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Embeddings of SL(2, 27) in Complex Exceptional Algebraic Groups

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1. Introduction

We classify embeddings of SL(2, 27) in $E_8(\mathbb{C})$: there are twelve equivalence classes, and all embeddings factor through a natural $2E_7(\mathbb{C})$ subgroup. This result contributes to the program, initiated in the early 1980s, to study embeddings of finite groups into an exceptional complex algebraic group—that is, one of $G_2(\mathbb{C})$, $F_4(\mathbb{C})$, $E_6(\mathbb{C})$, $E_7(\mathbb{C})$, $E_8(\mathbb{C})$. In fact, this result on SL(2, 27) removes the final obstruction to achieving *the classification of all QE-pairs*, that is, pairs (S, G), where S is a finite quasisimple group and G is a complex exceptional algebraic group such that there exists an embedding of S in G. The classification of QE-pairs is discussed in [GR4], which updates the survey [GR2].

The methods we use to construct and analyze embeddings represent some innovations. We mention (1) a new strategy in searching for invariant Lie algebras, given a representation of a finite group, and (2) a computational problem of searching for tensor squares of elements in a given linear subspace of a tensor square of a vector space; this leads to the concept of relative eigenvalues and relative eigenvectors.

Earlier computer constructions of a particular finite subgroup in an exceptional group of Lie type have followed one of two strategies: either giving generating elements of the finite group as words in explicit generators of the algebraic group, or determining an invariant Lie algebra on a module for the finite group. The idea of our new approach is to start with a natural invariant symplectic Lie algebra for SL(2, 27) and then find an invariant subalgebra of type E_7. Our search for the invariant subalgebras is exhaustive; hence it determines conjugacy classes of embeddings.

It is known that the simple group PSL(2, 27) embeds into $F_4(\mathbb{C})$ [CoW] and hence into the algebraic groups $3E_6(\mathbb{C})$, $2E_7(\mathbb{C})$, and $E_8(\mathbb{C})$ (see [GR4, Table QE] or [GR2, Table PE]). Our goal in this article is to exhibit and classify embeddings of the covering group SL(2, 27) into $E_8(\mathbb{C})$. We first note that such an embedding could arise only from a Lie primitive embedding of SL(2, 27) into $2E_7(\mathbb{C})$. (A finite subgroup of a connected algebraic group is *Lie primitive* if there is no infinite intermediate Zariski closed subgroup; according to [CoW], there is

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no embedding into E_6 or $3E_6$ and there are no small representations that could yield an embedding into A_8, D_8, C_7, or B_7.)

We recall the definition of the *Ad-order* of an element g in a connected algebraic group: it is the smallest integer $n > 0$ such that g^n is in the center.

We use the term *EFO theory* ("elements of finite order") to indicate the standard theory of classification of finite order semisimple elements of a connected quasisimple algebraic group, analysis of the spectra on highest weight modules, and so on. This is a body of standard results that is surveyed in [G; GR2; GR4]. For a systematic search of elements of a given order in a connected algebraic group of adjoint type, we mention the computationally useful procedure of labeling the extended Dynkin diagram (see [K]).

Here is our main result.

THEOREM 1.1. *There are exactly twelve conjugacy classes of embeddings of* $\mathrm{SL}(2,27)$ *into* $2E_7(\mathbb{C})$. *If* M *is the* 133*-dimensional adjoint module or the* 56*-dimensional irreducible module for* $2E_7(\mathbb{C})$, *then the embeddings are associated to exactly six characters of* $\mathrm{SL}(2,27)$ *for the representation on* M. *These characters form a set of algebraic conjugates, and to each is associated two of the twelve embeddings. Each embedding gives a faithful action of* $\mathrm{SL}(2,27)$ *on the* 56*-dimensional irreducible module for* $2E_7(\mathbb{C})$ *and a faithful action of* $\mathrm{PSL}(2,27)$ *on the* 133*-dimensional irreducible module for* $2E_7(\mathbb{C})$.

