Michigan Math. J. 50 (2002)

Near-Field Behavior of Static Spherically Symmetric Solutions of Einstein SU(2)–Yang/Mills Equations

Alexander N. Linden

1. Background

1.1. Introduction

Static spherically symmetric solutions of the Einstein SU(2)–Yang/Mills equations both with and without cosmological constant have been studied extensively over the last decade. Among the few analytical and several numerical results is a rigorous proof of the existence of a 1-parameter family of such solutions in which each solution in the family is smooth at the origin of spherical symmetry (see [14]). These solutions give boundary conditions at the center of spherical symmetry that must be satisfied for solutions to be smooth in a neighborhood of the origin. Many studies have considered the global behavior of such solutions.

Smoller and Wasserman [7] proved the existence of a discrete family of solutions that are everywhere smooth. These solutions exist only with vanishing cosmological constant. Breitenlohner, Forgács, and Maison [1], also considering only the case of vanishing cosmological constant, classified solutions that are smooth near the origin. References [5] and [6] describe classes of solutions that exist in the case of a positive cosmological constant.

Other studies have considered solutions of the equations without cosmological constant with given boundary conditions at infinity. For example, Wasserman [16] describes the number of singularities such solutions can have. It has already been mentioned that, in the case of zero cosmological constant, there exist conditions at infinity that yield solutions that are globally smooth in Schwarzschild coordinates. Winstanley [17] proved the existence of such solutions in the case of negative cosmological constant. However, no such solutions can exist in the presence of a positive cosmological constant (see [5]). Specifically, with a positive cosmological constant present, each solution that is smooth at the origin gives rise to a singularity at some finite Schwarzschild radius. Reference [3] describes the nature of such a singularity. Smoller, Wasserman, and Yau [13], considering only the case of vanishing cosmological constant, analyzed the behavior of solutions that satisfy certain conditions at this singularity.

In this paper we consider the Einstein SU(2)–Yang/Mills equations with arbitrary cosmological constant in Schwarzschild coordinates and take arbitrary

Received March 8, 2001. Revision received June 14, 2001.

boundary conditions not at infinity or at the center of spherical symmetry or at a singularity but rather at arbitrary positive finite radius $\bar{r}$ so as to make the equations nonsingular in a neighborhood of $\bar{r}$. We then analyze the behavior of such solutions as r decreases from $\bar{r}$.

We begin with Einstein's equations

$$R_{ij} - \frac{1}{2} R g_{ij} = \frac{8\pi G}{c^4} T_{ij} + \Lambda g_{ij} \tag{1}$$

coupled to the Yang/Mills equation

$$D * D\mathcal{A} = 0. \tag{2}$$

In equation (1), g_{ij} is the metric and R_{ij}, R are (respectively) the metric Ricci and scalar curvatures; G, c, and T_{ij} are physical quantities—respectively Newton's gravitational constant, the speed of light in vacuum, and the stress energy tensor. We use Λ to denote the cosmological constant. In equation (2), $\mathcal{A}$ is the Yang/Mills connection 1-form, D is the covariant derivative with respect to this connection, and $*$ is the Hodge star operator. The Yang/Mills and gravitational fields are coupled both through the stress energy tensor in equation (1) and, because the Hodge star operator depends on the metric, through equation (2).

In Schwarzschild coordinates a spherically symmetric metric assumes the form

$$\mathrm{d}s^2 = C^2 A\,\mathrm{d}t^2 - \frac{1}{A}\,\mathrm{d}r^2 - r^2(\mathrm{d}\phi^2 + \sin^2\phi\,\mathrm{d}\theta^2). \tag{3}$$

With suitable gauge under the assumption of a static magnetic field, a spherically symmetric SU(2)–Yang/Mills connection assumes the form

$$\mathcal{A} = \mathrm{w}\tau_2\,\mathrm{d}\phi + (\cos\phi\,\tau_3 - \mathrm{w}\sin\phi\,\tau_1)\,\mathrm{d}\theta. \tag{4}$$

Here τ_i are the following matrices, which form a basis of SU(2):

$$\tau_1 = \frac{i}{2}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \tau_2 = \frac{i}{2}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \tau_3 = \frac{i}{2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The Einstein–Yang/Mills equations with this metric and Yang/Mills connection are as follows:

$$rA' + 2A\mathrm{w}'^2 = \Phi, \tag{5a}$$

$$r^2 A\mathrm{w}'' + r\Phi\mathrm{w}' + \mathrm{w}(1-\mathrm{w}^2) = 0, \tag{5b}$$

$$\frac{C'}{C} = \frac{2\mathrm{w}'^2}{r}, \tag{5c}$$

where

$$\Phi = 1 - A - \frac{(1-\mathrm{w}^2)^2}{r^2} - \Lambda r^2; \tag{6}$$

A, w, and C are unknown functions of r. Here and throughout the paper, a prime denotes a derivative with respect to r.

As previously stated, we choose boundary conditions at positive $\bar{r}$ such that equations (5a)–(5c) are nonsingular in a neighborhood of $\bar{r}$. If there exists some $r < \bar{r}$ where equations (5a)–(5c) become singular, we take r_0 to be the largest such r. Otherwise, we take $r_0 = 0$. We analyze the local geometry of spacetime in a neighborhood of r_0. In particular, we prove that each solution describes a spacetime of one of the three following types.

(1) The spacetime manifold is regular in a neighborhood of the center of spherical symmetry. We call such solutions *smooth*. These are the solutions that satisfy special boundary conditions at $r = 0$.
(2) The spacetime manifold is regular near the center of spherical symmetry and has a Reissner–Nordström singularity at this center. We call such solutions *Reissner–Nordström-like.*
(3) The spacetime manifold has a black hole surrounding the center of spherical symmetry. We call such solutions *Schwarzschild-like.*

In [9] can be found a proof that, for $\Lambda = 0$, every solution describes a spacetime of one of these types. However, the arguments used therein do not apply to the general case $\Lambda \neq 0$. This paper provides a proof that does not depend on the value of Λ. In particular, we prove that the value of Λ has no qualitative effect on the spacetime geometry in the region where r is small. Thus, a cosmological constant has only a far-field effect. This is as expected, because Λ appears in equations (5a)–(5c) only in a term Λr^2.

Before presenting the proof, we describe these geometries as they relate to explicit well-known solutions of the Einstein–Yang/Mills equations. Setting $\mathrm{w} \equiv 1$ and integrating equation (5a) yields the metric

$$\begin{aligned} \mathrm{d}s^2 = {} & \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) \mathrm{d}t^2 \\ & - \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^{-1} \mathrm{d}r^2 - r^2(\mathrm{d}\phi^2 + \sin^2\phi\, \mathrm{d}\theta^2), \end{aligned} \tag{7}$$

where M is an arbitrary constant. Note that $M > 0$ yields Schwarzschild solutions. Of particular interest is that, in these coordinates, there exists a positive r_0 such that metric (7) becomes singular at $r = r_0$. The singularity occurs because $A(r_0) = 1 - 2M/r_0 - \Lambda r_0^2/3 = 0$. When $M = 0$, we have the metric either of deSitter, Minkowski, or anti-deSitter space—depending on whether the cosmological constant Λ is positive, zero, or negative (respectively). All three of these spaces are notable in that the metric is smooth near $r = 0$ and can be extended to be smooth at $r = 0$. Finally, $M < 0$ yields a Reissner–Nordström solution. Relevant here is the fact that the metric (7) becomes singular at $r = 0$. The singularity occurs because $\lim_{r\searrow 0} A(r) = \infty$. In all cases, $\lim_{r\searrow\infty} \mathrm{w}'(r) = 0$ and the qualitative behavior for small r does not depend on the value of Λ.

We prove that these phenomena are quite general. In particular, we choose any point $(\bar r, \bar A, \bar C, \bar{\mathrm{w}}, \bar{\mathrm{v}}) \in \mathbb{R}^3_+ \times \mathbb{R}^2$ and consider the unique solution of equations (5a)–(5c) that satisfies $(A(\bar r), C(\bar r), \mathrm{w}(\bar r), \mathrm{w}'(\bar r)) = (\bar A, \bar C, \bar{\mathrm{w}}, \bar{\mathrm{v}})$. In Theorem 1 we prove that this solution is either a Schwarzschild-like solution or smooth for all $r \in (0, \bar r)$. In other words, Theorem 1 precludes the possibility of any other types of singularities at positive r. In Theorem 2 we consider solutions that are not Schwarzschild-like. We prove that, as long as A is bounded, any such solution must be a smooth solution. Theorem 3 states that all solutions with unbounded A are Reissner–Nordström-like with w' vanishing at $r = 0$.

1.2. Preliminaries

We begin by stating some basic and simple facts regarding solutions to equations (5a)–(5c).

Fact 1: Equation (5c) separates from equations (5a)–(5c). Once a solution to the latter two is known, equation (5c) yields

$$C(r) = C(\bar{r}) \exp\left\{\int_{\bar{r}}^{r} (2\mathrm{w}'^2/s)\,\mathrm{d}s\right\}. \tag{8}$$

Because C and C' are both finite whenever $A, \mathrm{w}, \mathrm{w}'$ are finite and $r > 0$, we can ignore equation (5c) and restrict our analysis to the system of equations (5a)–(5b).

Fact 2: Equations (5a)–(5b) are invariant under the transformation $(r, A, \mathrm{w}) \to (r, A, -\mathrm{w})$.

Fact 3: Any solution of equations (5a)–(5b) with constant w must be either $A = 1 - 2M/r - \Lambda r^2/3$ with $\mathrm{w}^2 \equiv 1$ or $A = 1 - 2M/r - \Lambda r^2/3 + 1/r^2$ with $\mathrm{w} \equiv 0$. These solutions are also the only possible solutions that satisfy, for some $\hat{r} > 0$, $\mathrm{w}(1 - \mathrm{w}^2)(\hat{r}) = 0$ and $\mathrm{w}'(\hat{r}) = 0$.

Fact 3 follows from equation (5b), integrating equation (5a), and standard uniqueness theorems.

Fact 4: Given any Λ and λ, there exists an interval $I_\lambda = [0, r_\lambda)$ in which can be found a solution of equations (5a)–(5b) with the following properties.

(I) $(A, \mathrm{w}^2, \mathrm{w}') \to (1, 1, 0)$ as $r \searrow 0$ and $\lim_{r\searrow 0} \mathrm{w}''(r) = -\lambda$.

(II) The solution is analytic in the interior of I_λ and $\mathcal{C}^{2+\alpha}$ in I_λ for a small $\alpha > 0$.

(III) The solutions depend continuously on λ.

A proof of Fact 4 in the case $\Lambda = 0$ can be found in [14]. The same proof is valid with minor modification in the general case $\Lambda \neq 0$. Solutions that satisfy properties (I), (II), and (III) are smooth.

We define the region

$$\Gamma = \{(r, A, \mathrm{w}, \mathrm{w}') : r > 0,\ A > 0,\ \mathrm{w}^2 \le 1, \text{ and } (\mathrm{w}, \mathrm{w}') \neq (0, 0)\}.$$

We then have the following.

