partial q}{\partial s}, \quad \frac{\partial p}{\partial n} = - \rho q^2 \kappa .$$

The formulas (4.19) are well known in the two dimensional plane flow case[6].

To discuss the compressible flow of fluids, we first consider the isentropic case. The equations of motion (4.8) become

$$(4.20) \qquad q \frac{\partial q}{\partial s} t_k + q^2 \kappa n_k = - \frac{c^2}{\rho} \partial_k \rho ;$$

the equation of continuity (4.2) reduces to

$$(4.21) \qquad q \frac{\partial \rho}{\partial s} + \rho \left[ \frac{\partial q}{\partial s} + q (h_1 + h_2) \right] = 0 ;$$

and the energy relation (4.7) reduces to

$$(4.22) \qquad \frac{\partial}{\partial s} \left( \frac{c^2}{\gamma - 1} + \frac{q^2}{2} \right) = 0 .$$

Again, we see that the pressure (and the density) do

6) Theoretical Hydrodynamics, L. M. Milne-Thomson, loc. cit., p. 101. Note, only the Frenet formulas need be used in deriving (4.19). Thus, equations (4.19) must be known, though the present author cannot find any reference to them.

not vary in the binormal direction. Instead of (4.19), we have the relations

$$(4.23) \quad c^2 \frac{\partial \rho}{\partial s} = -\rho q \frac{\partial q}{\partial s}, \quad c^2 \frac{\partial \rho}{\partial n} = -\rho q^2 K .$$

Eliminating the derivative of $\rho$ in (4.21) by use of the first relation in (4.23), we find that

$$(4.24) \quad (q^2 - c^2) \frac{\partial q}{\partial s} - c^2 q (h_1 + h_2) = 0 .$$

Thus, the variation of q along the stream lines is determined by the sum of the principal values of the tensor $s_{jk}$.

Finally, we consider the general non-isentropic case of a compressible fluid. In this case, the first equation of (4.7) replaces (4.8) and instead of (4.23) we must write

$$(4.25) \quad \frac{\partial}{\partial s}(\rho c^2) = -\gamma \rho q \frac{\partial q}{\partial s} ,$$
$$\frac{\partial}{\partial n}(\rho c^2) = -\gamma \rho q^2 K .$$

Through use of the energy relation (4.22), the first of the above relations reduces to the first relation of (4.23), or

$$(4.26) \quad c^2 \frac{\partial \rho}{\partial s} = -\rho q \frac{\partial q}{\partial s} .$$

Thus, for the isentropic and non-isentropic cases, the rate of change of density along a stream line is determined by the same law (namely, 4.26). Further, in the non-isentropic case, the variation of the magnitude of the velocity follows the same law (4.24) as in the isentropic case. The essential difference between these two cases must lie in the variation of $\rho$, q, c along the principal normal and binormal to the stream lines.

Some additional insight into this last topic is given by discussing the intrinsic form of the following relation[7]

$$\partial_k \left(\frac{c^2}{\gamma - 1} + \frac{q^2}{2}\right) - T\,\partial_k S = e_{kj\ell}\, v^j \omega^\ell .$$

Forming the cross product of $v^j$ and $\omega^\ell$ in (2.19), we find that the above relation reduces to

$$(4.27)\quad \partial_k \left(\frac{c^2}{\gamma - 1} + \frac{q^2}{2}\right) - T\,\partial_k S$$

$$= q\left(\frac{\partial q}{\partial n} - q\kappa\right) n_k + q\,\frac{\partial q}{\partial b}\, b_k .$$

In terms of components along the principal normal and binormal to the stream lines, this last equation shows that

$$(4.28)\quad \frac{\partial}{\partial n}\left(\frac{c^2}{\gamma - 1}\right) - T\,\frac{\partial S}{\partial n} = -q^2 \kappa ,$$

$$\frac{\partial}{\partial b}\left(\frac{c^2}{\gamma - 1}\right) - T\,\frac{\partial S}{\partial b} = 0 .$$

These relations show that the rate of change of c along the principal normal direction is completely determined by the rate of change of S along this direction, and similarly for the rate of change of c along the binormal direction. If two mutually orthogonal directions $r^j$, $m^j$ which are perpendicular to $t^j$ are introduced, then (4.28) must be replaced by

7) Supersonic Flow and Shock Waves, R. Courant and K. O. Friedrichs, loc. cit., equation (14.03). This relation is easily obtained from the equations of motion and is the basis of the work in section 3.

$$(4.29)\quad \begin{aligned} \frac{\partial}{\partial r}\left(\frac{c^2}{\gamma - 1}\right) - T\frac{\partial S}{\partial r} &= -q^2 a K, \\ \frac{\partial}{\partial m}\left(\frac{c^2}{\gamma - 1}\right) - T\frac{\partial S}{\partial m} &= -q^2 b K. \end{aligned}$$