2. The Story of SL(2, 27) in $2E_7(\mathbb{C})$

We shall work extensively with irreducible characters of $\mathrm{SL}(2,27)$; we name each character by its degree, with an alphabetic subscript to distinguish the different characters of a given degree. We use lowercase subscripts to denote irreducible characters of the simple group $\mathrm{PSL}(2,27)$ and uppercase subscripts to denote characters of *faithful* irreducible representations of $\mathrm{SL}(2,27)$. The alphabetic position of the subscript corresponds to the position of a character as displayed in the *Atlas of Finite Groups* [CCNPW]. Thus, 28_A denotes the first faithful 28-dimensional character of $\mathrm{SL}(2,27)$ and 26_c denotes the third 26-dimensional irreducible character of $\mathrm{PSL}(2,27)$ in *Atlas* order. The unique 27-dimensional irreducible is therefore written 27_a.

LEMMA 2.1. *Let* χ_{133} *and* χ_{56} *denote the irreducible characters of* $2E_7(\mathbb{C})$ *with degrees* 133 *and* 56*, respectively. Then, associated to embeddings of* $\mathrm{SL}(2,27)$ *in* $2E_7(\mathbb{C})$ *are exactly six pairs of restrictions of* χ_{133} *and* χ_{56} *to* $\mathrm{SL}(2,27)$. *One such pair is* $26_a + 26_b + 26_c + 27_a + 28_a$ *and* $28_D + 28_E$, *and the other five pairs are obtained from these by applying algebraic conjugacy to* 13th *roots of unity.* (*These irrationalities occur here in the degree-*28 *irreducibles.*)

Proof. By algebraic conjugacy, it suffices to assume that there is an embedding and then show that it has a pair of restrictions equal to one of the six pairs.

An element of order 4 (say, f) in SL(2, 27) must correspond to an element of $2E_7(\mathbb{C})$ that maps to an involution of $E_7(\mathbb{C})$ (i.e., has Ad-order 2). From [CoG] we deduce that $\chi_{133}(f) \in \{-7, 25\}$. However, χ_{133} contains no copies of the trivial character (since SL(2, 27) is Lie primitive in $E_7(\mathbb{C})$) and therefore $\chi_{133}(f) \leq \deg(\chi_{133})/13 < 25$ (since $\psi(f)/\deg(\psi) \leq 1/13$ for all nontrivial ordinary irreducibles ψ). We deduce that $\chi_{133}(f) = -7$. Now, $\deg(\chi_{133}) = 133$ and $\chi_{133}(f) = -7$ imply that χ_{133} decomposes either as (a) a sum of three characters from the collection $26_a, 26_b, 26_c$ together with 27_a and a 28-dimensional character or (b) as a sum of three copies of 27_a and two characters from the set $26_a, 26_b, 26_c$. If the decomposition involves a 28-dimensional character, we can apply an algebraic conjugacy to ensure that it is 28_a.

The character χ_{56} restricts to a sum of irreducible representations of SL(2, 27) of degrees 14 and 28. (The only other faithful irreducibles of SL(2, 27) have degree 26, and we cannot decompose 56 as 26 plus a nonnegative integer linear combination of 14, 26, and 28.) It follows that, in the eigenvalue spectrum of the action of an element of order 28 in SL(2, 27) on the 56-dimensional module for $2E_7(\mathbb{C})$, each primitive 28th root of unity has multiplicity 4. The EFO theory gives us the conjugacy classes of elements of order 28 in $2E_7(\mathbb{C})$. A search of these classes shows that there are just two classes of elements of order 14 in $E_7(\mathbb{C})$ that lift to elements of order 28 in $2E_7(\mathbb{C})$ with this spectrum, and both possibilities are rational on the adjoint module. It follows that the three characters $26_a, 26_b, 26_c$ appear with equal multiplicity in the restriction of χ_{133} to SL(2, 27). Hence χ_{133} restricts to the sum of $26_a, 26_b, 26_c, 27_a$, and 28_a.

