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REEH-SCHLIEDER THEOREM FOR ULTRAHYPERFUNCTIONAL WIGHTMAN THEORY*

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Abstract. It will be shown that the Reeh-Schlieder property holds for states of quantum fields for ultrahyperfunctional Wightman theory. As by product, it is shown that the Reeh-Schlieder property also holds for states of quantum fields on a non-commutative Minkowski space in the setting ultrahyperfunctional.

Key words. Reeh-Schlieder theorem, tempered ultrahyperfunctions, non-commutative theory.

AMS subject classifications. 46F12, 46F15, 46F20, 81T05

1. Introduction. In recent years a considerable effort has been made to clarify the structural aspects of non-commutative quantum field theories (NCQFT). The first paper on quantum field theory by exploring the non-commutativity of a space-time manifold was proposed a long time ago as a generalization of the phase space of quantum mechanics by Snyder [1], who used this idea to give a solution for the problem of ultraviolet divergences which had plagued quantum field theories from very beginning. Since then, due to the success of the renormalization theory, this subject was abandoned. Only recently the plan of investigating field theories on non-commutative space-times has been revived. In a fundamental paper Doplicher-Fredenhagen-Roberts [2] have shown that a model quantum space-time can be described by a non-commutative algebra whose commutation relations do imply uncertainty relations motivated by Heisenberg's uncertainty principle and by Einstein's theory. Later, in a different context, NCQFT appear directly related with the string theory [3], when was found that a non-commutative Yang-Mills theory induced by the Moyal product can be seen as a vestige, in the low-energy limit, of open strings in the presence of a constant magnetic field, $B_{\mu\nu}$ (for a review see [4, 5]).

From an axiomatic standpoint, a language has been developed which, in principle, ought to enable one to extend the Wightman axioms to this context [6]-[12]. However, the axiomatic approach to **local** quantum field theory built up by Streater-Wightman [13], Jost [14], Bogoliubov *et al.* [15], Haag [16] and others turned out to be too narrow for theoretical physicists, who are interested in handling situations involving a NCQFT. In particular, some very important evidences to expect that the traditional Wightman axioms must be somewhat modified for the setting of NCQFT are:

- NCQFT incorporate **nonlocal** effects, but in a controllable way. This is reminiscent of its stringy origin where the gravitational sector was decoupled but still left some traces through the non-commutativity.

- The existence of hard infrared singularities in the non-planar sector of the theory can destroy the **tempered** nature of the Wightman functions.

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- The commutation relations $[x_\mu, x_\nu] = i\theta_{\mu\nu}$ also imply uncertainty relations for space-time coordinates $\Delta x_\mu \Delta x_\nu \sim |\theta_{\mu\nu}|$, indicating that the notion of space-time point loses its meaning. Space-time points are replaced by cells of area of size $|\theta_{\mu\nu}|$. This suggests the existence of a finite lower limit to the possible resolution of distance. The **nonlocal** structure of NCQFT manifests itself in a indeterminacy of the interaction regions, which spread over a space-time domain whose size is determined by the existence of a **fundamental length** ℓ related to the scale of nonlocality $\ell \sim \sqrt{|\theta_{\mu\nu}|}$.

In Ref. [12] has been suggested that tempered ultrahyperfunctions corresponding to tubular radial domains are well adapted for their use in the axiomatic description of NCQFT. The space of tempered ultrahyperfunctions has the advantage of being representable by means of holomorphic functions. It is the dual space of the space of entire functions rapidly decreasing in any horizontal strip and generalizes the notion of hyperfunctions on $\mathbb{R}^n$, but *can not* be localized as hyperfunctions. In the framework of this approach, fundamental results, as the CPT and Spin-Statistics theorems, the Borchers class of a non-commutative field and the Reconstruction theorem, were proven [12].

In this article we prove that the Reeh-Schlieder-type property [17] holds for states of quantum fields for ultrahyperfunctional Wightman theory. Then, as by product, it is shown that the Reeh-Schlieder property also holds for states of quantum fields on a non-commutative Minkowski space in the setting ultrahyperfunctional. According to the standard arguments, the Reeh-Schlieder property concerns with the cyclicity and separability of the vacuum sector in the context of local quantum field theories in Minkowski space-time. However, it holds equally well for a quantum field theory on curved space-times [18]-[20], as well as for thermal states [21] as a direct consequence of locality, additivity and the relativistic KMS condition. Once one has the concept of fundamental length incorporated in NCQFT, a natural problem is to recognize whether the Reeh-Schlieder property can also be established for a non-commutative quantum field theory. We show that this is feasible since a crucial mathematical tool leading to the Reeh-Schlieder property in the case of NCQFT is a tempered ultrahyperfunction version of Edge of the Wedge theorem [36].