5. The Case Where $v^j$ is Orthogonal to $\infty^1$ Surfaces. This is a generalization of the case of irrotational flow. In the latter case, the $\infty^1$ surfaces are the velocity equipotentials and $v^j$ is the gradient of a scalar function; in the present case, $v^j$ is proportional to the gradient of a scalar function. From the integrability condition

$$(5.1)\quad e^{ijk} v_i \partial_j v_k = v_j \omega^j = 0,$$

we see that $v^j$ is orthogonal to $\omega^j$, and conversely. Use of (2.19) shows that a necessary and sufficient condition for $v^j$ to be orthogonal to $\omega^j$ is

$$(5.2)\quad \omega^j = \frac{\partial q}{\partial b} n^j + \left(q K - \frac{\partial q}{\partial n}\right) b^j .$$

The condition (5.1) or (5.2) implies that, $w = 0$ (see (4.12) and (2.21)), and hence that $L_{jk}$ is a symmetric tensor. Further, the principal values $h_1$, $h_2$ of this tensor become the two principal curvatures of the $\infty^1$ surfaces orthogonal to $t^j$. Thus, only the mean curvature

$$M = h_1 + h_2$$

of these $\infty^1$ surfaces enters into relations (4.17) and (4.24).

For the incompressible case, we can state the following theorem: the magnitude of the velocity, q, varies along the stream lines according to the law

$$\frac{\partial q}{\partial s} = - M q .$$

In particular, a necessary and sufficient condition for q to be constant along a stream line is that M, the mean curvature, vanish along the stream line. At such points, the $\infty^1$ surfaces consist of hyperbolic points in the sense of differential geometry. The necessary and sufficient condition for q to be constant along any stream line is that the $\infty^1$ surfaces be minimal.

For the compressible case (isentropic and non-isentropic), relation (4.24) or

$$(q^2 - c^2) \frac{\partial q}{\partial s} - c^2 q M = 0$$

holds. In this case, we see that for either subsonic or supersonic flows, similar theorems to those of the preceding paragraphs are valid. Further, at those points where a transition between subsonic $(q < c)$ and supersonic $(q > c)$ occurs, the mean curvature of the $\infty^1$ surfaces orthogonal to the stream line vanishes.

Finally, the equations (4.29) furnish an interesting result. Consider the case where the stream lines are straight lines and the entropy depends only upon the temperature, then (4.29) shows that

$$(5.3) \qquad \frac{c^2}{\gamma - 1} - \int^T T \, d S = \text{constant},$$

for the surface elements determined by the directions $r_k$, $m_k$. The surfaces defined by (5.3) consists of the planar elements determined by the directions $r_k$, $m_k$, and these surfaces are perpendicular to the rectilinear congruence of stream lines. Thus, if the stream lines consist of a rectilinear congruence and entropy is a function of termperature only, then the stream lines are orthogonal to $\infty^1$ surfaces.

## Added in Proof

1. Presented to the American Mathematical Society, December 1952, St. Louis, Missouri.

2. The theorems of section 5 involving M, the mean curvature, are known for the cases of the incompressible and compressible isentropic fluids. It should be noted that the fact that (4.26) is valid for non-isentropic as well as for isentropic flows leads to the result that the above theorems remain valid in the compressible non-isentropic case. The following articles contain proofs of these theorems in the incompressible and compressible isentropic cases.

   a. the incompressible case: L. Castoldi, Sopra una proprieta dei moti permanenti di fluidi incomprimible..., Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8), 3, 333-347 (1947).

   b. the compressible isentropic case: S. S. Byušgens, The critical surface of an adiabatic flow, Doklady Akad. Nauk, SSSR (N.S.) 58, 365-368 (1948) also M. Giqueaux, Sur la gemoetrie des ecoulements permanentes des fluids compressibles, C. R. Acad. Sci. Paris, 226, 222-224, (1948).

3. In the theorem on rectilinear congruences of section 5, the restriction that the entropy be a function of temperature only is not necessary. By letting the curvature vanish in (4.29), replacing $c^2$ by $c_p T$, and integrating, we find that, $S - c_p \ln T =$ constant, along each of the $\infty^1$ surfaces orthogonal to $t^j$. By a change of thermodynamic variables, these surfaces can be written in the form, p = constant. This can be seen, more directly, by use of the intrinsic form of the equations of motion. For incompressible flow, this theorem is due to: S. S. Byušgens, The geometry of the stationary flow of an ideal incompressible fluid, Izvestiya Akad. Nauk, S.S.S.R. Ser. Nat. 12, 481-512 (1948).

4. It should be remarked that the $h_1$, $h_2$ of section 4 are known when the stream line pattern is given. In fact, the relations (4.17), (4.24) show that only that component of the stream line pattern which determines $h_1$, $h_2$ is essential in determining , q along a stream line.