The multiplicity of any eigenvalue of an SL(2, 27)-element of order 13 on the 56-dimensional module for $2E_7(\mathbb{C})$ must be between 4 and 8. This is because the multiplicity of any given eigenvalue of such an element on an irreducible representation is either 2 or 3 (for a 28-dimensional representation) or 1 or 2 (for a 14-dimensional representation). From EFO theory, such an element is either rational on the adjoint module or has a power t with $\chi_{133}(t) = 1 + y13$ and $\chi_{56}(t) = y13 * 5 + y13 * 6$. (We use *Atlas* notation for irrational character values, so that $y13$ is the sum of an inverse pair of primitive 13th roots of unity.) Our earlier description of the restriction of χ_{133} shows that the first of these possibilities does not occur, and the lemma follows. □

We remark that the same character-theoretic analysis applies to an embedding of SL(2, 27) into a group $2E_7(k)$ whenever k is a field whose characteristic is coprime to the order of SL(2, 27) (this is an obvious application of Larsen's $(0, p)$ correspondence [GR1]). Indeed, in our computer work, where we work over a finite field to have exact arithmetic, we choose to perform calculations in characteristic 1093. We note that 1093 does not divide |SL(2, 27)|, so the character analysis of Lemma 2.1 applies. Furthermore, the character irrationalities $y13$ and $y7$ that can arise in the characters 26_a, 26_b, 26_c, 27_a, 28_a, 28_D, and 28_E belong to the prime field $\mathbb{F}_{1093}$, so that there are matrix representations of SL(2, 27) over $\mathbb{F}_{1093}$ with these characters.

We write k for the field $\mathbb{F}_{1093}$, k^* for the multiplicative group of k, and $\widehat{k}$ for the algebraic closure of k. Let W be a 56-dimensional $k\mathrm{SL}(2,27)$-module with character $28_D + 28_E$. Write $\widehat{W}$ for the module obtained from W by extending scalars to $\widehat{k}$.

We can readily construct an explicit matrix representation of $\mathrm{SL}(2,27)$ on W (or $\widehat{W}$). In general, all irreducible $(q+1)$-dimensional representations of $\mathrm{SL}(2,q)$ are induced from 1-dimensional representations of the Borel subgroup. Hence, they can be written down as cases of the following recipe.

RECIPE 2.2. Let q be a prime power and l the field of order q. Write l^+ for the projective line over l (with $q+1$ points), l^* for the set of nonzero elements of l, and α for a multiplicative generator of l^*. Let K be a field (in most applications, K is different from l) and let ζ be a $(q-1)$th root of unity in K.

Let V be a $(q+1)$-dimensional vector space over K with basis $v_0, v_1, \dots, v_{q-1}, v_\infty$ parameterized by l^+. Then we can define the $\mathrm{GL}(V)$-elements f_x $(x \in l)$, g, and h that generate an image of $\mathrm{SL}(2,q)$ by the following formulas:

$$v_i f_x = v_{i+x};$$
$$v_i g = \frac{1}{\zeta} v_{i/\alpha^2} \ \ (i \neq \infty), \quad v_\infty g = \zeta v_\infty;$$
$$v_{\alpha^r} h = \zeta^{-r} v_{-1/\alpha^r}, \quad v_0 h = \zeta^{(q-1)/2} v_\infty, \quad v_\infty h = v_0.$$

We can think of the matrices f_x, g, and h as images of the $\mathrm{SL}(2,q)$-elements represented by the respective 2×2 matrices $\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$.

If q is odd, then the bilinear form $(\cdot,\cdot)$ defined by

$$(v_\infty, v_\infty) = (v_i, v_i) = 0, \qquad (v_\infty, v_i) = 1,$$
$$(v_i, v_\infty) = \zeta^{(q-1)/2}, \qquad (v_i, v_{i-\alpha^r}) = \zeta^{-r}$$

is invariant under our representation of $\mathrm{SL}(2,q)$.

The 2×2 matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right)$ in $\mathrm{GL}(2,q)$ acts as a diagonal outer automorphism of $\mathrm{SL}(2,q)$. If the element ζ has a square root $\sqrt{\zeta}$ in K, then we can extend our $(q+1)$-dimensional representation of $\mathrm{SL}(2,q)$ by representing this outer automorphism by a matrix d with

$$v_i d = \sqrt{\zeta}\, v_{i\alpha} \ \ (i \neq \infty), \quad v_\infty d = \frac{1}{\sqrt{\zeta}} v_\infty.$$

Moreover, the matrix d preserves the bilinear form $(\cdot,\cdot)$.

