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SECANT-LIKE METHOD FOR SOLVING GENERALIZED EQUATIONS*

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Abstract. In [2], [3], Argyros introduced a new derivative–free quadratically convergent method for solving a nonlinear equation in Banach space. In this paper, we extend this method to generalized equations in order to approximate a locally unique solution. The method uses only divided differences operators of order one. Under some Lipschitz–type conditions on the first and second order divided differences operators and Lipschitz–like property of set–valued maps, an existence–convergence theorem and a radius of convergence are obtained. Our method has the following advantages: we extend the applicability of this method than all the previous ones [2]–[5], [7], and we do not need to evaluate any Fréchet derivative. We provide also an improvement on the radius of convergence for our algorithm, under some center–condition and less computational cost. Numerical examples are also provided.

Key words. Fréchet derivative, Banach space, divided differences, Secant method, generalized equation, set–valued map, Aubin's continuity, radius of convergence.

AMS subject classifications. 65K10, 65G99, 65H10, 65B05, 47H04, 49M15, 47H17

1. Introduction.

A large number of problems in applied mathematics and engineering are solved by finding the solutions of generalized equation introduced by Robinson [13], [14]:

$$0 \in F(x) + G(x), \tag{1.1}$$

where F is a continuous function from an open subset $\mathcal{D}$ of $\mathcal{X}$ into $\mathcal{Y}$, G is a set–valued map from $\mathcal{X}$ to the subsets of $\mathcal{Y}$ with closed graph, and $\mathcal{X}$, $\mathcal{Y}$ are Banach spaces.

Generalized equation is an abstract model of a wide variety problems including:

(a) If G is the positive orthant in $\mathcal{X} := \mathbb{R}^m$, (1.1) is a system of inequalities.

(b) If $\mathcal{X} = \mathcal{Y}$ is a Hilbert space with inner product $(.;.)$, C is a convex subset of $\mathcal{X}$, and $G : \mathcal{X} \rightrightarrows \mathcal{Y}$ is a set–valued mapping defined by

$$G(x) = \begin{cases} \{z \,:\, (z; y-x) \ \leq 0 \text{ for all } y \in \mathcal{X}\} & \text{if } x \in C \\ \emptyset & \text{otherwise,} \end{cases} \tag{1.2}$$

then, a variational inequality problem consisting to

$$\text{find } c^\star \text{ in } C \text{ such that } (F(c^\star); c - c^\star) \ \geq \ 0, \text{ for all } c \in \mathcal{X}, \tag{1.3}$$

is equivalent to generalized equation (1.1) in the following form

$$\text{find } c^\star \text{ in } C \text{ such that } 0 \in F(c^\star) + G(c^\star).$$

For example, the last equation typically describes some (mechanical, economic) equilibrium.

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(c) If $G = \{0\}$, (1.1) is a nonlinear equation in the form

$$F(x) = 0. \tag{1.4}$$

For example, dynamic systems can be modelled by difference equations, and their solutions usually represent the states of the systems, which are determined by solving equation (1.4).

This paper considers the problem of approximating a locally unique solution $x^\star$ of (1.1) using an iterative method as follows

$$0 \in F(x_n) + [2\, x_n - x_{n-1}, x_{n-1}; F]\ (x_{n+1} - x_n) + G(x_{n+1}), \quad (x_0, x_1 \in \mathcal{D}), \quad (n \geq 1) \tag{1.5}$$

where $[x, y; F] \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ is called a divided difference of F of order one at the points x and y (to be defined later).

In the case of nonlinear equations (1.4), the method (1.5) is reduced to the following algorithm ([2], [3]):

$$x_{n+1} = x_n - [2\, x_n - x_{n-1}, x_{n-1}; F]^{-1}\ F(x_n), \quad (x_0, x_1 \in \mathcal{D}), \quad (n \geq 1). \tag{1.6}$$

Case $G \neq \{0\}$**.** Using Lipschitz and Hölder conditions on the first order divided differences operators, some convergence results of an uniparametric Secant–type method for solving (1.1) are developed in [5]. Using some ideas introduced by us in [3] for nonlinear equations, a Newton–like method is used in [4] for solving perturbed generalized equation under some condition on the second order divided difference operator. A family of Steffensen–type methods is presented in [5], [6], [10] for solving (1.1) under ω–conditioned divided differences operator, where ω is a continuous nondecreasing function.