Fact 5: Suppose equations (5a)–(5b) are nonsingular for all $r \in (r_0, r_c)$. Suppose also that $(r_c, A(r_c), \mathrm{w}(r_c), \mathrm{w}'(r_c)) \in \Gamma$ but $(r_e, A(r_e), \mathrm{w}(r_e), \mathrm{w}'(r_e)) \notin \Gamma$ for some $r_e \in (r_0, r_c)$. Then $(r, A(r), \mathrm{w}(r), \mathrm{w}'(r)) \notin \Gamma$ for all $r \in (r_0, r_c)$.

Fact 5 follows easily from the fact that $\mathrm{w}\mathrm{w}''(r) \ge 0$ whenever $\mathrm{w}^2(r) > 1$ and $\mathrm{w}'(r) = 0$.

In order to avoid repeating the same argument several times, we state the following lemma of basic calculus.

LEMMA 1. *Suppose f is differentiable for all $r > 0$ and $\underline{\lim}_{r\searrow 0}(rf')(r) > 0$. Then $\lim_{r\searrow 0} f(r) = -\infty$. Similarly, if $\overline{\lim}_{r\searrow 0}(rf') < 0$ then $\lim_{r\searrow 0} f(r) = \infty$.*

2. Extending Solutions

In this section we prove that solutions of equations (5a)–(5b) become nonsingular at $r_0 > 0$ only when $A(r_0) = 0$. This is the content of the following theorem.

THEOREM 1. *Let* $(\bar{r}, \bar{A}, \bar{\mathrm{w}}, \bar{\mathrm{v}}) \in \mathbb{R}^2_+ \times \mathbb{R}^2$ *be arbitrary. Let* (A, w) *be the unique solution of* (5a)–(5b) *that satisfies* $(A(\bar{r}), \mathrm{w}(\bar{r}), \mathrm{w}'(\bar{r})) = (\bar{A}, \bar{\mathrm{w}}, \bar{\mathrm{v}})$. *Then, for arbitrary* $r_0 \in (0, \bar{r})$, *equations* (5a)–(5b) *are nonsingular at* r_0 *whenever* $\overline{\lim}_{r \searrow r_0} A(r) > 0$.

Proof. It follows from standard theorems that equations (5a)–(5b) are singular at r_0 only if one of the following holds:

(A) $\underline{\lim}_{r \searrow r_0} A(r) \leq 0$;
(B) $\overline{\lim}_{r \searrow r_0} \mathrm{w}^2(r) = \infty$;
(C) $\overline{\lim}_{r \searrow r_0} \mathrm{w}'^2(r) = \infty$;
(D) $\overline{\lim}_{r \searrow r_0} A(r) = +\infty$.

We eliminate all of these possibilities in each of the following cases:

1. $\overline{\lim}_{r \searrow r_0} A(r) > 1$;
2. $\lim_{r \searrow r_0} A(r)$ exists and $0 < \lim_{r \searrow r_0} A(r) \leq 1$;
3. $\underline{\lim}_{r \searrow r_0} A(r) < \overline{\lim}_{r \searrow r_0} A(r) \leq 1$.

Case 1. In Lemma 6 we will prove that $\lim_{r \searrow r_0} A(r) = A_0$ exists and that $A_0 > 1$. Thus, equations (5a)–(5b) cannot become singular at r_0 on account of condition (A). The other possibilities are eliminated according to the scheme shown in Figure 1 in the case $A_0 = \infty$ and as shown in Figure 2 for the case $A_0 < \infty$. Figures 1 and 2 should be read as follows: at each node of the tree, we assume that everything beginning with and including the root is true. What is cited in parentheses excludes the possibility that, under these assumptions, equations (5a)–(5b) are singular at r_0. It is clear that the leaves of these trees exhaust all of conditions (B), (C), and (D).

Case 2. All of the possibilities are eliminated, as shown in Figure 2.

Case 3. We claim that this case cannot occur. Indeed otherwise, because (by hypothesis) $\underline{\lim}_{r \searrow r_0} A(r) < \overline{\lim}_{r \searrow r_0} A(r)$, $\overline{\lim}_{r \searrow r_0} A(r) > 0$, and $r_0 > 0$, the mean value theorem gives $\rho > r_0$ that satisfy $0 < A(\rho) \leq 1$ and $\rho A'(\rho) > 1$. Therefore,

$$\left[(rA' - 1) + 2A\mathrm{w}'^2 + A + \frac{(\mathrm{w}^2 - 1)^2}{r^2} + \Lambda r^2 \right] \Bigg|_{(r=\rho)} > 0,$$

as each term on the left side is positive. However, this contradicts equation (5a). $\square$

It remains to prove Lemmas 2, 3, 4, 5, and 6.

LEMMA 2. *Suppose* $\lim_{r \searrow \tilde{r}_0} A(r) > 0$ *and* $\overline{\lim}_{r \searrow \tilde{r}_0} \mathrm{w}(r) = \pm\infty$. *Then* $\tilde{r}_0 = 0$.

$$
\begin{tikzcd}
 & \lim_{r\searrow r_0} A(r)=\infty \arrow[dl] \arrow[dr] & \\
\overline{\lim}_{r\searrow r_0} \mathrm{w}'^2(r)=\infty \arrow[d] \arrow[dr] & & \begin{array}{c}\overline{\lim}_{r\searrow r_0} \mathrm{w}'^2(r)<\infty \\ \text{(Lemma 5)}\end{array} \\
\lim_{r\searrow r_0} \mathrm{w}(r) \ \text{exists} \arrow[d] \arrow[dr] & \begin{array}{c}\lim_{r\searrow r_0} \mathrm{w}(r) \ \text{does not exist} \\ \text{(Lemma 3)}\end{array} & \\
\begin{array}{c}\lim_{r\searrow r_0} \mathrm{w}(r)<\infty \\ \text{(Lemma 4)}\end{array} & \begin{array}{c}\lim_{r\searrow r_0} \mathrm{w}(r)=\infty \\ \text{(Lemma 2)}\end{array} &
\end{tikzcd}
$$

Figure 1

$$
\begin{tikzcd}
 & 0<\lim_{r\searrow r_0} A(r)<\infty \arrow[dl] \arrow[dr] & \\
\overline{\lim}_{r\searrow r_0} \mathrm{w}^2(r)<\infty \arrow[d] \arrow[dr] & & \begin{array}{c}\overline{\lim}_{r\searrow r_0} \mathrm{w}^2(r)=\infty \\ \text{(Lemma 2)}\end{array} \\
\overline{\lim}_{r\searrow r_0} \mathrm{w}'^2(r)=\infty \arrow[d] \arrow[dr] & \begin{array}{c}\overline{\lim}_{r\searrow r_0} \mathrm{w}'^2(r)<\infty \\ \text{(standard theorems)}\end{array} & \\
\begin{array}{c}\lim_{r\searrow r_0} \mathrm{w}(r) \ \text{exists} \\ \text{(Lemma 4)}\end{array} & \begin{array}{c}\lim_{r\searrow r_0} \mathrm{w}(r) \ \text{does not exist} \\ \text{(Lemma 3)}\end{array} &
\end{tikzcd}
$$

Figure 2

Proof. We prove that the assumption $\tilde{r}_0 > 0$ leads to a contradiction. Because of Fact 2, we may assume that $\underline{\lim}_{r\searrow\tilde{r}_0} \mathrm{w}(r) = +\infty$. Equation (5b) gives $\mathrm{w}''(\rho) > 0$ for any $\rho \in (r_0, \bar{r})$ that satisfies $\mathrm{w}(\rho) > 1$ and $\mathrm{w}'(\rho) = 0$. Consequently, $\lim_{r\searrow\tilde{r}_0} \mathrm{w}(r) = \infty$, $\underline{\lim}_{r\searrow\tilde{r}_0} \mathrm{w}'(\rho) = -\infty$, and $\mathrm{w}'(\rho) < 0$ for all ρ in some neighborhood $U = (\tilde{r}_0, \tilde{r}_0 + \varepsilon)$.

We next prove that

$$\lim_{r\searrow\tilde{r}_0} \mathrm{w}'(r) = -\infty. \tag{9}$$

Toward this end we note that, for any solution of equations (5a)–(5b), $A\mathrm{w}'$ satisfies the following:

$$r(A\mathrm{w}')' + 2\mathrm{w}'^2(A\mathrm{w}') + \frac{\mathrm{w}(1-\mathrm{w}^2)}{r} = 0. \tag{10}$$

It is clear from equation (10) that $(A\mathrm{w}')' > 0$ for all $r \in U$. This implies that $\lim_{r\searrow\tilde r_0} A\mathrm{w}'(r)$ exists. Because $\lim_{r\searrow\tilde r_0} A(r)$ exists and is nonzero, $\lim_{r\searrow\tilde r_0} \mathrm{w}'(r)$ also exists. The only possibility is that $\lim_{r\searrow\tilde r_0} \mathrm{w}'(r) = -\infty$. This is equation (9).

To complete the proof, we write equation (5b) as

$$\frac{r^2A\mathrm{w}''}{\mathrm{w}(\mathrm{w}^2-1)} = 1 - \mathrm{w}'r\left[\frac{1-A-\Lambda r^2}{\mathrm{w}(\mathrm{w}^2-1)} - \frac{(\mathrm{w}^2-1)}{\mathrm{w}r^2}\right]. \tag{11}$$

For all $r \in U$, the term inside the square brackets in equation (11) is negative. Equation (9) gives $\eta > 0$ that satisfy $\mathrm{w}'(r) < -\eta < 0$ for all $r \in U$. It is now clear that U can be chosen sufficiently small so that the dominant term on the right side of equation (11) is $\mathrm{w}\mathrm{w}'(\mathrm{w}^2-1)/(r\mathrm{w})$; that is, $\mathrm{w}''(r) < 0$ for all $r \in U$. But this contradicts equation (9). □

LEMMA 3. *Suppose* $\underline{\lim}_{r\searrow\tilde r_0} A(r) > 0$ *and* $\underline{\lim}_{r\searrow\tilde r_0} \mathrm{w}(r) < \overline{\lim}_{r\searrow\tilde r_0} \mathrm{w}(r)$. *Then* $\tilde r_0 = 0$.

Proof. We assume $\tilde r_0 > 0$ and reach a contradiction. As in Fact 5, equation (5b) implies $\overline{\lim}_{r\searrow\tilde r_0} \mathrm{w}^2(r) \le 1$. We now claim that, for any $\varepsilon > 0$ and $M > 0$, there exist $\hat r(\varepsilon, M)$ close to $\tilde r_0$ that satisfy $|\mathrm{w}'(\hat r)| < \varepsilon$ and $|\mathrm{w}''(\hat r)| > M$. Because $\lim_{r\searrow\tilde r_0} \mathrm{w}(r)$ does not exist, there exists a sequence $\{r_n\} \searrow \tilde r_0$ that satisfies $\mathrm{w}'(r_n) = 0$ and $\lim_{n\nearrow\infty} \mathrm{w}(r_n) = \underline{\lim}_{r\searrow\tilde r_0} \mathrm{w}(r)$. There also exists another sequence $\{s_n\} \searrow \tilde r_0$ that satisfies $\mathrm{w}'(s_n) = 0$ and $\lim_{n\nearrow\infty} \mathrm{w}(s_n) = \overline{\lim}_{r\searrow\tilde r_0} \mathrm{w}(r)$. Without loss of generality, we assume $r_n < s_n < r_{n-1}$. The mean value theorem gives $t_n \in (r_n, s_n)$ that satisfy

$$\mathrm{w}'(t_n) = \frac{\mathrm{w}(s_n) - \mathrm{w}(r_n)}{s_n - r_n}.$$

Clearly, $\lim_{n\nearrow\infty} \mathrm{w}'(t_n) = \infty$. Now, for any $\varepsilon > 0$, we define $b_n(\varepsilon)$ to be the smallest $r > r_n$ that satisfies $\mathrm{w}'(b_n(\varepsilon)) = \varepsilon$ and define $V_n(\varepsilon) = [r_n, b_n(\varepsilon))$. Clearly, $\mathrm{w}'(r) < \varepsilon$ for all $r \in V_n(\varepsilon)$. For any ε, each V_n is nonempty. Also, there exist $N(\varepsilon)$ such that $t_n \notin V_n(\varepsilon)$ for all $n > N(\varepsilon)$. Thus, $\lim_{n\nearrow\infty}(b_n - r_n) = 0$. The mean value theorem now gives, for each ε, some $u_n(\varepsilon) \in V_n(\varepsilon)$ that satisfy

$$\mathrm{w}''(u_n) = \frac{\mathrm{w}'(t_n) - \mathrm{w}'(r_n)}{(t_n - r_n)} = \frac{\varepsilon}{(t_n - r_n)}.$$

Clearly, $\lim_{n\nearrow\infty} \mathrm{w}''(u_n) = +\infty$. This proves the claim.