The following lemma makes clear that, in general, if field extensions are available then we can adjust a representation of a group automorphism so as to preserve an invariant bilinear form.

LEMMA 2.3. *Let V be an irreducible module for KG, where K is an algebraically closed field and G is a finite group, and suppose that f is a nondegenerate G-invariant bilinear form on V. Suppose that the automorphism a fixes the representation. Then there is a finite group H containing G as a normal subgroup and an* *element $b \in H$ for which conjugation on G by b induces the action of a, so that the representation of G on V extends to a representation of H on V that preserves the form f.*

Proof. Form the semidirect product $J := G\langle a\rangle$. The action of J on the group algebra kG by conjugation on G preserves the two-sided ideal (a matrix algebra over k) associated to V, so by Skolem–Nöther there is a projective representation ψ of J in $\mathrm{GL}(V)$. Since V is given as a G-module, ψ may be assumed to restrict to a homomorphism on G. Let $b := a^\psi \in \mathrm{GL}(V)$. If a has order n, then b^n is a scalar matrix. By algebraic closure, we may replace b by b times a scalar to assume $b^n = 1$, so in particular b has finite order. The action of b on the 1-dimensional space of invariant bilinear forms is by a root of unity $c \in K^*$. Take $d \in K$, a square root of c. Then $d^{-1}b$ preserves f and induces the action of a on $G^\psi \cong G$. The group $\langle G^\psi, b\rangle$ is finite. □

We use Recipe 2.2 to compute explicit matrices giving the action of SL(2, 27) on its faithful 28-dimensional modules with characters 28_D and 28_E. (Note that the matrix g in the 28-dimensional representation of Recipe 2.2 has order dividing 26 and has trace $\zeta + 1/\zeta$; thus we obtain representations with characters 28_D and 28_E by applying the recipe with ζ chosen to be any particular primitive 26th root of unity and its 9th power, respectively.) The direct sum of these matrix representations gives an explicit realization of the action of SL(2, 27) on W. We will fix this choice of matrix representation for the remainder of the paper, and we will write S for the group of 56×56 matrices giving the representation. We write C (resp. $\widehat{C}$) for the centralizer of S in $\mathrm{GL}(W)$ (resp. $\mathrm{GL}(\widehat{W})$). The group C has structure $k^* \times k^*$. Matrix generators for C are readily available as direct sums of scalar multiples of the identity acting on each of the summands of W.

The bilinear forms given by Recipe 2.2 are alternating on the representations of SL(2, 27) with characters 28_D and 28_E. Hence they provide a family of S-invariant symplectic forms on W. Each of these invariant symplectic forms is specified by giving two parameters from k^*. We write $\langle\cdot,\cdot\rangle$ for the particular form on W which restricts to (a) the form given by Recipe 2.2 on the submodule with character 28_D and (b) the negative of the form given by Recipe 2.2 on the submodule with character 28_E. We observe that the S-invariant forms on W belong to a single orbit of $\widehat{C}$, so our choice of bilinear form is equivalent to any other if we are prepared to allow field extensions. However, the bilinear forms on W fall into four orbits under the action of C (corresponding to elements of $k^*/k^{*2} \times k^*/k^{*2}$). Our choice of orbit for the form is convenient in avoiding later need for field extensions.

We write L for the Lie algebra of derivations of the symplectic form $\langle\cdot,\cdot\rangle$ (similarly, we write $\widehat{L}$ for the Lie algebra of derivations of the form when viewed as a pairing on $\widehat{W}$). As Lie algebras, L and $\widehat{L}$ have type C_{28}. It is clear that these Lie algebras are S-invariant. Let Γ be the general linear group $\mathrm{GL}(\widehat{W})$, and let $\Sigma \cong \mathrm{Sp}(56, \widehat{k})$ be the subgroup of Γ that preserves $\langle\cdot,\cdot\rangle$. Let C^- be the subgroup of C that fixes $\langle\cdot,\cdot\rangle$, so that $C^- \cong 2 \times 2$. The group C^- is the centralizer of S in Σ.