Case $G = \{0\}$**.** Using a cubic scalar majorizing polynomial instead of majorizing sequences, Argyros [2] provided a local as well as a semilocal convergence analysis (quadratic convergence) for method (1.6) for solving (1.4). In [2], the method (1.6) is also compared with Steffensen–type method considered by Amat, Busquier and Candela [1]. Some variants of method (1.6) and applications can be found in [3].

Here, we study method (1.5) motivated by the work in [2] for nonlinear equations. Under some condition on the first and the second order divided differences introduced for nonlinear equation in [7], and Lipschitz–like property of set–valued map G^{-1} around $(-F(x^\star), x^\star)$, we provide a convergence analysis of method (1.5). Our approach has the following advantages: we extend the applicability of this method than all the previous know ones [2]–[5], [7], and we do not need to evaluate any Fréchet derivative.

The structure of this paper is the following. In section 2, we collect a number of basic definitions and recall a fixed points theorem for set–valued maps. In section 3, we show an existence–convergence theorem of sequence given by (1.5). Finally, we provide also an improvement of the ratio of our algorithm under some center–conditions and less computational cost. Some remarks and numerical examples are also presented.

2. Background material. In order to make the paper as self–contained as possible we reintroduce some definitions and some results on fixed point theorems [2]–[16]. Let us begin with some notations that will used throughout this paper. We let $\mathcal{Z}$ be a Banach space equiped with the norm $\| \,.\, \|$. The distance from a point x to a set A in $\mathcal{Z}$ is defined by $\operatorname{dist}(x, A) = \inf_{y \in A} \| x - y \|$, with the convention $\operatorname{dist}(x, \emptyset) = +\infty$ (according to the general convention $\inf \emptyset = +\infty$). Given a subset C of $\mathcal{Z}$, we denote by $e(C, A)$ the Hausdorff–Pompeiu excess of C into A, defined by:

$$e(C, A) = \sup_{x \in C} \operatorname{dist}(x, A),$$

with the conventions $e(\emptyset, A) = 0$ and $e(C, \emptyset) = +\infty$ whenever $C \neq \emptyset$. For a set–mapping $\Lambda : \mathcal{X} \rightrightarrows \mathcal{Y}$, we denote by $\operatorname{gph} \Lambda$ the set $\{(x, y) \in \mathcal{X} \times \mathcal{Y},\ y \in \Lambda(x)\}$ and $\Lambda^{-1}(y)$ the set $\{x \in \mathcal{X},\ y \in \Lambda(x)\}$. The norms in both the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ will be denoted by $\| \,.\, \|$ and the closed ball centered at x with radius r by $\mathbb{B}_r(x)$.

Definition 2.1. *An operator $[.,.;F]$ belonging $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is called the first order divided difference of F at the points x and y in $\mathcal{X}$ $(x \neq y)$ if the following holds*

$$[x, y; F]\,(y - x) = F(y) - F(x).$$

If F is Fréchet differentiable at x, then $[x, x; F] = \nabla F(x)$.

Definition 2.2. *An operator $[.,.,.;F]$ belonging $\mathcal{L}(\mathcal{X}, \mathcal{L}(\mathcal{X}, \mathcal{Y}))$ is called the second order divided difference of F at the points x, y, and z in $\mathcal{X}$ if the following holds*

$$[x, y, z; F] = [x, y; F] - [x, z; F].$$

If F is two Fréchet differentiable at x, then $[x, x, x; F] = \dfrac{1}{2}\,\nabla^2 F(x)$.

We also need to define the pseudo–Lipschitzian concept of set–valued maps, introduced by Aubin [8] and also known as Lipschitz–like property [12]:

Definition 2.3. *A set–valued Λ is pseudo–Lipschitz around $(\overline{x}, \overline{y}) \in \operatorname{gph} \Lambda$ with modulus M if there exist constants a and b such that*

$$\sup_{z \in \Lambda(y') \cap \mathbb{B}_a(\overline{y})} \operatorname{dist}(z, \Lambda(y'')) \leq M \| y' - y'' \|, \quad \text{for all } y' \text{ and } y'' \text{ in } \mathbb{B}_b(\overline{x}). \tag{2.1}$$

In the term of excess, we have an equivalent definition of pseudo–Lipschitzian property replacing the inequality (2.1) by

$$e(\Lambda(y') \cap \mathbb{B}_a(\overline{y}), \Lambda(y'')) \leq M \| y' - y'' \|, \quad \text{for all } y' \text{ and } y'' \text{ in } \mathbb{B}_b(\overline{x}). \tag{2.2}$$

Let us note that the Lipschitz–like of Λ is equivalent to the metric regularity of Λ^{-1} which is a basic well–posedness property in optimization problems. The Lipschitz–like property play a crucial role in many aspects of variational analysis and applications [12], [16]. Other characterization of this property is given by Mordukhovich [11], [12] via the concept of coderivative of set–valued maps. For some characterizations and applications of the Lipschitz–like property the reader could be referred to [8], [9], [11], [15], [16] and the references given there.