We outline the content of this contribution as follows. In Section 2, for the convenience of the reader, we shall present briefly some definitions and basic properties of the tempered ultrahyperfunction space of Sebastião e Silva [22, 23] and Hasumi [24] (we indicate the Refs. [22]-[36] for more details). Section 3 contains some needed results concerning with the proof of the Reeh-Schlieder theorem for ultrahyperfunctional Wightman Theory. In Section 4, we give Reeh-Schlieder theorem for ultrahyperfunctional Wightman Theory and for NCQFT. We consider for simplicity a theory with only one basic field, a neutral scalar field. Section 5 contains the final considerations.

2. Tempered ultrahyperfunctions: Some basic properties. Tempered ultrahyperfunctions were introduced in papers of Sebastião e Silva [22, 23] and Hasumi [24] (orginally called *tempered ultradistributions*) as the strong dual of the space of test functions of rapidly decreasing entire functions in any horizontal strip. While Sebastião e Silva [22] used extension procedures for the Fourier transform combined with holomorphic representations and considered the 1-dimensional case, Hasumi [24] used duality arguments in order to extend the notion of tempered ultrahyperfunctions for the case of n dimensions (see also [23, Section 11]). In a brief tour, Marimoto [26, 27] gave some more precise informations concerning the work of Hasumi.

More recently, the relation between the tempered ultrahyperfuntions and Schwartz distributions and some major results, as the kernel theorem and the Fourier-Laplace transform have been established by Brüning and Nagamachi in [33]. Earlier, some precisions on the Fourier-Laplace transform theorem for tempered ultrahyperfunctions were given by Carmichael [30] (see also [35, 36]), by considering the theorem in its simplest form, *i.e.*, the equivalence between support properties of a distribution in a closed convex cone and the holomorphy of its Fourier-Laplace transform in a suitable tube with conical basis. In this more general setting, which includes the results of Sebastião e Silva and Hasumi as special cases, Carmichael obtained new representations of tempered ultrahyperfunctions which were not considered by Sebastião e Silva [22, 23] or Hasumi [24]. In this section, we include the definitions and basic properties of the tempered ultrahyperfunction space which are the most important in applications to quantum field theory.

Next, we shall introduce briefly here some definitions and basic properties of the tempered ultrahyperfunction space of Sebastião e Silva [22, 23] and Hasumi [24] (we indicate the Refs. for more details). To begin with, we introduce the following multi-index notation. Let $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) be the real (resp. complex) n-space whose generic points are denoted by $x = (x_1, \ldots, x_n)$ (resp. $z = (z_1, \ldots, z_n)$), such that $x + y = (x_1 + y_1, \ldots, x_n + y_n)$, $\lambda x = (\lambda x_1, \ldots, \lambda x_n)$, $x \geq 0$ means $x_1 \geq 0, \ldots, x_n \geq 0$, $\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n$ and $\|x\|^2 = \langle x, x\rangle$. Moreover, we define $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_o^n$, where $\mathbb{N}_o$ is the set of non-negative integers, such that the length of α is the corresponding ℓ^1-norm $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha + \beta$ denotes $(\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$, $\alpha \geq \beta$ means $(\alpha_1 \geq \beta_1, \ldots, \alpha_n \geq \beta_n)$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, and

$$D^\alpha \varphi(x) = \frac{\partial^{|\alpha|}\varphi(x_1, \ldots, x_n)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \ .$$

Let Ω be a set in $\mathbb{R}^n$. Then we denote by Ω° the interior of Ω and by $\overline{\Omega}$ the closure of Ω. For $r > 0$, we denote by $B(x_o; r) = \big\{x \in \mathbb{R}^n \mid |x - x_o| < r\big\}$ a open ball and by $B[x_o; r] = \big\{x \in \mathbb{R}^n \mid |x - x_o| \leq r\big\}$ a closed ball, with center at point x_o and of radius r, respectively.