Finally, we evaluate equation (5b) at $\hat r$:

$$\left[\hat r^2\mathrm{w}'' + \frac{\hat r\mathrm{w}'}{A}\left(1 - \frac{(\mathrm{w}^2-1)^2}{\hat r^2} - \Lambda\hat r^2\right) - \hat r\mathrm{w}' + \frac{\mathrm{w}(1-\mathrm{w}^2)}{A}\right]_{r=\hat r(\varepsilon,M)} = 0. \tag{12}$$

Since w is bounded and since (by assumption) $\hat r(\varepsilon, M) > \tilde r_0 > 0$ for all ε and M, it follows that we can choose ε sufficiently small and M sufficiently large so that the first term in equation (12) dominates. Hence the left side of equation (12) is nonzero, contradicting equation (5b). The result follows. □

LEMMA 4. *Suppose that* $\lim_{r\searrow\tilde r_0} A(r)$ *exists and is positive. Suppose also that* $\overline{\lim}_{r\searrow\tilde r_0} \mathrm{w}'^2(r)=\infty$ *and that* $\lim_{r\searrow\tilde r_0}\mathrm{w}(r)=\mathrm{w}_0$ *is finite. Then* $\tilde r_0 = 0$.

Proof. We assume $\tilde r_0 > 0$ and arrive at a contradiction. Clearly,

$$\overline{\lim_{r\searrow\tilde r_0}}\ \Phi(r) < 1.$$

Making use of Fact 2, we assume that $\overline{\lim}_{r\searrow\tilde r_0}\mathrm{w}'(r)=+\infty$. It follows that

$$\overline{\lim_{r\searrow\tilde r_0}}\ \ln(\mathrm{w}')=+\infty.$$

This implies that $\underline{\lim}_{r\searrow\tilde r_0}(\ln(\mathrm{w}))'=-\infty$; that is, for any $M>0$ there exist ρ near $\tilde r_0$ that satisfy $\mathrm{w}'(\rho)>1$ and $\mathrm{w}''(\rho)/\mathrm{w}'(\rho)<-M$. If M is sufficiently large, then

$$\left[r^2A\frac{\mathrm{w}''}{\mathrm{w}'}+r\Phi+\frac{\mathrm{w}(1-\mathrm{w}^2)}{\mathrm{w}'}\right]_{r=\rho}<0. \tag{13}$$

This is because the first term on the left side of equation (13) is large and negative while the second term is at most ρ and the third term is at most 1. However, inequality (13) contradicts equation (5b). The result follows. □

LEMMA 5. *Suppose* $\lim_{r\searrow\tilde r_0} A(r)=+\infty$ *and* $\overline{\lim}_{r\searrow\tilde r_0}\mathrm{w}'^2(r)<+\infty$. *Then* $\tilde r_0=0$.

Proof. We rewrite equation (5a) as

$$\frac{rA'}{A}+1+2\mathrm{w}'^2=\frac{1}{A}-\frac{(\mathrm{w}^2-1)^2}{r^2A}-\frac{\Lambda r^2}{A}. \tag{14}$$

Next, we assume $\tilde r_0>0$ and arrive at a contradiction. As $r\searrow\tilde r_0$, the right side of equation (14) approaches 0. On the other hand, there exists some positive M that satisfies $1+2\mathrm{w}'^2<\tilde r_0 M$ in a neighborhood $U=(\tilde r_0,\ \tilde r_0+\varepsilon)$. Thus,

$$A'>-MA\quad\text{for all }\ r\in U. \tag{15}$$

Integrating inequality (15) on any interval $(r,r_2)\subset U$ gives

$$A(r)<A(r_2)e^{M(r_2-r)}\quad\text{for all }\ r\in U. \tag{16}$$

Taking the limit in inequality (16) as $r\searrow\tilde r_0$ yields

$$\lim_{r\searrow\tilde r_0}A(r)<A(r_2)e^{M\varepsilon}<\infty.$$

However, this contradicts our hypothesis. □

LEMMA 6. *Suppose there exists some* $\rho\in(r_0,\bar r)$ *that satisfies* $A(\rho)=\bar A\geq 1-\Lambda\rho^2/3$. *Then* $\lim_{r\searrow r_0}A(r)$ *exists and* $\lim_{r\searrow r_0}A(r)\geq 1-\Lambda r_0^2/3$.

Proof. We define

$$\mu=r(1-A-\Lambda r^2/3). \tag{17}$$

For any solution of equations (5a)–(5b), μ satisfies

$$\mu' = 1 - A - rA' - \Lambda r^2 = \frac{(\mathrm{w}^2-1)^2}{r^2} + 2A\mathrm{w}'^2 \geq 0. \tag{18}$$

If $(\mathrm{w}^2(\rho), \mathrm{w}'(\rho)) = (1, 0)$, then

$$A = 1 - \frac{\Lambda r^2}{3} + \frac{\rho}{r}\left[\bar{A} + \frac{\Lambda\rho^2}{3} - 1\right], \quad \mathrm{w}^2 \equiv 1, \tag{19}$$

is the unique solution of equations (5a) and (5b) that satisfies $\mathrm{w}(\rho) = 1$, $\mathrm{w}'(\rho) = 0$, and $A(\rho) = \bar{A}$. However, our assumptions imply that the term in the square brackets of equation (19) is nonnegative. This term obviously does not depend on r. Therefore, with the solution equation (19), equations (5a)–(5b) are nonsingular for all $r \in (0, \rho)$. Moreover, $\lim_{r\searrow 0} A(r)$ exists, and either $\lim_{r\searrow 0} A(r) = 1$ or $\lim_{r\searrow 0} A(r) = +\infty$.

To finish the proof, we need consider only the case that there exists some $\rho \in (r_0, \bar{r})$ that satisfies

$$\mathrm{w}(1-\mathrm{w}^2)(\rho) \neq 0.$$

Equation (18) gives $\mu'(\rho)/\rho > 0$. Also, our assumptions imply that $\mu(\rho)/\rho^2 \leq 0$. Therefore,

$$\left(\frac{\mu}{r}\right)'(\rho) = \frac{\mu'(\rho)}{\rho} - \frac{\mu(\rho)}{\rho^2} > 0.$$

We now suppose that there exists some $\hat{r} \in (r_0, \rho)$ that satisfies $(\mu/r)'(\hat{r}) = 0$. Because $\hat{r}$ can always be chosen so that $\mu(\hat{r})/\hat{r} < 0$, we have

$$\mu'(\hat{r}) = \left(\frac{r\mu}{r}\right)'(\hat{r}) = \hat{r}\left(\frac{\mu}{r}\right)'(\hat{r}) + \frac{\mu}{\hat{r}}(\hat{r}) < 0. \tag{20}$$

Equation (20) contradicts equation (18). Hence, in the interval (r_0, ρ), $(\mu'/r)' > 0$; that is,

$$A' = -\frac{2\Lambda r}{3} - \left(\frac{\mu}{r}\right)' < -\frac{2\Lambda r}{3}. \tag{21}$$

Since A' is bounded from above, it follows that $\lim_{r\searrow r_c} A(r)$ exists. Also, from equation (21) and the fact that $A(\rho) \geq 1 - \Lambda\rho^2/3$ it is clear that $\lim_{r\searrow r_c} A(r) \geq 1 - \Lambda r_0^2/3$. □

3. Behavior at the Origin

3.1. *Case* $A < 1$

In this section we prove that any solution of equations (5a)–(5b) that has bounded A and is not a Schwarzschild-like solution is smooth. We state this precisely as follows.

THEOREM 2. *Let* $(\bar{r}, \bar{A}, \bar{\mathrm{v}}) \in \mathbb{R}^2_+ \times \mathbb{R}$ *and let* $(A(r), \mathrm{w}(r))$ *be the solution of equations* (5a)–(5b) *that satisfies* $(A(\bar{r}), \mathrm{w}(\bar{r}), \mathrm{w}'(\bar{r})) = (\bar{A}, \bar{\mathrm{w}}, \bar{\mathrm{v}})$. *Suppose that, with this solution, equations* (5a)–(5b) *are nonsingular for all* $r \in (0, \bar{r})$. *Suppose also that there exist* A_0 *and* A_1 *that satisfy*

$$0 \le A_0 = \varliminf_{r \searrow 0} A(r) \le \varlimsup_{r \searrow 0} A(r) = A_1 \le 1.$$

Then

$$\lim_{r \searrow 0} (A(r), \mathrm{w}^2(r), \mathrm{w}'(r)) = (1, 1, 0).$$

We first prove that $\lim_{r \searrow 0} \mathrm{w}^2(r) = 1$ in the case $A_0 = 0$. We then prove that $\lim_{r \searrow 0} \mathrm{w}^2(r) = 1$ in the case $A_0 > 0$. We then use this limit to prove that $\lim_{r \searrow 0} \mathrm{w}'(r)$ exists. The rest of Theorem 2 will follow.

LEMMA 7. *Suppose* (A, w) *is a solution of equations* (5a)–(5b) *that satisfies the hypotheses of Theorem 2. Then either* $\lim_{r \searrow 0} \mathrm{w}^2(r) = 1$ *or* $\varliminf_{r \searrow 0} A(r) > 0$.