We observe that $\mathbb{F}_{1093}$ contains square roots of its primitive 26th roots of unity (since $1093 \equiv 1 \pmod 4$). Hence, the automorphisms specified in Recipe 2.2 provide an element d in $\mathrm{GL}(W)$ that acts as a graph automorphism of S and preserves the S-invariant bilinear form $\langle\cdot,\cdot\rangle$.

The computations that we describe later in this section will establish the following theorem.

COMPUTATIONAL THEOREM 2.4. *The Lie algebra $\widehat{L}$ contains exactly four S-invariant subalgebras of type E_7. These algebras fall into a single orbit under the normalizer of S in Σ and into two orbits* (*of size* 2) *under C^-. The four S-invariant subalgebras of type E_7 are spanned* (*as vector spaces*) *by elements in the subalgebra $L \subset \widehat{L}$.*

COROLLARY 2.5. *The group* $\mathrm{SL}(2,27)$ *has twelve Lie primitive embeddings into* $2E_7(\mathbb{C})$. *For each of the six algebraically conjugate characters given by Lemma 2.1, there are two embeddings and these are conjugate by the action of* $\mathrm{GL}(2,27)$.

Proof. According to Larsen's $(0,p)$-correspondence (see [GR1]), we can enumerate Lie primitive embeddings of $\mathrm{SL}(2,27)$ into $2E_7(\mathbb{C})$ by enumerating embeddings into $2E_7(\widehat{k})$. It is clear from Theorem 2.4 that such embeddings exist, but we must now settle their number.

Let Φ be a copy of $2E_7(\widehat{k})$ with $\Phi \le \Sigma$. Write E for a type E_7-subalgebra of L that is invariant under Φ. Suppose now that S is a copy of $\mathrm{SL}(2,27)$ in Φ. Then, by Theorem 2.4, S preserves exactly four E_7-subalgebras of L. One of these is E (since $S \le \Phi$) and all four have the form E^θ, where $\theta \in N_\Sigma(S)$.

Now suppose that $S_1 \le \Phi$ is Γ-conjugate to S, say $S = S_1^\gamma$. We will show that S and S_1 are setwise conjugate in Φ.

We begin by showing that S and S_1 are Σ-conjugate. The form $\langle\cdot,\cdot\rangle$ is S_1-invariant because $S_1 \le \Phi \le \Sigma$. We deduce that γ transforms $\langle\cdot,\cdot\rangle$ to an S-invariant form (since γ takes S_1 to S). However, as we noted, the S-invariant bilinear forms on W are images of $\langle\cdot,\cdot\rangle$ under elements of $\widehat{C}$. Hence there is an element $\delta \in \widehat{C}$ such that $\sigma = \gamma\delta$ preserves $\langle\cdot,\cdot\rangle$. But $S_1^\sigma = S$.

Similarly, E is S_1-invariant because $S_1 \le \Phi$, so E^σ is an S-invariant subalgebra of $\widehat{L}$. We deduce that $E^\sigma = E^\theta$ for some $\theta \in N_\Sigma(S)$. Hence $E^{\sigma\theta^{-1}} = E$, so that $\sigma\theta^{-1}$ is an element of Φ that conjugates S_1 to S. Thus, two $\mathrm{SL}(2,27)$ subgroups of Φ are conjugate if and only if they are conjugate in Γ.

The $\mathrm{SL}(2,27)$-decompositions of the characters χ_{133} and χ_{56} given in Lemma 2.1 fall into two orbits under $\mathrm{Aut}(\mathrm{SL}(2,27))$. Hence there are two orbits of Γ on $\mathrm{SL}(2,27)$-subgroups in Φ. Because we have just established that Φ controls fusion in Γ of $\mathrm{SL}(2,27)$ subgroups, we can now deduce that there are two conjugacy classes of $\mathrm{SL}(2,27)$ subgroups in Φ. We further observe that no embedding of $\mathrm{SL}(2,27)$ into Φ is stabilized by a field automorphism of $\mathrm{SL}(2,27)$, since the character decompositions of Lemma 2.1 are not stabilized by field automorphisms. Moreover, no embedding of $\mathrm{SL}(2,27)$ into Φ is stabilized by an outer diagonal automorphism, since Theorem 2.4 shows that outer diagonal involutions fix no