We need also the following fixed point theorem [9].

Lemma 2.4. *Let ϕ be a set–valued map from $\mathcal{X}$ into the closed subsets of $\mathcal{X}$. We suppose that for $\eta_0 \in \mathcal{X}$, $r \geq 0$ and $0 \leq \lambda < 1$ the following properties hold*

(i) dist $(\eta_0, \phi(\eta_0)) \leq r(1-\lambda)$;
(ii) $e(\phi(x) \cap \mathbb{B}_r(\eta_0), \phi(z)) \leq \lambda \ \| x - z \|, \ \forall x, z \in \mathbb{B}_r(\eta_0)$.
Then ϕ *has a fixed point in* $\mathbb{B}_r(\eta_0)$. *That is, there exists* $x \in \mathbb{B}_r(\eta_0)$ *such that* $x \in \phi(x)$. *If* ϕ *is single–valued, then* x *is the unique fixed point of* ϕ *in* $\mathbb{B}_r(\eta_0)$.

3. Local convergence of method (1.5). In this section we will be concerned with the existence and the convergence of the sequence defined by (1.5) to the solution $x^\star$ of (1.1). The main result of this study is as follows.

THEOREM 3.1. *Let* $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ *be an operator such that for every distinct points* x, y *and* z *in* $\mathcal{D}_0 \subseteq \mathcal{D}$, *there exist a first and second order divided differences of* F $[x, y; F]$ *and* $[x, y, z; F]$ *respectively at these points.*

Assume:

$(\mathcal{H}0)$ *There exists* $\alpha > 0$, *such that for all* $(x, y, u, v) \in \mathcal{D}_0^4$

$$\| [x, y; F] - [u, v; F] \| \leq \alpha \left(\| x - u \| + \| y - v \| \right);$$

$(\mathcal{H}1)$ *There exists* $\beta > 0$ *such that:*

$$\text{for all } (x, y) \in \mathcal{D}_0^2 \Longrightarrow 2y - x \in \mathcal{D}_0,$$

and

$$\| [y, x, y; F] - [2\,y - x, x, y; F] \| \leq \beta \ \| x - y \|;$$

$(\mathcal{H}2)$ *The set–valued map* G^{-1} *is pseudo–Lipschitz around* $(-F(x^\star), x^\star)$, *with constants* M, a *and* b *(These constants are given by Definition 2.3);*

$(\mathcal{H}3)$ *There exists* $\kappa > 0$, *such that for all* $x, y \in \mathcal{D}_0$, *we have*

$$\| [x, y; F] \| \leq \kappa \qquad \text{and} \qquad M \ \kappa < 1;$$

Then, for every constant C *such that*

$$C \geq C_0 = \frac{M \ (2\ \beta + 5\ \alpha)}{1 - M\ \kappa}, \tag{3.1}$$

exists $\delta > 0$, *satisfying*

$$\mathbb{B}_\delta(x^\star) \subseteq \mathcal{D}_0,$$

where,

$$\delta < \delta_0 = \min \left\{ a;\ \frac{1}{C};\ \frac{b}{2\ \kappa};\ \sqrt{\frac{b}{2\ (2\ \beta + 5\ \alpha)}} \right\} \tag{3.2}$$

such that, for every distinct starting points x_0 *and* x_1 *in* $\mathbb{B}_\delta(x^\star)$ *(with* $x_0 \neq x^\star$ *and* $x_1 \neq x^\star$*), and a sequence* (x_k) *defined by (1.5), which is convergent to* $x^\star$, *i.e.;*

$$\| x_{k+1} - x^\star \| \leq C \ \| x_k - x^\star \| \ \max \left\{ \| x_k - x^\star \|, \| x_{k-1} - x^\star \| \right\}. \tag{3.3}$$

To prove Theorem 3.1, we need the following lemma.