Proof. As in [1], we introduce the new variables

$$N = -\sqrt{A}, \tag{22}$$

$$U = N\mathrm{w}', \tag{23}$$

$$\kappa = \frac{1}{2N}(\Phi + 2U^2 + 2N^2) \tag{24}$$

and a new parameter τ defined by $\mathrm{d}r/\mathrm{d}\tau = rN$. Equations (5a)–(5c) then transform into

$$\dot{r} = rN, \tag{25a}$$

$$\dot{\mathrm{w}} = rU, \tag{25b}$$

$$\dot{N} = (\kappa - N)N - 2U^2, \tag{25c}$$

$$\dot{U} = -\frac{\mathrm{w}(1 - \mathrm{w}^2)}{r} - (\kappa - N)U, \tag{25d}$$

$$\dot{CN} = (\kappa - N)CN, \tag{25e}$$

where an overdot ($\dot{}$) here and elsewhere denotes $\mathrm{d}/\mathrm{d}\tau$. We also have the auxiliary equation

$$\dot{\kappa} = 1 + 2U^2 - \kappa^2 - 2\Lambda r^2. \tag{25f}$$

The metric (3) transforms into

$$\mathrm{d}s^2 = C^2 N^2\, \mathrm{d}t^2 - r^2(\tau)(\mathrm{d}\tau^2 + \mathrm{d}\phi^2 + \sin^2\phi\, \mathrm{d}\theta^2). \tag{26}$$

As expected, equation (25e) separates from the others. Hence, as in Fact 1, it can be ignored. Here N is defined to be negative, so that r decreases with increases in the new parameter τ. The solution (A, w) of equations (5a)–(5b) is equivalent to the solution (r, N, w, U) of equations (25a)–(25f) that satisfies $r(0) = \bar{r}$, $N(0) = -\sqrt{\bar{A}}$, $\mathrm{w}(0) = \bar{\mathrm{w}}$, $U(0) = -\sqrt{\bar{A}}\bar{\mathrm{v}}$, and

$$\kappa(0) = (1 + \bar{A} - (1 - \bar{\mathrm{w}}^2/\bar{r}^2 - \Lambda\bar{r}^2 + 2\bar{A}\bar{\mathrm{v}}^2)/\big(-2\sqrt{\bar{A}}\big).$$

We are free to choose $\tau = 0$ for this solution because equations (25a)–(25f) are autonomous.

Integrating equation (25a) yields, for any τ,

$$r(\tau) = \bar{r}\exp\left\{\int_0^\tau N(\tilde{\tau})\,\mathrm{d}\tilde{\tau}\right\}. \tag{27}$$

By assumption and because of Lemma 6, equation (27) implies that

$$\tau(r = 0) = \infty. \tag{28}$$

We now consider the following three cases separately:

1. $0 < \underline{\lim}_{\tau\nearrow\infty}\,\kappa(\tau)$;
2. $-3 - 2/\sqrt{3} \le \underline{\lim}_{\tau\nearrow\infty}\,\kappa(\tau) \le 0$;
3. $\underline{\lim}_{\tau\nearrow\infty}\,\kappa(\tau) < -3 - 2/\sqrt{3}$.

Case 1. We choose any $\tilde{\kappa}$ that satisfies $0 < \tilde{\kappa} < \underline{\lim}_{\tau\nearrow\infty}\,\kappa(\tau)$ and then choose $\tilde{\tau}$ sufficiently large so that $\kappa(\tau) > \tilde{\kappa}$ whenever $\tau > \tilde{\tau}$. Equation (25c) then gives, for all $\tau > \tilde{\tau}$,

$$\dot{N} < \tilde{\kappa}N. \tag{29}$$

Integrating equation (29) yields

$$\overline{\lim_{\tau\nearrow\infty}}\, N(\tau) < N(\tilde{\tau}) \lim_{\tau\nearrow\infty} e^{\tilde{\kappa}(\tau-\tilde{\tau})} < 0,$$

as desired.

Case 2. It suffices to assume that $\overline{\lim}_{\tau\nearrow\infty}\,N(\tau) = 0$ and then to establish, under this assumption, that $\lim_{\tau\nearrow\infty}\mathrm{w}^2(\tau) = 1$. We establish this latter limit by eliminating the following two possibilities:

a. $\lim_{\tau\nearrow\infty}\mathrm{w}(\tau)$ does not exist;
b. $\lim_{\tau\nearrow\infty}\mathrm{w}(\tau)$ exists but $\lim_{\tau\nearrow\infty}\mathrm{w}^2(\tau) \ne 1$.

Case 2a. It follows easily from equation (25f) that, if $\{\tau_n\} \nearrow \infty$ is an arbitrary sequence that satisfies $\kappa(\tau_n) = 0$, then $\dot{\kappa}(\tau_n) > 0$ for sufficiently large n. Therefore, $\overline{\lim}_{\tau\nearrow\infty}\,\kappa(\tau) \le 0$. In other words, κ is bounded.

We now rewrite equation (25f) as

$$\dot{\kappa} = -(\kappa - N)^2 + \frac{(1-\mathrm{w}^2)^2}{r^2} - \Lambda r^2. \tag{30}$$

Because $\lim_{\tau\nearrow\infty}\mathrm{w}(\tau)$ does not exist, there are two sequences $\{s_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ that satisfy the following conditions:

(i) $\lim_{n\nearrow\infty} s_n = \lim_{n\nearrow\infty} t_n = \infty$;
(ii) $s_n < t_n < s_{n+1}$ for each $n > 0$;
(iii) there exist $\varepsilon > 0$ that satisfy $\mathrm{w}(t_n) - \mathrm{w}(s_n) > \varepsilon$ for all $n > 0$;
(iv) for each $n > 0$ and for all $\tau \in (s_n, t_n)$, $\mathrm{w}(s_n) \le \mathrm{w}(\tau) \le \mathrm{w}(t_n)$;
(v) there exist $\tilde{\varepsilon}$ that satisfy, for all $n > 0$,

$$-1 + \tilde{\varepsilon} < \mathrm{w}(s_n) < 0 < \mathrm{w}(t_n) < 1 - \tilde{\varepsilon}.$$

Conditions (i)–(iv) are a restatement of the fact that w has no limit as $\tau \nearrow \infty$. Condition (v) follows easily from equation (5b) because no τ can exist that satisfies either all of $\mathrm{w}^2(\tau) > 1$, $\mathrm{w}'(\tau) = 0$, and $\mathrm{w}\mathrm{w}''(\tau) < 0$ or all of $0 < \mathrm{w}^2(\tau) < 1$, $\mathrm{w}'(\tau) = 0$, and $\mathrm{w}\mathrm{w}''(\tau) > 0$. As a result, we must have

$$-1 \le \varliminf_{\tau \nearrow \infty} \mathrm{w}(\tau) < 0 < \varlimsup_{\tau \nearrow \infty} \mathrm{w}(\tau) \le 1.$$

We claim that

$$\lim_{n \nearrow \infty} (t_n - s_n) = 0. \tag{31}$$

Indeed, integrating equation (30) yields

$$\kappa(t_n) > \kappa(s_n) - \left(-3 - 2\sqrt{3}\right)^2 (t_n - s_n) - \Lambda r_{s_n}^2 (t_n - s_n) + \frac{4\tilde{\varepsilon}^2}{r_{s_n}^2}(t_n - s_n), \tag{32}$$

where $r_{s_n} = r(s_n)$. Since $\lim_{\tau \nearrow \infty} r(\tau) = 0$ it follows that, if there exists some $\eta > 0$ and a subsequence n_j that satisfies $(t_n - s_n) > \eta$, then from inequality (32) we easily obtain that $\lim_{n_j \nearrow \infty} \kappa(t_{n_j}) = +\infty$. This contradicts the fact that κ is bounded. Therefore, no such subsequence can exist and equation (31) must hold.

The mean value theorem now implies the existence of a sequence $\{\tau_n\} \nearrow \infty$ that satisfies $\lim_{n \nearrow \infty} \dot{\mathrm{w}}(\tau_n) = \infty$. Therefore,

$$r^2 \kappa = \frac{r^2 + (rN)^2 - (1 - \mathrm{w}^2) + 2(rU)^2 - \Lambda r^4}{2N} \tag{33}$$

evaluated at τ_n gives $\varliminf_{n \nearrow \infty} r^2 \kappa(\tau_n) < 0$. However, because $\lim_{n \nearrow \infty} r(\tau_n) = 0$, this contradicts the fact that κ is bounded. The result in Case 2a follows.

Case 2b. By assumption, $A(r) > 0$ for all $r \in (0, \bar{r})$. The result now follows immediately from equation (5a).

Case 3. It follows easily from equations (25c) and (25f) that, for any solution of equations (25a)–(25f), $(\kappa + N)/2$ satisfies

$$2 \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\kappa + N}{2}\right) = -\left(\frac{\kappa + N}{2}\right)^2 - 3\left(\frac{\kappa - N}{2}\right)^2 + 1 - 2\Lambda r^2. \tag{34}$$

We claim that there exists some $\tilde{\tau}$ that satisfies

$$\kappa(\tau) < -1 - 2/\sqrt{3} \quad \text{for all } \tau > \tilde{\tau}.$$

Indeed, we choose any $\tilde{\tau}$ that satisfies $\kappa(\tilde{\tau}) < -1 - 2/\sqrt{3}$. From Lemma 6 it follows that $\inf_{\tau > \tilde{\tau}} N(\tau) > -1$. Now, if there exists some $\tau_{\kappa+N}^{-2-2/\sqrt{3}} > \tilde{\tau}$ that satisfies $(\kappa + N)\left(\tau_{\kappa+N}^{-2-2/\sqrt{3}}\right) = -2 - 2\sqrt{3}$ or some $\tau_{\kappa}^{-1-2/\sqrt{3}}$ that satisfies $\kappa\left(\tau_{\kappa}^{-1-2/\sqrt{3}}\right) = -1 - 2/\sqrt{3}$, then $\tau_{\kappa+N}^{-2-2/\sqrt{3}} < \tau_{\kappa}^{-1-2/\sqrt{3}}$. However, equation (34) gives $(\dot{\kappa} + \dot{N}) < 0$ for all $\tau \in \left(\tilde{\tau}, \tau_{\kappa}^{-1-2/\sqrt{3}}\right)$. This establishes the claim.

Equation (34) now gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\kappa + N}{2}\right) < -\frac{1}{2}\left(\frac{\kappa + N}{2}\right)^2 \tag{35}$$

for all $\tau > \tilde{\tau}$. Integrating inequality (35) then yields a finite τ_0 that satisfies $\lim_{\tau \nearrow \tau_0} \kappa(\tau) = -\infty$; that is, equations (25a)–(25f) become singular at τ_0. Integrating equation (25a) yields equation (27), from which it follows that equations (5a)–(5b) become singular at some strictly positive r_0. However, this is contrary to assumption. □

LEMMA 8. *Suppose* (A, w) *is a solution of equations* (5a)–(5b) *that satisfies the hypotheses of Theorem 2. Suppose also that* $\underline{\lim}_{r\searrow 0} A(r) > 0$. *Then* $\lim_{r\searrow 0} \mathrm{w}^2(r) = 1$.

Proof. It follows easily from equation (5b) that $\lim_{r\searrow 0} \mathrm{w}(r)$ exists whenever $\overline{\lim}_{r\searrow 0} \mathrm{w}^2(r) > 1$. In this case, Lemma 1 gives $\lim_{r\searrow 0} A(r) = \infty$, contrary to assumption. Similarly, Lemma 1 also gives $\lim_{r\searrow 0} A(r) = \infty$ whenever $\overline{\lim}_{r\searrow 0} \mathrm{w}^2(r) < 1$. Hence we may assume that

$$\overline{\lim_{r\searrow 0}}\, \mathrm{w}^2(r) = 1.$$

To complete the proof, it suffices to eliminate the possibility that

$$-1 \le \underline{\lim_{r\searrow 0}}\, \mathrm{w}(r) < \overline{\lim_{r\searrow 0}}\, \mathrm{w}(r) \le 1.$$

We prove that this inequality leads to a contradiction.