We consider two n-dimensional spaces – x-space and ξ-space – with the Fourier transform defined

$$\widehat{f}(\xi) = \mathscr{F}[f(x)](\xi) = \int_{\mathbb{R}^n} f(x) e^{i\langle \xi, x\rangle} d^n x \ ,$$

while the Fourier inversion formula is

$$f(x) = \mathscr{F}^{-1}[\widehat{f}(\xi)](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{-i\langle \xi, x\rangle} d^n \xi \ .$$

The variable ξ will always be taken real while x will also be complexified – when it is complex, it will be noted $z = x + iy$. The above formulas, in which we employ the symbolic "function notation," are to be understood in the sense of distribution theory.

We shall consider the function

$$h_K(\xi) = \sup_{x \in K} |\langle \xi, x\rangle| \ , \quad \xi \in \mathbb{R}^n \ ,$$

where K is a compact set in $\mathbb{R}^n$. One calls $h_K(\xi)$ the *supporting function* of K. We note that $h_K(\xi) < \infty$ for every $\xi \in \mathbb{R}^n$ since K is bounded. For sets $K = \big[-k, k\big]^n$,

$0 < k < \infty$, the supporting function $h_K(\xi)$ can be easily determined [24]:

$$h_K(\xi) = \sup_{x \in K} |\langle \xi, x \rangle| = k|\xi| \ , \quad \xi \in \mathbb{R}^n \ , \quad |\xi| = \sum_{i=1}^{n} |\xi_i| \ .$$

Let K be a convex compact subset of $\mathbb{R}^n$, then $H_b(\mathbb{R}^n; K)$ (b stands for bounded) defines the space of all functions $\in C^\infty(\mathbb{R}^n)$ such that $e^{h_K(\xi)} D^\alpha \varphi(\xi)$ is bounded in $\mathbb{R}^n$ for any multi-index α. One defines in $H_b(\mathbb{R}^n; K)$ seminorms

$$\|\varphi\|_{K,N} = \sup_{\substack{\xi \in \mathbb{R}^n \\ \alpha \leq N}} \left\{ e^{h_K(\xi)} |D^\alpha \varphi(\xi)| \right\} < \infty \ , \quad N = 0, 1, 2, \ldots \ . \tag{2.1}$$

If $K_1 \subset K_2$ are two compact convex sets, then $h_{K_1}(\xi) \leq h_{K_2}(\xi)$, and thus the canonical injection $H_b(\mathbb{R}^n; K_2) \hookrightarrow H_b(\mathbb{R}^n; K_1)$ is continuous. Let O be a convex open set of $\mathbb{R}^n$. To define the topology of $H(\mathbb{R}^n; O)$ it suffices to let K range over an increasing sequence of convex compact subsets $K_1, K_2, \ldots$ contained in O such that for each $i = 1, 2, \ldots$, $K_i \subset K_{i+1}^\circ$ and $O = \bigcup_{i=1}^\infty K_i$. Then the space $H(\mathbb{R}^n; O)$ is the projective limit of the spaces $H_b(\mathbb{R}^n; K)$ according to restriction mappings above, *i.e.*

$$H(\mathbb{R}^n; O) = \lim_{K \subset O} \text{proj} \ H_b(\mathbb{R}^n; K) \ , \tag{2.2}$$

where K runs through the convex compact sets contained in O.

THEOREM 2.1 ([24, 26, 33]). *The space $\mathscr{D}(\mathbb{R}^n)$ of all C^∞-functions on $\mathbb{R}^n$ with compact support is dense in $H(\mathbb{R}^n; K)$ and $H(\mathbb{R}^n; O)$. Moreover, the space $H(\mathbb{R}^n; \mathbb{R}^n)$ is dense in $H(\mathbb{R}^n; O)$ and $H(\mathbb{R}^m; \mathbb{R}^m) \otimes H(\mathbb{R}^n; \mathbb{R}^n)$ is dense in $H(\mathbb{R}^{m+n}; \mathbb{R}^{m+n})$.*

From Theorem 2.1 we have the following injections [26]:

$$H'(\mathbb{R}^n; K) \hookrightarrow H'(\mathbb{R}^n; \mathbb{R}^n) \hookrightarrow \mathscr{D}'(\mathbb{R}^n) \ ,$$

and

$$H'(\mathbb{R}^n; O) \hookrightarrow H'(\mathbb{R}^n; \mathbb{R}^n) \hookrightarrow \mathscr{D}'(\mathbb{R}^n) \ .$$