LEMMA 3.2. *Define the set–valued maps $\Lambda : \mathcal{X} \rightrightarrows \mathcal{Y}$, and $\Theta_k : \mathcal{X} \rightrightarrows \mathcal{X}$ by*

$$\Lambda(x) = F(x^\star) + G(x), \qquad \Theta_k(x) = \Lambda^{-1}(\Xi_k(x)), \quad k \geq 1, \tag{3.4}$$

where Ξ_k is a mapping from $\mathcal{X}$ to $\mathcal{Y}$ defined by

$$\Xi_k(x) = F(x^\star) - F(x_k) - [2\,x_k - x_{k-1}, x_{k-1}; F]\ (x - x_k), \qquad k \geq 1. \tag{3.5}$$

Assume that the assumptions of Theorem 3.1 hold. Then, for every distinct starting points x_0 and x_1 in $\mathbb{B}_\delta(x^\star)$ (with $x_0 \neq x^\star$ and $x_1 \neq x^\star$), the set–valued map Θ_1 has a fixed point x_2 in $\mathbb{B}_\delta(x^\star)$, and the inequality (3.3) is satisfied for $k = 1$.

Proof of Lemma 3.2. Let us note that the point x_2 is a fixed point of Θ_1 if and only if

$$0 \in F(x_1) + [2\,x_2 - x_1, x_1; F]\ (x_2 - x_1) + G(x_2).$$

Step 1. We prove that the first assumption in Lemma 2.4 is satisfied.

By hypothesis ($\mathcal{H}2$), we have

$$e(\Lambda^{-1}(y') \cap \mathbb{B}_a(x^\star), \Lambda^{-1}(y'')) \leq M \ \parallel y' - y'' \parallel,\ \forall y', y'' \in \mathbb{B}_b(0). \tag{3.6}$$

According to the definition of excess e:

$$\text{dist}\ (x^\star, \Theta_1(x^\star)) \leq e\bigg(\Lambda^{-1}(0) \cap \mathbb{B}_\delta(x^\star), \Theta_1(x^\star)\bigg). \tag{3.7}$$

By Definition 2.2, we obtain

$$[x_1, x_0, x_1; F] = [x_1, x_0; F] - [x_1, x_1; F] \tag{3.8}$$

and

$$[2\,x_1 - x_0, x_0, x_1; F] = [2\,x_1 - x_0, x_0; F] - [2\,x_1 - x_0, x_1; F]. \tag{3.9}$$

Using (3.8), (3.9), we have:

$$\begin{aligned}
\parallel \Xi_1(x^\star) \parallel \ &= \ \parallel F(x^\star) - F(x_1) - [2\,x_1 - x_0, x_0; F]\ (x^\star - x_1) \parallel \\
&= \ \parallel \Big([x^\star, x_1; F] - [2\,x_1 - x_0, x_0; F]\Big)\ (x^\star - x_1) \parallel \\
&= \ \parallel \Big([x^\star, x_1; F] + [x_1, x_0; F] - [x_1, x_1; F] - \\
&\quad [x_1, x_0; F] + [x_1, x_1; F] - [2\,x_1 - x_0, x_0; F] + \\
&\quad [2\,x_1 - x_0, x_1; F] - [2\,x_1 - x_0, x_1; F]\Big)\ (x^\star - x_1) \parallel \qquad (3.10) \\
&= \ \parallel \Big([x^\star, x_1; F] + [x_1, x_0, x_1; F] - [2\,x_1 - x_0, x_0, x_1; F] - \\
&\quad [x_1, x_0; F] + [x_1, x_1; F] - [2\,x_1 - x_0, x_1; F]\Big)\ (x^\star - x_1) \parallel
\end{aligned}$$