Because w has no limit, it follows that there exists a sequence $\{r_n\} \searrow 0$ that satisfies $\lim_{n\nearrow\infty} \mathrm{w}(r_n) = \overline{\lim}_{r\searrow 0} \mathrm{w}(r)$ and $\mathrm{w}'(r_n) = 0$. There also exists a sequence $\{s_n\}$ that satisfies

$$(A\mathrm{w}')(s_n) \searrow -\infty \quad \text{and} \quad (A\mathrm{w}')'(s_n) = 0. \tag{36}$$

We choose any $\delta \in (0, 1)$, any $\tilde{A}_1 > A_1$, and any c that together satisfy

$$c > \max\left\{\sqrt{\tilde{A}_1/(3\delta A_0)}, \sqrt{1 - A_0}, 1/\sqrt{A_0}\right\}. \tag{37}$$

Also, for each n, we define

$$r_n^0 = \min\{r > r_n : \mathrm{w}(r) = 0\},$$

$$r_n^\delta = \min\{r > r_n : \mathrm{w}(r) = \delta\}.$$

It follows easily from equation (5b) that r_n^0 is well-defined for all n, as is r_n^δ for sufficiently small δ. Next, if $\mathrm{w}(r_n) > 1 - cr_n$ then we define

$$t_n = \min\{r > r_n : \mathrm{w}(r) = 1 - cr\},$$

whereas if $\mathrm{w}(r_n) \le 1 - cr_n$ then we set $t_n = r_n$ (see Figure 3). We will prove that, for sufficiently large n, there can be no s_n. This will be our contradiction.

From equation (10) it is clear that for each n, $s_n \in [r_m, r_m^0]$ for some m. It is obvious that $t_n < r_n^\delta$ for any δ and sufficiently large n. We now consider the three intervals in which s_n could possibly lie:

1. $s_n \in [r_m, t_m]$ for some m;
2. $s_n \in (t_m, r_m^\delta)$ for some m;
3. $s_n \in [r_m^\delta, r_m^0]$ for some m.

We shall prove that, for large n, s_n cannot lie in any of these intervals.

Intervals of type 1. Because $A(s_n) < 1$ for all n, we have

$$\lim_{n\nearrow\infty} 2\mathrm{w}'^2(A\mathrm{w}')(s_n) = -\infty.$$

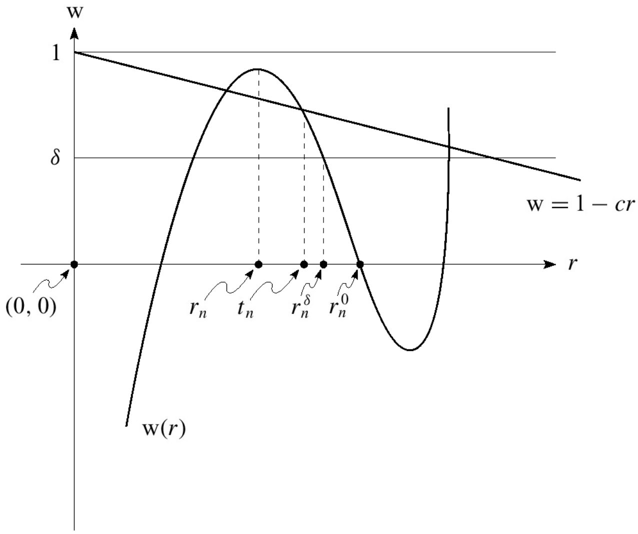

Figure 3

Equation (10) then gives $\lim_{n\nearrow\infty} \mathrm{w}(s_n)(1-\mathrm{w}^2(s_n))/s_n = +\infty$. However, in the interval $[r_m, t_m]$ we have $\mathrm{w}(1-\mathrm{w}^2)/r = \mathrm{w}(1+\mathrm{w})(1-\mathrm{w})/r < 2c$ for any m. It follows that there exist $M > 0$ such that, whenever $n > M$, $s_n \notin \bigcup_m [r_m, t_m]$.

Intervals of type 2. We note that the definition of t_m gives

$$\mathrm{w}'(t_m) \le -c. \tag{38}$$

Also, for sufficiently large m and all $r \in (t_m, r_m^\delta)$,

$$\Phi(r) = 1 - A - \frac{(1-\mathrm{w}^2)^2}{r^2} - \Lambda r^2 < 1 - A_0 - c^2 \le 0.$$

Substituting this into equation (5b) gives

$$r\mathrm{w}''(r) \le -\frac{\mathrm{w}(1-\mathrm{w}^2)}{rA} = -\mathrm{w}(1+\mathrm{w})\frac{1-\mathrm{w}}{rA} < -\frac{\delta c}{\tilde{A}_1}. \tag{39}$$

We now consider the function

$$q(r) = 2rA_0\mathrm{w}'^3 + \mathrm{w}(1-\mathrm{w}^2). \tag{40}$$

A simple calculation yields

$$q'(r) = \mathrm{w}'(2A_0\mathrm{w}'^2 + 6A_0 r\mathrm{w}'\mathrm{w}'' + 1 - 3\mathrm{w}^2). \tag{41}$$

Since $\mathrm{w}'' < 0$ for all $r \in (t_m, r_m^\delta)$, we have

$$\mathrm{w}'(r) < \mathrm{w}'(t_m). \tag{42}$$

Substituting equations (38), (39), and (42) into equation (41) yields

$$q'(r) < \mathrm{w}'(r)[6A_0 r\mathrm{w}''(r)\mathrm{w}'(t_m) - 2] \le \mathrm{w}'(r)[6A_0\delta c^2/\tilde{A}_1 - 2] < 0. \tag{43}$$

The last inequality follows from inequality (37). Equations (10), (40), (43), (38), and (37) now yield, for any $r \in (t_n, r_n^\delta)$,

$$\begin{aligned} r^2(A\mathrm{w}')' &= -2rA\mathrm{w}'^3 - \mathrm{w}(1-\mathrm{w}^2) \\ &> -2rA_0\mathrm{w}'^3 - \mathrm{w}(1-\mathrm{w}^2) \\ &= -q(r) > -q(t_m) \\ &> 2A_0rc^3 - 2rc \\ &= 2rc(A_0c^2 - 1) > 0. \end{aligned} \tag{44}$$

Thus, there exists some $M > 0$ such that $\left(\bigcup_n \{s_n\}\right) \cap \left(\bigcup_{m>M} (t_m, r_m^\delta)\right) = \emptyset$. Clearly, there exist $\varepsilon > 0$ such that $\bigcup_{m \le M}(t_m, r_m^\delta) \cap (0, \varepsilon) = \emptyset$. There also exist M_n such that $\bigcup_{n>M_n}\{s_n\} \subset (0, \varepsilon)$. It follows that $\left(\bigcup_{n>M_n}\{s_n\}\right) \cap \left(\bigcup_m (t_m, r_m^\delta)\right) = \emptyset$.

Intervals of type 3. For m sufficiently large, equation (5b) gives

$$\begin{aligned} \mathrm{w}'(r) &= \mathrm{w}'(t_m) + \int_{t_n}^r \mathrm{w}''(\rho)\,\mathrm{d}\rho \\ &= \mathrm{w}'(t_n) + \int_{t_m}^r \left(\frac{-\mathrm{w}(1-\mathrm{w}^2)}{\rho^2 A} - \frac{\Phi \mathrm{w}'}{\rho A} \right) \mathrm{d}\rho \\ &\le \int_{t_n}^r -\frac{\Phi \mathrm{w}'}{\rho A}\,\mathrm{d}\rho \\ &\le \frac{\varepsilon}{rA_1} \int_{t_n}^r \mathrm{w}'\,\mathrm{d}\rho \\ &= \frac{\varepsilon}{rA_1}[\mathrm{w}(r) - \mathrm{w}(t_m)] \end{aligned} \tag{45}$$

for any $r \in [r_m^\delta, r_m^0]$. The last inequality follows for arbitrary $\varepsilon \in (0, c^2 - (1 - A_0))$ from inequality (37). This is because, for any such ε and for sufficiently large m, the following inequality holds throughout the interval $[t_m, r_m^0]$:

$$\Phi = 1 - A - \frac{(1-\mathrm{w}^2)^2}{r^2} - \Lambda r^2 < 1 - A_0 - c^2 < -\varepsilon.$$

We have also used the fact that $\mathrm{w}' < 0$ in this same interval.

We now choose an arbitrary $\tilde{\delta} \in (\delta, 1)$. Since $\lim_{m \nearrow \infty} \mathrm{w}(t_m) = 1$ it follows that, for sufficiently large m and arbitrary $r \in [r_m^\delta, r_m^0]$,

$$\mathrm{w}(r) - \mathrm{w}(t_m) < -(1 - \tilde{\delta}). \tag{46}$$

Substituting equation (46) into inequality (45) yields

$$\mathrm{w}'(r) < -\frac{\varepsilon(1-\tilde{\delta})}{rA_1}. \tag{47}$$

Finally, we substitute inequality (47) into equation (10) to get, for large m,

$$r(A\mathrm{w}')'(r) = -2A\mathrm{w}'^3(r) - \frac{\mathrm{w}(1-\mathrm{w}^2)}{r} > \frac{2A(r)\varepsilon^3(1-\tilde{\delta})^3}{r^3A_1^3} - \frac{1}{r}. \tag{48}$$

It is clear that, for sufficiently large m, the first term on the right side of inequality (48) dominates. Thus, there exist $M > 0$ such that, whenever $m > M$,

$$(A\mathrm{w}')' > 0 \quad \text{for all } \ r \in [r_m^\delta, r_m^0];$$

that is, $\left(\bigcup_n \{s_n\}\right) \cap \left(\bigcup_{m>M} (t_m, r_m^\delta)\right) = \emptyset$. As with intervals of type 2, there exist $\varepsilon > 0$ such that $\bigcup_{m \le M} (t_m, r_m^\delta) \cap (0, \varepsilon) = \emptyset$. There also exist M_n such that $\bigcup_{n>M_n} \{s_n\} \subset (0, \varepsilon)$. It follows that $\left(\bigcup_{n>M_n} \{s_n\}\right) \cap \left(\bigcup_m (t_m, r_m^\delta)\right) = \emptyset$. □

Having established that $\lim_{r\searrow 0} \mathrm{w}^2(r) = 1$, we next establish the existence of $\lim_{r\searrow 0} \mathrm{w}'(r)$. Because of Fact 2 and Fact 3, we may assume $\lim_{r\searrow 0} \mathrm{w}(r) = 1$ and one of the following possibilities:

(1) for any $\varepsilon > 0$ there exists a $\rho > 0$ such that, for all $r \in (0, \rho)$, $1 - \varepsilon < \mathrm{w}(r) < 1$ and $\mathrm{w}'(r) < 0$;
(2) for any $\varepsilon > 0$ there exists a $\rho > 0$ such that, for all $r \in (0, \rho)$, $1 < \mathrm{w}(r) < 1 + \varepsilon$ and $\mathrm{w}'(r) > 0$.

We prove only the first case; the proof for the second case is similar. The existence of $\lim_{r\searrow 0} \mathrm{w}'(r)$ is a consequence of the following lemma.

LEMMA 9. *Let (A, w) be a solution of equations* (5a)–(5b) *that satisfies the hypotheses of Theorem 2. Then there exists a $\rho \in (0, \bar{r})$ such that, for any $b \in (0, \rho)$ and any $\varepsilon > 0$,*

$$1 - \varepsilon b \le \mathrm{w}(b) < 1,$$
$$1 - \varepsilon r \le \mathrm{w}(r) < 1$$

for all $r \in (0, b)$.