A distribution $V \in H'(\mathbb{R}^n; O)$ may be expressed as a finite order derivative of a continuous function of exponential growth

$$V = D_\xi^\gamma [e^{h_K(\xi)} g(\xi)] \ ,$$

where $g(\xi)$ is a bounded continuous function. For $V \in H'(\mathbb{R}^n; O)$ the following result is known:

LEMMA 2.2 ([26]). *A distribution $V \in \mathscr{D}'(\mathbb{R}^n)$ belongs to $H'(\mathbb{R}^n; O)$ if and only if there exists a multi-index γ, a convex compact set $K \subset O$ and a bounded continuous function $g(\xi)$ such that*

$$V = D_\xi^\gamma [e^{h_K(\xi)} g(\xi)] \ .$$

In the space $\mathbb{C}^n$ of n complex variables $z_i = x_i + iy_i$, $1 \leq i \leq n$, we denote by $T(\Omega) = \mathbb{R}^n + i\Omega \subset \mathbb{C}^n$ the tubular set of all points z, such that $y_i = \operatorname{Im} z_i$ belongs to the domain Ω, *i.e.*, Ω is a connected open set in $\mathbb{R}^n$ called the basis of the tube $T(\Omega)$. Let K be a convex compact subset of $\mathbb{R}^n$, then $\mathfrak{H}_b(T(K))$ defines the space of all continuous functions φ on $T(K)$ which are holomorphic in the interior $T(K^\circ)$ of $T(K)$ such that the estimate

$$|\varphi(z)| \leq \mathbf{M}_{K,N}(\varphi)(1+|z|)^{-N} \tag{2.3}$$

is valid. The best possible constants in (2.3) are given by a family of seminorms in $\mathfrak{H}_b(T(K))$

$$\|\varphi\|_{K,N} = \sup_{z\in T(K)} \left\{(1+|z|)^N|\varphi(z)|\right\} < \infty\ , \quad N=0,1,2,\dots\ . \tag{2.4}$$

If $K_1 \subset K_2$ are two convex compact sets, then $\mathfrak{H}_b(T(K_2)) \hookrightarrow \mathfrak{H}_b(T(K_1))$. Given that the spaces $\mathfrak{H}_b(T(K_i))$ are Fréchet spaces, the space $\mathfrak{H}(T(O))$ is characterized as a projective limit of Fréchet spaces

$$\mathfrak{H}(T(O)) = \lim_{K\subset O} \operatorname{proj}\ \mathfrak{H}_b(T(K))\ , \tag{2.5}$$

where K runs through the convex compact sets contained in O and the projective limit is taken following the restriction mappings above.

For any element $U \in \mathfrak{H}'$, its Fourier transform is defined to be a distribution V of exponential growth, such that the Parseval-type relation

$$\langle V,\varphi\rangle = \langle U,\psi\rangle\ , \quad \varphi\in H\ ,\ \psi = \mathscr{F}[\varphi] \in \mathfrak{H}\ , \tag{2.6}$$

holds. In the same way, the inverse Fourier transform of a distribution V of exponential growth is defined by the relation

$$\langle U,\psi\rangle = \langle V,\varphi\rangle\ , \quad \psi\in \mathfrak{H}\ ,\ \varphi = \mathscr{F}^{-1}[\psi] \in H\ . \tag{2.7}$$

Proposition 2.3 ([26]). *If $f \in H(\mathbb{R}^n;O)$, the Fourier transform of f belongs to the space $\mathfrak{H}(T(O))$, for any open convex non-empty set $O\subset\mathbb{R}^n$. By the dual Fourier transform $H'(\mathbb{R}^n;O)$ is topologically isomorphic with the space $\mathfrak{H}'(T(-O))$.*

Definition 2.4. *A tempered ultrahyperfunction is a continuous linear functional defined on the space of test functions $\mathfrak{H}(T(\mathbb{R}^n))$ of rapidly decreasing entire functions in any horizontal strip.*