By assumptions ($\mathcal{H}0$), ($\mathcal{H}1$), and (3.10), we obtain the following estimate:

$$\begin{aligned}
\| \Xi_1(x^\star) \| &\leq \Big(\| [x_1, x_0, x_1; F] - [2\,x_1 - x_0, x_0, x_1; F] \| + \\
&\quad \| [x^\star, x_1; F] - [x_1, x_0; F] \| + \\
&\quad \| [x_1, x_1; F] - [2\,x_1 - x_0, x_1; F] \|\Big) \ \| x_1 - x^\star \| \\
&\leq \Big(\beta \ \| x_1 - x_0 \| + \alpha \ (\| x^\star - x_1 \| + \| x_1 - x_0 \|) + \\
&\quad \alpha \ \| x_1 - (2\,x_1 - x_0) \| \Big) \ \| x_1 - x^\star \| \\
&= \Big((\beta + 2\,\alpha) \ \| x_1 - x_0 \| + \alpha \ \| x^\star - x_1 \| \Big) \ \| x_1 - x^\star \| \\
&\leq \Big((\beta + 3\,\alpha) \ \| x_1 - x^\star \| + (\beta + 2\,\alpha) \ \| x^\star - x_0 \| \Big) \ \| x_1 - x^\star \| \\
&\leq (2\,\beta + 5\ \alpha) \ \| x_1 - x^\star \| \ \max\Big\{ \| x_1 - x^\star \|, \| x_0 - x^\star \| \Big\}.
\end{aligned} \tag{3.11}$$

By (3.2), we have $\Xi_1(x^\star) \in \mathbb{B}_b(0)$.

Hence from (3.6), one gets

$$\begin{aligned}
e\Big(\Lambda^{-1}(0) \cap \mathbb{B}_\delta(x^\star), \Theta_1(x^\star)\Big) &= e\Big(\Lambda^{-1}(0) \cap \mathbb{B}_\delta(x^\star), \Lambda^{-1}[\Xi_1(x^\star)]\Big) \\
&\leq (2\,\beta + 5\ \alpha) \ \| x_1 - x^\star \| \ \max\Big\{ \| x_1 - x^\star \|, \| x_0 - x^\star \| \Big\}.
\end{aligned} \tag{3.12}$$

Using (3.7), the following inequality holds:

$$\text{dist } (x^\star, \Theta_1(x^\star)) \quad \leq \quad M \ (2\,\beta + 5\ \alpha) \ \| x_1 - x^\star \| \ \max\Big\{ \| x_1 - x^\star \|, \| x_0 - x^\star \| \Big\}. \tag{3.13}$$

Since $C\ (1 - M\ \kappa) > M\ (2\,\beta + 5\ \alpha)$, there exists $\lambda \in [M\ \kappa, 1[$ such that:

$$C\ (1 - \lambda) \geq M\ (2\,\beta + 5\ \alpha),$$

and

$$\text{dist } (x^\star, \Theta_1(x^\star)) \leq C\ (1 - \lambda) \ \| x_1 - x^\star \| \ \max\Big\{ \| x_1 - x^\star \|, \| x_0 - x^\star \| \Big\}. \tag{3.14}$$

Identifying η_0, ϕ and r in Lemma 2.4 by

$$x^\star, \quad \Theta_1 \quad \text{and} \quad r_1 = C \ \| x_1 - x^\star \| \ \max\Big\{ \| x_1 - x^\star \|, \| x_0 - x^\star \| \Big\}$$

respectively, we can deduce from the inequality (3.14) that the first assumption in Lemma 2.4 is satisfied.

Step 2. We prove now that the second assumption of Lemma 2.4 is verified.

Using (3.2), we have $r_1 \leq \delta \leq a$, and moreover for $x \in \mathbb{B}_\delta(x^\star)$ we get in turn:

$$\begin{array}{rcl} \| \Xi_1(x) \| & = & \| F(x^\star) - F(x_1) - [2\,x_1 - x_0, x_0; F]\ (x - x_1) \| \\ & = & \| [x^\star, x_1; F]\ (x^\star - x + x - x_1) - [2\,x_1 - x_0, x_0; F]\ (x - x_1) \| \\ & \leq & \| [x^\star, x_1; F] \| \; \| x - x^\star \| + \\ & & \| [x^\star, x_1; F] - [2\,x_1 - x_0, x_0; F] \| \; \| x - x_1 \| . \end{array} \tag{3.15}$$

Using (3.11), assumptions $(\mathcal{H}0)$, $(\mathcal{H}1)$, and $(\mathcal{H}3)$, we obtain:

$$\begin{array}{rcl} \| \Xi_1(x) \| & \leq & \kappa\ \| x - x^\star \| + \Big((\beta + 2\,\alpha)\ \| x_1 - x_0 \| + \alpha\ \| x^\star - x_1 \| \Big)\ \| x - x_1 \| \\ & \leq & \kappa\ \delta + (2\,\beta + 5\,\alpha)\ \delta^2. \end{array} \tag{3.16}$$

Then by (3.2), we deduce that for all $x \in \mathbb{B}_\delta(x^\star)$, we have $\Xi_1(x) \in \mathbb{B}_b(0)$.