It is worth noting that ρ in Lemma 9 is independent of b and ε.

Proof. For any $\varepsilon > 0$ we define

$$U_\varepsilon = \{r \in [0, b] : \mathrm{w}(s) \ge 1 - \varepsilon s \ \text{for all } s \in [0, r]\}$$

as well as

$$a_\varepsilon = \sup\{r \in U_\varepsilon\}.$$

Because it contains 0, U_ε is nonempty. It is also clear that U_ε is closed; that is, $a_\varepsilon \in U_\varepsilon$. We claim that $a_\varepsilon = b$. In the interval $[a_\varepsilon, b]$ we define

$$g(\varepsilon, r) = 1 - \varepsilon r - \mathrm{w}. \tag{49}$$

Since ε is constant, we denote $g(\varepsilon, r)$ also by $g(r)$.

Now, if $a_\varepsilon < b$ then there exist $\tilde{c} \in (a_\varepsilon, b)$ that satisfy $g(\tilde{c}) > 0$. We let $c \in [a_\varepsilon, b]$ be where g assumes its maximum. Since $g(a_\varepsilon) = g(b) = 0$ and $g(\tilde{c}) > 0$, it follows that $c \in (a_\varepsilon, b)$. Consequently, $g'(c) = 0$ and $g''(c) \le 0$; that is, $\mathrm{w}(c) < 1 - \varepsilon c$, $\mathrm{w}'(c) = -\varepsilon$, and $\mathrm{w}''(c) \ge 0$. Equation (5b) now yields

$$\begin{aligned} 0 &= [r^2 A\mathrm{w}'' + r\Phi\mathrm{w}' + \mathrm{w}(1-\mathrm{w}^2)]_{r=c} \\ &\geq [c\mathrm{w}'\Phi + \mathrm{w}(1-\mathrm{w}^2)]_{r=c} \\ &\geq [-c\varepsilon + \mathrm{w}(1-\mathrm{w}^2)]_{r=c} \\ &= c\left[-\varepsilon + \mathrm{w}(1+\mathrm{w})\frac{(1-\mathrm{w})}{c}\right]_{r=c} \\ &> c\varepsilon[-1 + \mathrm{w}(1+\mathrm{w})]_{r=c} > 0, \end{aligned} \tag{50}$$

provided ρ is small enough that $\mathrm{w}(1+\mathrm{w}) > 1$ for all $r \in (0, \rho)$. We have also used the fact that $\Phi(r) < 1$ for all $r \in (0, \rho)$. Inequality (50) is a contradiction from which it follows that $a_\varepsilon = b$. $\square$

LEMMA 10. *Suppose* (A, w) *is a solution of equations* (5a)–(5b) *that satisfies the hypotheses of Theorem 2. Then* $\lim_{r\searrow 0} \mathrm{w}'(r)$ *exists and is finite.*

Proof. Equation (49) is defined on $[0, \infty] \times (0, \bar{r})$. We now define the set

$$O = \{\varepsilon \geq 0 : \text{there exist } \rho_\varepsilon > 0 \text{ such that } g(\varepsilon, r) > 0 \text{ for all } r \in (0, \rho_\varepsilon)\}.$$

We also define

$$\bar{\varepsilon} = \sup\{\varepsilon \in O\}. \tag{51}$$

If there exist ε and ρ_ε such that $g(\varepsilon, r) \equiv 0$ in $(0, \rho_\varepsilon)$, then there is nothing to prove. Consequently, we assume this is not the case.

We first prove that $\bar{\varepsilon}$ is well-defined. O is nonempty, since $0 \in O$. Also, for any $\varepsilon \in O$, if $\varepsilon > 1$ then, for all $r \in (0, \rho_\varepsilon)$,

$$\frac{(1-\mathrm{w}^2)^2}{r^2} > 1.$$

As a consequence of equation (5a), there exist $\eta > 0$ such that, in the same interval,

$$rA' < -\eta.$$

Lemma 1 then implies $\lim_{r\searrow 0} A(r) = \infty$, contrary to our hypothesis. We conclude that $\bar{\varepsilon} \leq 1$. In particular, $\bar{\varepsilon} < \infty$ and is therefore well-defined.

We claim also that O is closed. To prove this, we choose arbitrary $\varepsilon_0 \notin O$. The definition of O gives a sequence $\{r_n\} \searrow 0$ that satisfies, for each n, $g(\varepsilon_0, r_n) \leq 0$. Lemma 9 then gives some $\rho_0 \in (0, \bar{r})$ that satisfies $g(\varepsilon_0, r) \leq 0$ for all $r \in (0, \rho_0)$. Equation (5a) and Lemma 1 preclude the possibility that $\mathrm{w} = 1 - \varepsilon_0 r$ in a neighborhood of $r = 0$. Thus, there exist $\tilde{r} \in (0, \rho_0)$ that satisfy $g(\varepsilon_0, \tilde{r}) < 0$. Because $g(\varepsilon, r)$ is continuous, there exist $\eta > 0$ such that, whenever $|\varepsilon - \varepsilon_0| < \eta$, $g(\varepsilon, \tilde{r}) < 0$ also. Lemma 9 now gives, for such ε, $g(\varepsilon, r) \leq 0$ for all $r \in (0, \tilde{r})$. In other words, $(\varepsilon_0 - \eta, \varepsilon_0 + \eta) \cap O = \emptyset$. This proves that O is closed. As a consequence, $\bar{\varepsilon} \in O$.

We now consider the following possibilities:

1a. $\bar{\varepsilon} > 0$;
1b. $\bar{\varepsilon} = 0$.

Case 1a. Because $\Phi(r) < 1$ for all $r \in (0, \bar{r})$, equation (5b) gives, whenever $\mathrm{w}'(r) < 0$,

$$\begin{aligned} rA\mathrm{w}'' &= -\frac{\mathrm{w}(1-\mathrm{w}^2)}{r} - \mathrm{w}'\Phi \\ &\le -\mathrm{w}(1+\mathrm{w})\frac{1-\mathrm{w}}{r} - \mathrm{w}'. \end{aligned} \tag{52}$$

We prove that

$$\varlimsup_{r\searrow 0} \mathrm{w}'(r) \ge -\bar{\varepsilon}. \tag{53}$$

Now, because $\bar{\varepsilon} \in O$, it follows that $(1-\mathrm{w})/r \ge \bar{\varepsilon}$ for all $r \in (0, \rho_\varepsilon)$. This and the fact that $\lim_{r\searrow 0} \mathrm{w}(r) = 1$ yield, for any $\eta \in (0, \bar{\varepsilon}/3)$,

$$\varliminf_{r\searrow 0} \mathrm{w}(1+\mathrm{w})\frac{1-\mathrm{w}}{r} \ge 2\bar{\varepsilon} = \frac{3}{2}\left(\bar{\varepsilon} + \frac{\bar{\varepsilon}}{3}\right) > \frac{3}{2}(\bar{\varepsilon} + \eta).$$

Consequently, on any sequence $\{r_n\} \searrow 0$ that satisfies $\mathrm{w}'(r_n) > -(\bar{\varepsilon} + \eta)$, equations (52) and (53) give M such that $\mathrm{w}''(r_n) < 0$ whenever $n > M$. This proves that $\lim_{r\searrow 0} \mathrm{w}'(r)$ exists. Indeed, otherwise there exists a sequence $\{r_n\} \searrow 0$ that satisfies $\mathrm{w}'(r_n) > -(\bar{\varepsilon} + \eta)$ and $\mathrm{w}''(r_n) = 0$. Clearly, equation (53) implies that $\lim_{r\searrow 0} \mathrm{w}'(r)$ must be finite.

It remains to establish equation (53). Toward this end, we define $\delta(r)$ by

$$\mathrm{w}(r) = 1 - \bar{\varepsilon}r - \delta(r). \tag{54}$$

Since $\bar{\varepsilon} \in O$, we have

$$\delta(r) \ge 0 \quad \text{for all } \; r \in (0, \rho_0). \tag{55}$$

Indeed, otherwise there exist $\tilde{r} \in (0, \rho_0)$ that satisfy $\delta(r) < 0$. Substituting equation (54) into equation (49) yields

$$g(\bar{\varepsilon}, \tilde{r}) = -\delta(\tilde{r}) < 0.$$

Lemma 9 then implies that $g(\bar{\varepsilon}, r) \le 0$ for all $r \in (0, \tilde{r})$; that is, $\bar{\varepsilon} \notin O$, contradicting the fact that $\bar{\varepsilon} \in O$.

We next claim that

$$\varliminf_{r\searrow 0} \frac{\delta(r)}{r} = 0. \tag{56}$$

Indeed, equation (55) gives $\varliminf_{r\searrow 0} \delta(r)/r \ge 0$. Now we assume that there exist arbitrarily small $\tilde{\eta}$ such that

$$\varliminf_{r\searrow 0} \frac{\delta(r)}{r} \ge 3\tilde{\eta} > 0$$

and arrive at a contradiction. If so, then there exist $\xi > 0$ such that $\delta(r) > 2\tilde{\eta}r$ whenever $r \in (0, \xi)$. Hence

$$g(\bar{\varepsilon} + \tilde{\eta}, r) = -\tilde{\eta}r + \delta(r) > \tilde{\eta}r > 0;$$

that is, $\bar{\varepsilon} + \tilde{\eta} \in O$. However, this contradicts the definition of $\bar{\varepsilon}$. It follows that $\varliminf_{r\searrow 0} \delta(r)/r \le 0$. From equation (55) we deduce equation (56).

Finally, we choose a sequence $\{r_n\} \searrow 0$ that satisfies $\delta(r_n)/r_n \searrow 0$. The mean value theorem yields a sequence $\{s_n\} \to 0$ that satisfies

$$\mathrm{w}'(s_n) = \frac{\mathrm{w}(r_n) - 1}{r_n} = \frac{-\bar{\varepsilon} r_n - \delta(r_n)}{r_n} = -\bar{\varepsilon} - \frac{\delta(r_n)}{r_n}.$$

Clearly, $\mathrm{w}'(s_n) \to -\bar{\varepsilon}$. This establishes equation (53) and completes the proof of Lemma 10 in Case 1a.