S-invariant Lie algebras of type E_7. We deduce that the two classes of subgroups give rise to exactly twelve classes of embeddings. □

Proof of Theorem 2.4. We now describe our computer construction that establishes the Computational Theorem 2.4. Our goal is to classify S-invariant Lie algebras of type E_7 in $\widehat{L}$. As a Lie algebra, $\widehat{L}$ has type C_{28}, but we can also view it as an S-module, in which context it is isomorphic to $S^2(\widehat{W})$. This isomorphism and the translation of $[\cdot, \cdot]$ to an explicit invariant Lie product (also written as $[\cdot, \cdot]$) on $S^2(\widehat{W})$ are given in [R]. We carry out our computations in $S^2(\widehat{W})$ rather than in $\widehat{L}$. We used our own implementation of the MeatAxe to work in $S^2(\widehat{W})$, but the computations described here could also be carried out with the GAP and Magma systems.

The module $S^2(\widehat{W})$ has character $13_a + 13_b + 6 \times 26_a + 6 \times 26_b + 6 \times 26_c + 2 \times 26_d + 2 \times 26_e + 2 \times 26_f + 6 \times 27_a + 6 \times 28_a + 4 \times 28_b + 5 \times 28_c + 4 \times 28_d + 4 \times 28_e + 5 \times 28_f$. If π is an irreducible character of S, we write L_π for the submodule of $S^2(\widehat{W})$ spanned by all irreducible submodules of $S^2(\widehat{W})$ with character π. Thus L_{26_a} has character $6 \times 26_a$. It is a routine application of the MeatAxe to obtain bases for all of the modules of the form L_π. (Note that we perform this computation over the prime field and work inside the module $S^2(W)$.)

The irreducible submodules of L_{26_a} can be parameterized by the 1-dimensional subspaces of a 6-dimensional space, X say. Take six independent (isomorphic) irreducible submodules of L_{26_a}, and select a vector from the first of these submodules together with a corresponding vector (under an S-isomorphism) from each of the other independent submodules. We will call this 6-tuple of vectors $(w^1, w^2, \dots, w^6)$. Any nonzero vector $(x_1, x_2, \dots, x_6) \in X$ corresponds to the irreducible submodule of L_{26_a} spanned by the S-images of $\sum_i x_i w^i$. Moreover, all irreducible submodules of L_{26_a} are obtained in this way. It is easy to use the standard basis program of the MeatAxe to obtain an explicit 6-tuple of vectors from $S^2(W)$ as just described.

For each element $s \in S$, let $\phi_s : X \otimes X \to S^2(\widehat{W})$ be the linear transformation defined by

$$\phi_s((x_1, \dots, x_6) \otimes (y_1, \dots, y_6)) = \left[\textstyle\sum_i x_i w^i, \sum_i y_i w^i s\right].$$

Suppose that E is an S-invariant subalgebra of $S^2(\widehat{W})$ with type E_7. Then E has an S-invariant constituent with character 26_a. This constituent is an irreducible submodule of L_{26_a}; it corresponds to a vector $(x_1, x_2, \dots, x_6) \in X$ as before. Let s be an element of S; then we have $\left[\sum_i x_i w^i, \sum_i x_i w^i s\right] \in E \subset L_{26_a} + L_{26_b} + L_{26_c} + L_{27_a} + L_{28_a}$. Now $L_{26_a} + L_{26_b} + L_{26_c} + L_{27_a} + L_{28_a}$ is a proper subspace of $\widehat{L}$, so for each choice of s we have $(x_1, x_2, \dots, x_6) \otimes (x_1, x_2, \dots, x_6) \in \phi_s^{-1}(L_{26_a} + L_{26_b} + L_{26_c} + L_{27_a} + L_{28_a})$. We used a standard Gaussian elimination to compute this inverse image for a random nonidentity element $s \in S$. We obtained a 10-dimensional subspace, $Y = \phi_s^{-1}(L_{26_a} + L_{26_b} + L_{26_c} + L_{27_a} + L_{28_a})$.