The space of all tempered ultrahyperfunctions is denoted by $\mathscr{U}(\mathbb{R}^n)$. As a matter of fact, these objects are equivalence classes of holomorphic functions defined by a certain space of functions which are analytic in the 2^n octants in $\mathbb{C}^n$ and represent a natural generalization of the notion of hyperfunctions on $\mathbb{R}^n$, but are *non-localizable*. The space $\mathscr{U}(\mathbb{R}^n)$ is characterized in the following way [24]: Let $\mathscr{H}_{\boldsymbol{\omega}}$ be the space of all functions $f(z)$ such that (*i*) $f(z)$ is analytic for $\{z\in\mathbb{C}^n \mid |\operatorname{Im} z_1| > p, |\operatorname{Im} z_2| > p,\dots,|\operatorname{Im} z_n| > p\}$, (*ii*) $f(z)/z^p$ is bounded continuous in $\{z\in\mathbb{C}^n \mid |\operatorname{Im} z_1| \geqq p, |\operatorname{Im} z_2| \geqq p,\dots,|\operatorname{Im} z_n| \geqq p\}$, where $p=0,1,2,\dots$ depends on $f(z)$ and (*iii*) $f(z)$ is bounded by a power of z, $|f(z)|\leq \mathbf{M}(1+|z|)^N$, where $\mathbf{M}$ and N depend on $f(z)$. Define the *kernel* of the mapping $f:\mathfrak{H}(T(\mathbb{R}^n))\to\mathbb{C}$ by $\mathbf{\Pi}$, as the set of all z-dependent pseudo-polynomials, $z\in\mathbb{C}^n$ (a pseudo-polynomial is a function of z of the form $\sum_s z_j^s G(z_1,...,z_{j-1},z_{j+1},...,z_n)$, with $G(z_1,...,z_{j-1},z_{j+1},...,z_n)\in\mathscr{H}_{\boldsymbol{\omega}}$). Then, $f(z)\in\mathscr{H}_{\boldsymbol{\omega}}$ belongs to the kernel $\mathbf{\Pi}$ if and only if $\langle f(z),\psi(x)\rangle = 0$, with $\psi(x)\in\mathfrak{H}(T(\mathbb{R}^n))$ and $x=\operatorname{Re} z$. Consider the quotient space $\mathscr{U}=\mathscr{H}_{\boldsymbol{\omega}}/\mathbf{\Pi}$. The set $\mathscr{U}$ is the space of tempered ultrahyperfunctions. Thus, we have the

Theorem 2.5 (Hasumi [24], Proposition 5). *The space of tempered ultrahyperfunctions $\mathscr{U}$ is algebraically isomorphic to the space of generalized functions $\mathfrak{H}'$.*

Theorem 2.6 (Kernel theorem for tempered ultrahyperfunctions [33]). *Let M be a separately continuous multilinear functional on $[\mathfrak{H}(T(\mathbb{R}^4))]^n$. Then there is a unique functional $F \in \mathfrak{H}'(T(\mathbb{R}^{4n}))$, for all $f_i \in \mathfrak{H}(T(\mathbb{R}^4))$, $i = 1, \ldots, n$ such that $M(f_1, \ldots, f_n) = F(f_1 \otimes \cdots \otimes f_n)$.*

Theorem 2.7 ([26, 33]). *The space $\mathfrak{H}(T(\mathbb{R}^n))$ is dense in $\mathfrak{H}(T(O))$ and the space $\mathfrak{H}(T(\mathbb{R}^{m+n}))$ is dense in $\mathfrak{H}(T(O))$.*