Case 1b. We choose arbitrary finite $\tilde{A} > A_1$. For any $\varepsilon > 0$, there exists a sequence $\{r_n^\varepsilon\} \searrow 0$ such that $1 - (\varepsilon/\tilde{A}) r_n^\varepsilon < \mathrm{w}(r_n^\varepsilon) < 1$ for each n. From the mean value theorem it follows that, for any $\varepsilon > 0$, there exists another sequence $\{s_n^\varepsilon\} \searrow 0$ such that $-\varepsilon/\tilde{A} < \mathrm{w}'(s_n^\varepsilon) < 0$ for all n. Therefore,

$$-\varepsilon < \tilde{A}\mathrm{w}'(s_n^\varepsilon) < (A\mathrm{w}')(s_n^\varepsilon) < 0. \tag{57}$$

Also, Lemma 9 provides $\rho > 0$ such that, for all $r \in (0, \rho)$,

$$1 - \varepsilon r < \mathrm{w}(r) < 1. \tag{58}$$

Now, for any $r \in (0, \rho)$, whenever $A\mathrm{w}'(r) < -\sqrt[3]{\varepsilon}$ we have

$$A\mathrm{w}'^3 = (A\mathrm{w}')\mathrm{w}'^2 < -\sqrt[3]{\varepsilon}\left(\frac{\sqrt[3]{\varepsilon}}{A}\right)^2 < -\varepsilon. \tag{59}$$

Substituting inequalities (58) and (59) into equation (10) implies that any $r \in (0, \rho)$ that satisfies $A\mathrm{w}'(r) < -\sqrt[3]{\varepsilon}$ also satisfies $(A\mathrm{w}')'(r) > 0$. It follows from this and (57) that, because ε is arbitrarily small,

$$\lim_{r \searrow 0} A\mathrm{w}'(r) = 0. \tag{60}$$

The mean value theorem easily gives $\overline{\lim}_{r \searrow 0} \mathrm{w}'(r) = 0$. We now assume that $\underline{\lim}_{r \searrow 0} \mathrm{w}'(r) < 0$ and arrive at a contradiction. Indeed, this assumption implies the existence of some $\mathrm{v}_l < 0$ and a sequence $\{t_n\} \searrow 0$ that satisfies $\lim_{n \nearrow \infty} \mathrm{w}'(t_n) = \mathrm{v}_l$ and $\mathrm{w}''(t_n) = 0$. Equation (5b) easily gives $\lim_{n \nearrow \infty} A(t_n) = 1$. However, this implies that $\lim_{n \nearrow \infty} A\mathrm{w}'(t_n) = \mathrm{v}_l < 0$, which contradicts equation (60). □

We state the next lemma—even though it is a trivial application of the mean value theorem—because it eliminates any ambiguity in the definition of $\mathrm{w}'(0)$.

LEMMA 11. *If* $\lim_{r \searrow 0} \mathrm{w}'(r)$ *exists and is finite and also* w *is differentiable at* $r = 0$, *then* w' *is right-continuous at* 0.

Proof. By assumption, $\lim_{r \searrow 0} \mathrm{w}(r)$ exists and is finite. From the definition of a derivative, for any $\varepsilon > 0$ there exists a sequence $\{r_n\} \searrow 0$ that satisfies

$$\left| \frac{\mathrm{w}(r_n) - \mathrm{w}(0)}{r_n} - \mathrm{w}'(0) \right| < \varepsilon. \tag{61}$$

The mean value theorem yields a sequence $\{s_n\}$, $0 < s_n < r_n$, that satisfies

$$\mathrm{w}'(s_n) = \frac{\mathrm{w}(r_n) - \mathrm{w}(0)}{r_n}. \tag{62}$$

Substituting equation (62) into equation (61) yields

$$|w'(s_n) - w'(0)| < \varepsilon;$$

that is, $\lim_{n\nearrow\infty} w'(s_n) = w'(0)$. Since we assume $\lim_{r\searrow 0} w'(r)$ to exist, this limit must also equal $w'(0)$. □

Lemma 11 allows us to define, without ambiguity, $w'_0 = \lim_{r\searrow 0} w'(r) = w'(0)$.

Proof of Theorem 2. What remains is to prove that $\lim_{r\searrow 0} A(r) = 1$ and $w'_0 = 0$. We first prove that $\lim_{r\searrow 0} A(r)$ exists.

Equation (5a) implies that any sequence $\{r_n\} \searrow 0$ that satisfies $A'(r_n) = 0$ also satisfies

$$A(r_n) = \frac{1 - \dfrac{(1 - w^2(r_n))^2}{r_n^2} - \Lambda r_n^2}{1 + 2w'^2(r_n)}. \tag{63}$$

Because w'_0 is well-defined, it follows that, as $n \nearrow \infty$, the right side of equation (63) approaches

$$\frac{1 - 4w_0'^2}{1 + 2w_0'^2}.$$

Therefore, $\lim_{r\searrow 0} A(r)$ exists and obviously $\lim_{r\searrow 0} A(r) = A_0$.

Now, equation (5a) gives

$$\lim_{r\searrow 0}(rA') = 1 - A_0 - 4w_0'^2 - 2A_0 w_0'^2;$$

that is, $\lim_{r\searrow 0}(rA')(r)$ exists. Lemma 1 gives

$$\lim_{r\searrow 0}(rA')(r) = 0. \tag{64}$$

Since w'_0 is finite, we have

$$\lim_{r\searrow 0}(rA' w_0'^2)(r) = 0 \tag{65}$$

also. Clearly, $\Phi_0 = \lim_{r\searrow 0} \Phi(r) \le 1$. If $\Phi_0 < 1$ then equation (5b) yields

$$\lim_{r\searrow 0}(rA w'_0 w'')(r) = -\Phi_0 w_0'^2 + 2w_0'^2 > 0 \tag{66}$$

unless $w'_0 = 0$. Equations (66) and (65) imply that, whenever $w'_0 \neq 0$,

$$w'_0 \lim_{r\searrow 0}(rAw')'(r) > 0. \tag{67}$$

Lemma 1 now gives $\lim_{r\searrow 0}(Aw')(r) = \infty$. However, this is impossible. Hence it must be that $w'_0 = 0$ in the case $\Phi_0 < 1$. On the other hand, $\Phi_0 = 1$ only if $A_0 = 0$ and $w'_0 = 0$. Thus, in all cases, $w'_0 = 0$. It follows easily from equation (5a) and from Lemma 1 that $\Phi_0 = 0$. The definition of Φ then gives $A_0 = 1$. □

3.2. Case $A > 1$

In this section we prove that any solution that is not Schwarzschild-like or smooth is Reissner–Nordström-like. Specifically, we have the following theorem.

THEOREM 3. *Let* $(\bar{r}, \bar{A}, \bar{\mathrm{v}}) \in \mathbb{R}_+^2 \times \mathbb{R}$ *and let* $(A(r), \mathrm{w}(r))$ *be the solution of equations* (5a)–(5b) *that satisfies* $(A(\bar{r}), \mathrm{w}(\bar{r}), \mathrm{w}'(\bar{r})) = (\bar{A}, \bar{\mathrm{w}}, \bar{\mathrm{v}})$. *Suppose also that* $\bar{A} \geq 1 - \Lambda\bar{r}^2/3$. *Then* $\lim_{r\searrow 0} A(r) = \infty$ *and* $\lim_{r\searrow 0} \mathrm{w}'(r) = 0$.

We note that we already proved (in Section 2) that equations (5a)–(5b), with any solution that satisfies the hypotheses of Theorem 3, are nonsingular for all $r \in (0, \bar{r})$.

Proof. To prove that $\lim_{r\searrow 0} A(r) = \infty$, we recall equations (17) and (18):

$$\mu(r) = r\left(1 - A - \frac{\Lambda r^2}{3}\right),$$

$$\mu'(r) = 2A\mathrm{w}'^2 + \frac{(1-\mathrm{w}^2)}{r^2} \geq 0.$$

Clearly, μ is nondecreasing in the interval $(0, \bar{r})$. Thus $\lim_{r\searrow 0} \mu(r)$ exists. Since $\mu(\bar{r}) < 0$ (by assumption), it must be that $\lim_{r\searrow 0} \mu(r) < 0$; that is, $\lim_{r\searrow 0} rA > 0$. Lemma 1 now gives $\lim_{r\searrow 0} A(r) = \infty$.

In order to prove that $\lim_{r\searrow 0} \mathrm{w}'(r) = 0$, let us assume for the moment that $\lim_{r\searrow 0} \mathrm{w}'(r) = \mathrm{w}_0'$ exists. There are two cases to consider:

1. $\lim_{r\searrow 0} \mathrm{w}^2(r) = 1$;
2. $\lim_{r\searrow 0} \mathrm{w}^2(r) \neq 1$.

Case 1. We set $\mathrm{w}_0 = 1$ and apply L'Hôpital's rule to $(\mathrm{w}^2 - 1)/r$. The result is $\lim_{r\searrow 0}(\mathrm{w}^2(r) - 1)/r = 2\mathrm{w}_0'$. If $\mathrm{w}_0' = \infty$, then there exists an $\tilde{r}$ arbitrarily close to 0 that satisfies $\mathrm{w}(\tilde{r}) > 1$, $\mathrm{w}'(\tilde{r}) > 0$, and $\mathrm{w}''(\tilde{r}) < 0$. This contradicts equation (5b). Similarly, $\mathrm{w}_0' \neq -\infty$. We thus have $(\mathrm{w}^2 - 1)/r$ bounded near 0 and $(\mathrm{w}_0')^2 < \infty$.

We now assume that $\mathrm{w}_0' \neq 0$ and arrive at a contradiction. Indeed, this assumption together with Fact 3 and equation (5b) imply (as in Fact 5) that w' has only one sign near $r = 0$. Without loss of generality, we assume that $\mathrm{w}' > 0$. Since $\lim_{r\searrow 0} A(r) = \infty$, equation (5b) also gives $\varepsilon > 0$ such that, whenever $r \in (0, \varepsilon)$,

$$r\mathrm{w}''(r) = \mathrm{w}'(r) + \left(\frac{(\mathrm{w}^2-1)^2}{r^2} - 1 + \Lambda r^2\right)\frac{\mathrm{w}'}{A} + \frac{\mathrm{w}(\mathrm{w}^2-1)}{rA} > \frac{1}{2}\mathrm{w}'(r) > 0.$$

Lemma 1 now gives $\lim_{r\searrow 0} \mathrm{w}'(r) = -\infty$, which is impossible. It follows that $\lim_{r\searrow 0} \mathrm{w}'(r) = 0$.

Case 2. We prove that the assumption $\lim_{r\searrow 0} \mathrm{w}'^2(r) > 2\varepsilon > 0$ $(\varepsilon < 1)$ leads to a contradiction. Indeed, there exist η such that $\mathrm{w}'^2(r) > \varepsilon$ whenever $r \in (0, \eta)$. Also, $|\lim_{r\searrow 0} A\mathrm{w}'(r)| = \infty$. Multiplying equation (10) by $A\mathrm{w}'$ gives, for $r \in (0, \eta)$,

$$\begin{aligned}(A\mathrm{w}')(A\mathrm{w}')' &= \frac{-2r\mathrm{w}'^2(A\mathrm{w}')^2 - \mathrm{w}(1-\mathrm{w}^2)(A\mathrm{w}')}{r^2} \\ &\leq \frac{-2\varepsilon r(A\mathrm{w}')^2 - \mathrm{w}(1-\mathrm{w}^2)(A\mathrm{w}')}{r^2}. \end{aligned} \tag{68}$$

Now, equation (5a) implies that $\lim_{r\searrow 0} r(rA)' = -\infty$. Lemma 1 then gives

$$\lim_{r\searrow 0}(rA) = \infty.$$

Thus, for $r \in (0, \eta)$,

$$\begin{aligned}-2\varepsilon r(A\mathrm{w}')^2 - \mathrm{w}(1-\mathrm{w}^2)A\mathrm{w}' &= -\varepsilon r(A\mathrm{w}')^2 - \varepsilon r(A\mathrm{w}')^2 - \mathrm{w}(1-\mathrm{w}^2)A\mathrm{w}' \\ &= -r\varepsilon(A\mathrm{w}')^2 - (A\mathrm{w}')[\varepsilon(rA)\mathrm{w}' + \mathrm{w}(1-\mathrm{w}^2)] \\ &\le -r\varepsilon(A\mathrm{w}')^2;\end{aligned}$$

this and equation (68) yield $(A\mathrm{w}')'/(A\mathrm{w}') \le -\varepsilon/r$. Integrating gives $c > 0$ such that, for $r \in (0, \eta)$, $|A\mathrm{w}'(r)| \le cr^{-\varepsilon}$ or $r^{\varepsilon}|A\mathrm{w}'(r)| \le c$. Because $\varepsilon < 1$, this contradicts the fact that $\lim_{r\searrow 0} A(r) = \infty$. □