In order to locate candidates for the vector $(x_1, x_2, \dots, x_6)$, we are faced with determining which vectors of Y can be written as "tensor squares" of elements in X. This is a special case of the following general problem.

PROBLEM 2.6. Suppose that X is a vector space and Y is a given subspace of $X \otimes X$. Find an efficient procedure to determine all elements of Y that can be written in the form $x \otimes x$ for some $x \in X$.

We can view this as a problem of finding a relative eigenvector for a collection of matrices.

DEFINITION 2.7. Suppose that $A_1, A_2, \dots, A_r$ is a collection of (not necessarily square) matrices of the same dimensions. A *relative eigenvalue* is a projective point $(a_1 : a_2 : \cdots : a_r)$ for which there is a nonzero vector v such that $vA_1, vA_2, \dots, vA_n$ are linearly dependent vectors in the proportion $(a_1 : a_2 : \cdots : a_n)$; that is, $a_j vA_i = a_i vA_j$ for all i, j. We also say that v is a *relative eigenvector for the relative eigenvalue* $(a_1 : a_2 : \cdots : a_r)$.

We remark that if A_1 and A_2 are square matrices such that A_2 is invertible, then v is a relative eigenvector with eigenvalue $(a_1 : a_2)$ if and only if v is an eigenvector of $A_1 A_2^{-1}$ with eigenvalue a_1/a_2.

We transform Problem 2.6 into a relative eigenvector problem by selecting bases of X and $X \otimes X$ so that a vector of X corresponds to a row vector $\alpha = (x_1, x_2, \dots, x_n)$ with tensor square

$$\alpha \otimes \alpha = (x_1x_1, x_1x_2, \dots, x_1x_n, x_2x_1, x_2x_2, \dots, x_2x_n, \dots, x_nx_n).$$

Now, in Problem 2.6, if Y has dimension m then we can describe a basis of $Y \leq X \otimes X$ by giving an $m \times n^2$ matrix. This matrix is naturally partitioned into n blocks, $B_1, B_2, \dots, B_n$, of size $m \times n$. (The block B_i corresponds to the n columns of $\alpha \otimes \alpha$ of the form $x_i x_j$.) Any solution to Problem 2.6 gives a vector $\beta = (y_1, y_2, \dots, y_m)$ such that $\beta B_1 = x_1 \alpha, \beta B_2 = x_2 \alpha, \dots, \beta B_n = x_n \alpha$. Hence $(x_1 : x_2 : x_3 : \cdots : x_n)$ is a relative eigenvalue for the matrices $B_1, B_2, \dots, B_n$. Moreover, there is a relative eigenvector β for the relative eigenvalue $(x_1 : x_2 : x_3 : \cdots : x_n)$ such that the images βB_i are all multiples of $(x_1, x_2, x_3, \dots, x_n)$.

In the particular instance of Problem 2.6 that we face, with $\dim(X) = 6$ and $\dim(Y) = 10$, the space B is small enough to allow us to locate all relative eigenvalues quickly. In fact, the echelon form of our basis of Y gives the following six matrices for $B_1, B_2, \dots, B_6$ over $\mathbb{F}_{1093}$.

999	992	0	0	0	0
0	0	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	0
0	0	0	0	0	1
0	0	0	0	0	0
0	0	757	161	0	0
0	0	985	341	0	0
658	1034	0	0	0	0
1066	426	0	0	0	0

992	1	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	0
0	0	0	0	0	1
0	0	579	623	0	0
0	0	485	585	0	0
1034	0	0	0	0	0
426	0	0	0	0	0

0	0	0	0	0	0
0	0	884	1015	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
757	579	0	0	511	1
985	485	0	0	387	0
0	0	202	157	0	0
0	0	5	756	0	0

0	0	0	0	0	0
0	0	1015	1	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
161	623	0	0	425	0
341	585	0	0	180	1
0	0	157	0	0	0
0	0	756	0	0	0

0	0	0	0	0	0
0	0	0	0	0	0
1	0	0	0	0	0
0	1	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	511	425	0	0
0	0	387	180	0	0
0	0	0	0	691	1
0	0	0	0	357	0