Finally, for all $x', x'' \in \mathbb{B}_{r_1}(x^\star)$, we have:

$$e(\Theta_1(x') \cap \mathbb{B}_{r_1}(x^\star), \Theta_1(x'')) \leq e(\Theta_1(x') \cap \mathbb{B}_\delta(x^\star), \Theta_1(x'')),$$

which yields by (3.6)

$$\begin{array}{rcl} e(\Theta_1(x') \cap \mathbb{B}_{r_1}(x^\star), \Theta_1(x'')) & \leq & M\ \| \Xi_1(x') - \Xi_1(x'') \| \\ & = & M\ \| [2\,x_1 - x_0, x_0; F]\ (x'' - x') \| \end{array} \tag{3.17}$$

Using $(\mathcal{H}3)$, and the fact that $\lambda \geq M\ \kappa$, we obtain:

$$e(\Theta_1(x') \cap \mathbb{B}_{r_1}(x^\star), \Theta_1(x'')) \leq M\ \kappa\ \| x'' - x' \| \leq \lambda\ \| x'' - x' \| . \tag{3.18}$$

The second condition of Lemma 2.4 is satisfied.

By Lemma 2.4, we can deduce the existence of a fixed point $x_2 \in \mathbb{B}_{r_1}(x^\star)$ for the map Θ_1. The proof of Lemma 3.2 is complete. ☐

Proof of Theorem 3.1. The proof of theorem 3.1 is given by induction on k. For starting point x_0 and x_1, and by Lemma 3.2, the set–valued map Θ_1 has a fixed point x_2 in $\mathbb{B}_{r_1}(x^\star)$. We show that the function Θ_k has a fixed point x_{k+1} in $\mathcal{X}$. This process is useful to prove the existence of a sequence (x_k) satisfying (1.5).

Keep

$$\eta_0 = x^\star, \quad r := r_k = C\ \| x^\star - x_k \|\ \max\Big\{ \| x_k - x^\star \|, \| x_{k-1} - x^\star \| \Big\}.$$

Then, the application of Lemma 3.2 to the map Θ_k gives the desired result. ☐

Remark 3.3. *Delicate condition*

$$(x, y) \in \mathcal{D}_0^2 \Longrightarrow 2y - x \in \mathcal{D}_0, \tag{3.19}$$

certainly holds if $\mathcal{D}_0 = \mathcal{D} = \mathcal{X}$, but not only in this case. In particular, this condition can also be replaced by a stronger but more practical which we decided not to introduce *originally in Theorem 3.1, so as to leave this theorem as uncluttered as possible. Indeed, define ball* $\mathbb{B}_{\delta_1}(x^\star)$*, where* $\delta_1 = 3\,\delta$*. Then*

$$\| 2y - x \| \leq \| y - x^\star \| + \| y - x \| \leq 2\ \| y - x^\star \| + \| x - x^\star \| \leq 3\ \delta = \delta_1.$$

Then, we choose x_0, x_1 *in* $\mathbb{B}_\delta(x^\star)$*, but we require* $\mathbb{B}_{\delta_1}(x^\star) \subseteq \mathcal{D}_0$ *instead of* $\mathbb{B}_\delta(x^\star) \subseteq \mathcal{D}_0$ *in Theorem 3.1, so that condition (3.19) can be dropped. Note also that condition (3.19) suffices to hold only for the iterates of method (1.5).*

Example 3.4. *(see [2]) Let* $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$*,* $G = \{0\}$*,* $x_0 = (3.9, .9)$*,* $x_1 = (4, 1)$*, and define operator* F *on* $\mathcal{X}$ *by*

$$F(x, y) = (y^2 - 4, x^2 - 2\,y - 21).$$

Our method (1.5) generates the solution $x^\star = (5, 2)$ *of (1.1) after 5 iterations.*

Note that for $\theta_n = \theta^\star = 10^{-8}$ *fixed and small in Steffensen–type method considered in [1]:*

$$\begin{aligned} y_n &= x_n + \theta_n\ F(x_n), \qquad \theta_n \in [0, 1) \\ x_{n+1} &= y_n - [x_n, y_n; F]^{-1}\ F(y_n). \end{aligned} \tag{3.20}$$

We cannot compute the iterates of method (3.20). Then method (1.5) can serve as an alternative.