All that remains is to establish the existence of $\lim_{r\searrow 0} \mathrm{w}'(r)$. Toward this end we define, for any solution of equations (5a) and (5b),

$$\theta(r) = \arctan \frac{\mathrm{w}'(r)}{\mathrm{w}(r)}. \tag{69}$$

For any solution of equations (5a)–(5b), $\theta(r)$ satisfies

$$\theta' = \frac{1}{r^2 A}[(\mathrm{w}^2 - 1)\cos^2\theta - r\Phi\cos\theta\sin\theta - r^2 A\sin^2\theta]. \tag{70}$$

LEMMA 12. *Suppose (A, w) is a solution of equations* (5a)–(5b) *that satisfies the hypotheses of Theorem 3. Suppose also that there exists an $\hat{r} \in (0, \bar{r})$ such that, for all $r \in (0, \hat{r})$, $\mathrm{w}^2(r) < 1$. Then there exist $r_0 \in (0, \hat{r})$ such that, for all $r \in (0, r_0)$, $\mathrm{w}' \neq 0$.*

Proof. The lemma follows once it is shown that, for sufficiently small r, $\theta'|_{\theta=0} < 0$ and $\theta'|_{\theta=\pi/4} > 0$. The first inequality follows immediately from equation (70):

$$\theta'_{\theta=\pi/4} = \frac{1}{2r^2A}\left[(\mathrm{w}^2-1) - r + A(r - r^2) + \frac{(\mathrm{w}^2-1)^2}{r} + \Lambda r^3\right]. \tag{71}$$

We choose ε sufficiently small so that, whenever $r \in (0, \varepsilon)$, $A(r) > 3$ and $2r - 3r^2 > 0$. Then, for any r in this interval such that $(1 - \mathrm{w}^2) \ge r$,

$$\begin{aligned}\theta'_{\theta=\pi/4} &> \frac{1}{2r^2A}\left[(\mathrm{w}^2-1) - r + \frac{(\mathrm{w}^2-1)^2}{r} + 3(r - r^2)\right] \\ &= \frac{1}{2r^2A}\left[(1-\mathrm{w}^2)\left(\frac{1-\mathrm{w}^2}{r} - 1\right) + 2r - 3r^2\right] \ge 0.\end{aligned}$$

For any r in this interval such that $(1 - \mathrm{w}^2) < r$, because $r < 1/2$ and $A(r) > 2$ we have

$$\theta'_{\theta=\pi/4} > \frac{1}{2rA}[A(1-r) - 2] > 0. \qquad \square$$

LEMMA 13. *Suppose (A, w) is a solution of equations* (5a)–(5b) *that satisfies the hypotheses of Theorem 3. Then* $\lim_{r\searrow 0} \mathrm{w}'(r) = \mathrm{w}'_0$ *exists.*

Proof. We choose an $\tilde{r} > 0$ such that $\Phi(r) < 0$ whenever $r \in (0, \tilde{r})$. Considering Fact 3, we then choose an $r_0 \in (0, \tilde{r})$ such that $(1 - \mathrm{w}^2)\mathrm{w}(r_0) \neq 0$. Equation (5b) implies (as in Fact 5) that r_0 can be chosen so that one of the following holds:

1. for all $r \in (0, r_0)$, $\mathrm{w}^2(r) > 1$;
2. for all $r \in (0, r_0)$, $\mathrm{w}^2(r) < 1$.

Case 1. Using Fact 2, we assume that $\mathrm{w}(r) > 1$ for all $r \in (0, r_0)$. Equation (5b) and Fact 3 allow us to assume (by choosing a smaller r_0, if necessary) that one of the following holds:

a. $\mathrm{w}'(r) \leq 0$ for all $r \in (0, r_0)$;
b. $\mathrm{w}'(r) > 0$ for all $r \in (0, r_0)$.

Case 1a. $\lim_{r\searrow 0} \mathrm{w}(r)$ exists and exceeds 1. We assume that $\lim_{r\searrow 0} \mathrm{w}'(r)$ does not exist and then arrive at a contradiction. With this assumption, there exists an $\tilde{r}$ arbitrarily close to 0 that satisfies $\mathrm{w}(\tilde{r}) > 1$, $\mathrm{w}'(\tilde{r}) < 0$, $\mathrm{w}''(\tilde{r}) = 0$, and $\mathrm{w}'''(\tilde{r}) \leq 0$. Differentiating equation (5b) yields

$$(r^2 A)'\mathrm{w}'' + (r^2 A)\mathrm{w}''' + (r\Phi)\mathrm{w}'' + [(r\Phi)' + 1 - 3\mathrm{w}^2]\mathrm{w}' = 0. \qquad (72)$$

Equation (6) gives

$$(r\Phi)' = 2A\mathrm{w}'^2 + \frac{2(\mathrm{w}^2-1)^2}{r^2} - \frac{4\mathrm{w}\mathrm{w}'(\mathrm{w}^2-1)}{r} - 2\Lambda r^2. \qquad (73)$$

Now, for any $\bar{\mathrm{w}} > 1$, there exists a ρ such that, whenever $r \in (0, \rho)$ and $\mathrm{w} > \bar{\mathrm{w}}$,

$$\frac{2(\mathrm{w}^2-1)^2}{r^2} - 2\Lambda r^2 + 1 - 3\mathrm{w}^2 > 0. \qquad (74)$$

We choose any $\bar{\mathrm{w}} \in (1, \lim_{r\searrow 0} \mathrm{w}(r))$. It follows from inequality (74) and equation (73) that $\tilde{r}$ can be chosen so that $[(r\Phi)'(\tilde{r}) + 1 - 3\mathrm{w}^2(\tilde{r})]\mathrm{w}'(\tilde{r}) < 0$. The left side of equation (72), evaluated at $\tilde{r}$, is negative. This is a contradiction.

Case 1b. Equation (5b) gives $\mathrm{w}''(r) \geq 0$ for all $r \in (0, r_0)$. Consequently, $\lim_{r\searrow 0} \mathrm{w}'(r)$ exists and is both finite and nonnegative.

Case 2. Lemma 12 implies that w' has only one sign. Therefore, $\lim_{r\searrow 0} \mathrm{w}(r)$ must exist. We take r sufficiently small so that $\mathrm{w}' \neq 0$ in $(0, r)$. In this interval, $\mathrm{w}\mathrm{w}'(s) = 0$ if and only if $\mathrm{w}(s) = 0$; in this case $(\mathrm{w}\mathrm{w}')'(s) = \mathrm{w}'^2(s) > 0$. It follows that there exist $\hat{r}$ such that, for all $r \in (0, \hat{r})$, $\mathrm{w}\mathrm{w}'$ has only one sign.

If $\underline{\lim}_{r\searrow 0} \mathrm{w}'(r) < \overline{\lim}_{r\searrow 0} \mathrm{w}'(r)$, then there exists a sequence $\{r_n\} \searrow 0$ that satisfies $\mathrm{w}''(r_n) = 0$. If we multiply equation (5b) by w', then Fact 3 and $\Phi(r_n) < 0$ give $\mathrm{w}\mathrm{w}'(r_n) > 0$. It follows that $\mathrm{w}\mathrm{w}' > 0$ in $(0, \hat{r})$. Now equation (5b), multiplied by w and evaluated at any nonzero $\bar{\mathrm{w}}' \in (\underline{\lim}_{r\searrow 0} \mathrm{w}', \overline{\lim}_{r\searrow 0} \mathrm{w}')$, yields $(\mathrm{w}\mathrm{w}'')|_{\mathrm{w}'(r)=\bar{\mathrm{w}}'} > 0$, provided r is sufficiently small. However, if w' has no limit then there must be arbitrarily small r and nonzero $\bar{\mathrm{w}}'$ that satisfy $\mathrm{w}'(r) = \bar{\mathrm{w}}$ and $(\mathrm{w}\mathrm{w}'')(r) \leq 0$. It follows that w' has a limit. □

References

[1] P. Breitenlohner, P. Forgács, and D. Maison, *Static spherically symmetric solutions of the Einstein–Yang–Mills equations,* Comm. Math. Phys. 163 (1994), 141–172.

[2] P. Hartman, *Ordinary differential equations,* Wiley, New York, 1973.

[3] A. Linden, *Horizons in spherically symmetric static Einstein* SU(2)*–Yang Mills spacetimes*, Classical Quantum Gravity 18 (2001), 695–708.
[4] ———, *Far field behavior of noncompact static spherically symmetric solutions of Einstein* SU(2)*–Yang Mills equations,* J. Math. Phys. 42 (2001), 1196–1202.
[5] ———, *Existence of noncompact static spherically symmetric solutions of Einstein* SU(2)*–Yang Mills equations,* Comm. Math. Phys. 221 (2001), 525–547.
[6] ———, *Existence of oscillating solutions of Einstein* SU(2)*–Yang Mills equations,* unpublished manuscript.
[7] J. Smoller and A. Wasserman, *Existence of infinitely many smooth static solutions of the Einstein/Yang–Mills equations,* Comm. Math. Phys. 151 (1993), 303–325.
[8] ———, *An investigation of the limiting behavior of particle-like solutions to the Einstein–Yang/Mills equations and a new black hole solution*, Comm. Math. Phys. 161 (1994), 365–389.
[9] ———, *Regular solutions of the Einstein–Yang–Mills equations,* J. Math. Phys. 36 (1995), 4301–4323.
[10] ———, *Uniqueness of the extreme Reissner–Nordström solution in* SU(2) *Einstein–Yang–Mills theory for spherically symmetric spacetime,* Phys. Rev. D 52 (1995), 5812–5815.
[11] ———, *Reissner–Nordström-like solutions of the* SU(2) *Einstein–Yang/Mills equations,* J. Math. Phys. 38 (1997), 6522–6559.
[12] ———, *Extendability of solutions of the Einstein–Yang/Mills equations,* Comm. Math. Phys. 194 (1998), 707–732.
[13] J. Smoller, A. Wasserman, and S.-T. Yau, *Existence of black hole solutions for the Einstein–Yang–Mills equations,* Comm. Math. Phys. 154 (1993), 377–401.
[14] J. Smoller, A. Wasserman, S.-T. Yau, and J. B. McLeod, *Smooth static solutions of the Einstein/Yang–Mills equations,* Comm. Math. Phys. 143 (1991), 115–147.
[15] M. S. Volkov, N. Straumann, G. Lavrelashvili, M. Heusler, and O. Brodbeck, *Cosmological analogues of the Bartnik–McKinnon solutions,* Phys. Rev. D 54 (1996), 7243–7251.
[16] A. Wasserman, *Solutions of the spherically symmetric* SU(2) *Einstein Yang Mills equations defined in the far field,* J. Math. Phys. 41 (2000), 6930–6936.
[17] E. Winstanley, *Existence of stable hairy black holes in* SU(2) *Einstein–Yang–Mills theory with a negative cosmological constant,* Classical Quantum Gravity 16 (1999), 1963–1978.