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

These matrices are small enough and sparse enough to make easy the determination of relative eigenvalues $(x_1 : x_2 : x_3 : x_4 : x_5 : x_6)$ and corresponding relative eigenvectors $(y_1, \ldots, y_{10})$. For example, consideration of the third and fourth columns of the last two matrices shows that (y_7, y_8) is a relative eigenvector with relative eigenvalue $(x_5 : x_6)$ for the matrices: $\begin{pmatrix} 511 & 425 \\ 387 & 180 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This is just an ordinary eigenvector computation. Moreover given the sparse outer columns of the fourth matrix, knowledge of y_7 and y_8 determines the proportions $(x_1 : x_2 : x_5 : x_6)$—this because $(y_7, y_8)\begin{pmatrix} 161 & 623 & 0 & 0 & 425 & 0 \\ 341 & 585 & 0 & 0 & 180 & 1 \end{pmatrix}$ is a scalar multiple of $(x_1, x_2, x_3, x_4, x_5, x_6)$. After a short series of similar computations, we obtained four possibilities for $(x_1 : x_2 : \cdots : x_6)$ as follows:

(684 : 249 : 20 : 54 : 135 : 1) with relative eigenvector
(793, 730, 528, 825, 684, 249, 20, 54, 135, 1);
(684 : 249 : 1073 : 1039 : 135 : 1) with relative eigenvector
(793, 730, 528, 825, 684, 249, 1073, 1039, 135, 1);
(414 : 915 : 374 : 196 : 556 : 1) with relative eigenvector
(1080, 161, 654, 495, 414, 915, 374, 196, 556, 1);
(414 : 915 : 719 : 897 : 556 : 1) with relative eigenvector
(1080, 161, 654, 495, 414, 915, 719, 897, 556, 1).

We thus find that there are at most four S-invariant Lie subalgebras of $\widehat{L}$ that have type E_7. Moreover, we know explicit vectors spanning 26-dimensional subspaces of each of these four potential Lie subalgebras in L. In all four cases the 26-dimensional subspace generates a 133-dimensional subalgebra of L, which must be S-invariant because it is generated by an S-invariant space. In each case, an application of the MeatAxe shows that this S-invariant 133-dimensional algebra has character $26_a + 26_b + 26_c + 27_a + 28_a$ when viewed as an S-module. Moreover, for each of the four 133-dimensional algebras, we can check (as before) that each of its five irreducible submodules is a generating set. It follows that the algebra has no S-invariant subalgebra. Moreover, since the four 133-dimensional algebras have been obtained as algebras of 56×56 matrices, it is an easy matter to check (from its 56-dimensional representation) that each of them has a nonsingular trace form. Now consider one of the four 133-dimensional algebras that we have obtained, call it X. As in [GR3, Lemma 4], we can apply Block's theorem [B] to show that X is a direct sum $\bigoplus_i X_i$ of indecomposable ideals, each of which is either (a) 1-dimensional or (b) simple and having one of the types A, B, C, D, E, F, or G. By construction, we know that X is generated by the elements of one of its 26-dimensional subspaces. It follows that X cannot be abelian; hence at least one of its ideals X_i is nonabelian. The sum of the S-images of X_i must be X, since X has no S-invariant subalgebra. However, as in the proof of [GR3, Lemma 4], the sum of S-images of X_i is a direct sum of independent S-images of X_i. We deduce that $\dim(X_i)$ divides $\dim(X) = 133 = 7 \times 19$. But the only divisor of 133 that is the dimension of a simple Lie algebra of one of the types $A, \dots, G$ is 133 itself (we are in characteristic not 2 or 3, so the algebras with Chevalley bases for indecomposable root systems remain simple). It follows that X_i and X must both have type E_7, and thus our construction of S as automorphisms of a 56-dimensional representation of X gives an embedding $S \leq 2E_7(1093)$.

Finally, we verified the information in Theorem 2.4 about the action of $N_\Sigma(S)$ on the four S-invariant Lie algebras by checking that the algebras are paired up by the group C^- and that these pairs are interchanged by the explicit matrix d in $N_\Sigma(S) \setminus C^-$.

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