Example 3.5. *[2] Let* $\mathcal{X} = \mathcal{Y} = \mathbb{R}$*,* $G = \{0\}$*,* $x_0 = .6$*,* $x_1 = .7$*, and define operator* F *on* $\mathcal{X}$ *by*

$$F(x) = x^2 - 6\,x + 5.$$

Our method (1.5) becomes:

$$x_{n+1} = \frac{x_n^2 - 5}{2\ (x_n - 3)}, \tag{3.21}$$

and coincides with the usual Newton's method (NM) for solving nonlinear equation (1.4). Moreover, the Secant method (SM) is:

$$x_{n+1} = \frac{x_{n-1}\ x_n - 5}{x_{n-1} + x_n - 6}. \tag{3.22}$$

Then, we have the results:

Comparison table

n	*(3.21)=(NM)*	*(3.22)=(SM)*
1	.980434783	.96875
2	.999905228	.997835498
3	.999999998	.99998323
4	1=$x^\star$	.99999991
5		1

Example 3.6. *We consider* $\mathcal{X} = \mathcal{Y} = \mathbb{R}$*, and the generalized equation* $0 \in F(x) + G(x)$*, with* $F(x) = x$*;* $G(x) = |x| + x + (-\infty, 0]$*, and* $x^\star = 0$*. For all* $x, y \in \mathcal{X}$*, with* $x \neq y$*, we have* $[x, y; F] \equiv 1$*, and assumption* $(\mathcal{H}3)$ *is satisfied. For* y *in* $\mathbb{R}$*,* $(F + G)^{-1}(y)$ *is the set of solutions of inequality* $2\,x + |x| \geq y$*. The set–valued mapping* $(F(x^\star) + G(.))^{-1} = G^{-1}$ *is pseudo–Lipschitz around* $(0, x^\star)$*, and assumption* $(\mathcal{H}2)$ *is satisfied.*

4. An improved local convergence and remarks. In this section, we show by using more precise estimates that under less computational cost, and weaker hypothesis than $(\mathcal{H}3)$: the radius of convergence is enlarged. The idea is taken from the works on nonlinear equations [2].

Remark 4.1. *We can enlarge the radius of convergence in Theorem 3.1 even further as follows: using inequalities (3.16), (3.11), and* $(2\,\beta+5\,\alpha)\,\delta^2<\kappa\,\delta+(2\,\beta+5\,\alpha)\,\delta^2$*, we can improve* δ *given in (3.2) by considering the constant* δ'*:*

$$\delta'<\delta'_0=\min\left\{a;\ \frac{1}{C};\ \delta_1\right\},$$

where δ_1 *is the constant given by*

$$\delta_1=\max\{\eta>0\,:\,\kappa\,\eta+(2\,\beta+5\,\alpha)\,\eta^2-b<0\},$$

and the constants κ*,* α*,* β*,* b *are as given in Theorem 3.1.*

We can show the following result for the local convergence of method (1.5).

Proposition 4.2. *Let* $F\,:\,\mathcal{D}\subseteq\mathcal{X}\longrightarrow\mathcal{Y}$ *be an operator such that for every distinct points* x*,* y *and* z *in* $\mathcal{D}_0\subseteq\mathcal{D}$*, there exist a first and second order divided differences of* F $[x,y;F]$ *and* $[x,y,z;F]$ *respectively at these points.*

Assume:

$(\mathcal{H}0)$–$(\mathcal{H}2)$ *hold;*

$(\mathcal{H}3)^\star$ *There exist* $\kappa>0$*, and* $\overline{\kappa}>0$ *such that for all* $(x,y)\in\mathcal{D}_0^2$*, we have:*

$$\|\,[x,y;F]\,\|\leq\kappa,\qquad \|\,[x^\star,y;F]\,\|\leq\overline{\kappa}\qquad \text{and}\qquad M\ \kappa<1.$$

Then, for every constant C *given by (3.1), there exists* $\overline{\delta}>0$ *satisfying*

$$\overline{\delta}<\overline{\delta_0}=\min\left\{a;\ \frac{1}{C};\ \delta_2\right\},\tag{4.1}$$

where,

$$\delta_2=\max\{\eta>0\,:\,\overline{\kappa}\,\eta+(2\,\beta+5\,\alpha)\,\eta^2-b<0\}$$

such that the conclusions of Theorem 3.1 hold in $\mathbb{B}_{\overline{\delta}}(x^\star)$*.*

Remark 4.3. *In general,* κ *given in* $(\mathcal{H}3)$ *is not easy to compute. This is our motivation for introducing even weaker hypothesis* $(\mathcal{H}3)^\star$*.*

We clearly have:

$$\overline{\kappa}\leq\kappa,\tag{4.2}$$

$$\overline{\delta_0}\geq\delta_0,\tag{4.3}$$

and $\dfrac{\overline{\kappa}}{\kappa}$ *can be arbitrarily large [2]–[6].*

It follows using (4.3) that the radius of convergence is larger, and the convergence of method (1.5) is faster in Proposition 4.2 than the corresponding in Theorem 3.1. Hence, the claims made at the beginning of this section have been justified.

Conclusion. We provided a Secant–type method to approximate solutions for variational inclusions. This method extends the one related to the resolution of nonlinear equations [2]. Using some ideas given in [3] for nonlinear equations, we provided a local convergence analysis (see Theorem 3.1), with the following advantages:

1. Faster convergence to the solution than the corresponding ones in [10];
2. Our method does not need the evaluation of any Fréchet derivative;
3. Our method uses only divided differences operators of order one;
4. Our convergence result simplify the existing sufficient convergence conditions.

Using some observations (see Remarks 4.1 and 4.3), we provided under weaker hypotheses than used in Theorem 3.1, and less computational cost a local convergence analysis (see Proposition 4.2), with a larger radius of convergence, which allows a larger choice of initial guesses x_0 and x_1.

These observations are very important in computational mathematics [2], [3]. Finally, examples validating the results (see Examples 3.4 and 3.5) are given.

REFERENCES

[1] S. Amat, S. Busquier and V. F. Candela, *A class of quasi–Newton generalized Steffensen's methods on Banach spaces*, J. Comput. Appl. Math., 149 (2002), pp. 397–406.

[2] I. K. Argyros, *A Kantorovich–type analysis for a fast iterative method for solving nonlinear equations*, J. Math. Anal. Appl., 332 (2007), pp. 97–108.

[3] I. K. Argyros, *Convergence and applications of Newton–type iterations*, Springer Verlag Publ., New York, 2008.

[4] I. K. Argyros and S. Hilout, *Local convergence of Newton–like methods for generalized equations*, Appl. Math. Comput., 197 (2008), pp. 507–514.

[5] I. K. Argyros and S. Hilout, *Efficient methods for solving equations and variational inequalities*, Polimetrica Publisher, 2009.

[6] I. K. Argyros and S. Hilout, *Aspects of the computational theory for certain iterative methods*, Polimetrica Publisher, 2009.

[7] I. K. Argyros and S. Hilout, *Extending the applicability of a Secant–type method of order two using recurrent functions*, East Asian Math. J., to appear.

[8] J. P. Aubin and H. Frankowska, *Set–valued analysis*, Birkhäuser, Boston, 1990.

[9] A. L. Dontchev and W. W. Hager, *An inverse function theorem for set–valued maps*, Proc. Amer. Math. Soc., 121 (1994), pp. 481–489.

[10] S. Hilout, *Convergence analysis of a family of Steffensen–type methods for generalized equations*, J. Math. Anal. Appl., 339 (2008), pp. 753–761.

[11] B. S. Mordukhovich, *Stability theory for parametric generalized equations and variational inequalities via nonsmooth analysis*, Trans. Amer. Math. Soc., 343 (1994), pp. 609–657.

[12] B. S. Mordukhovich, *Variational analysis and generalized differentiation, I. Basic theory, II. Applications* (two volumes), Springer, Grundlehren Series, Vol. 330 and 331, 2006.

[13] S. M. Robinson, *Generalized equations and their solutions, part I: basic theory*, Math. Programming Study, 10 (1979), pp. 128–141.

[14] S. M. Robinson, *Generalized equations and their solutions, part II: applications to nonlinear programming*, Math. Programming Study, 19 (1982), pp. 200–221.

[15] R. T. Rockafellar, *Lipschitzian properties of multifunctions*, Nonlinear Analysis, 9 (1984), pp. 867–885.

[16] R. T. Rockafellar and R. J-B. Wets, *Variational analysis*, A Series of Comprehensives Studies in Mathematics, Springer, 317, 1998.