3. Tempered ultrahyperfunctions corresponding to a proper convex cone. In order to prove the theorem Reeh-Schlieder theorem for NCQFT in terms of tempered ultrahyperfuncions, we shall recall some needed results taken from Refs. [35, 36]. We now shall define the space of holomorphic functions with which this paper is concerned. We start by introducing some terminology and simple facts concerning cones. An open set $C \subset \mathbb{R}^n$ is called a cone if C (unless specified otherwise, all cones will have their vertices at zero) is invariant under positive homoteties, *i.e.*, if for all $\lambda > 0$, $\lambda C \subset C$. A cone C is an open connected cone if C is an open connected set. Moreover, C is called convex if $C + C \subset C$ and *proper* if it contains no any straight line. A cone C' is called compact in C – we write $C' \Subset C$ – if the projection $\mathsf{pr}\overline{C}' \overset{\text{def}}{=} \overline{C}' \cap S^{n-1} \subset \mathsf{pr}C \overset{\text{def}}{=} C \cap S^{n-1}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Being given a cone C in y-space, we associate with C a closed convex cone C^* in ξ-space which is the set $C^* = \bigl\{\xi \in \mathbb{R}^n \mid \langle \xi, y\rangle \geq 0, \forall\, y \in C\bigr\}$. The cone C^* is called the *dual cone* of C. In the sequel, it will be sufficient to assume for our purposes that the open connected cone C in $\mathbb{R}^n$ is an open convex cone with vertex at the origin and proper. By $T(C)$ we will denote the set $\mathbb{R}^n + iC \subset \mathbb{C}^n$. If C is open and connected, $T(C)$ is called the tubular radial domain in $\mathbb{C}^n$, while if C is only open $T(C)$ is referred to as a tubular cone. In the former case we say that $f(z)$ has a boundary value $U = BV(f(z))$ in $\mathfrak{H}'$ as $y \to 0$, $y \in C$ or $y \in C' \Subset C$, respectively, if for all $\psi \in \mathfrak{H}$ the limit

$$\langle U, \psi\rangle = \lim_{\substack{y\to 0\\ y\in C \text{ or } C'}} \int_{\mathbb{R}^n} f(x+iy)\psi(x) d^n x \ ,$$

exists. We will deal with tubes defined as the set of all points $z \in \mathbb{C}^n$ such that

$$T(C) = \Bigl\{x + iy \in \mathbb{C}^n \mid x \in \mathbb{R}^n, y \in C, |y| < \delta\Bigr\} \ ,$$

where $\delta > 0$ is an arbitrary number.

An important example of tubular radial domain used in quantum field theory is the forward light-cone

$$V_+ = \Bigl\{z \in \mathbb{C}^n \mid \operatorname{Im} z_1 > \Bigl(\sum_{i=2}^{n} \operatorname{Im}^2 z_i\Bigr)^{\frac{1}{2}}, \operatorname{Im} z_1 > 0\Bigr\} \ .$$

Let C be an open convex cone, and let $C' \Subset C$. Let $B[0; r]$ denote a **closed** ball of the origin in $\mathbb{R}^n$ of radius r, where r is an arbitrary positive real number. Denote $T(C'; r) = \mathbb{R}^n + i\bigl(C' \setminus \bigl(C' \cap B[0; r]\bigr)\bigr)$. We are going to introduce a space of holomorphic functions which satisfy certain estimate according to Carmichael [29]. We want to consider the space consisting of holomorphic functions $f(z)$ such that

$$\bigl|f(z)\bigr| \leq \mathbf{M}(C')(1 + |z|)^N e^{h_{C^*}(y)} \ , \quad z \in T(C'; r) \ , \tag{3.1}$$

where $h_{C^*}(y) = \sup_{\xi \in C^*} \langle \xi, y \rangle$ is the supporting function of C^*, $\mathbf{M}(C')$ is a constant that depends on an arbitrary compact cone C' and N is a non-negative real number. The set of all functions $f(z)$ which are holomorphic in $T(C'; r)$ and satisfy the estimate (3.1) will be denoted by $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$.

Remark 1. *The space of functions $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ constitutes a generalization of the space $\mathfrak{A}_{\omega}^{i}$ of Sebastião e Silva [22] and the space $\mathcal{A}_{\omega}$ of Hasumi [24] to arbitrary tubular radial domains in $\mathbb{C}^n$.*

Lemma 3.1 ([29, 35]). *Let C be an open convex cone, and let $C' \Subset C$. Let $h(\xi) = e^{k|\xi|} g(\xi)$, $\xi \in \mathbb{R}^n$, be a function with support in C^*, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$. Let y be an arbitrary but fixed point of $C' \setminus \big(C' \cap B[0; r]\big)$. Then $e^{-\langle \xi, y \rangle} h(\xi) \in L^2$, as a function of $\xi \in \mathbb{R}^n$.*

Definition 3.2. *We denote by $H'_{C^*}(\mathbb{R}^n; O)$ the subspace of $H'(\mathbb{R}^n; O)$ of distributions of exponential growth with support in the cone C^*:*

$$H'_{C^*}(\mathbb{R}^n; O) = \Big\{ V \in H'(\mathbb{R}^n; O) \mid \operatorname{supp}(V) \subseteq C^* \Big\} . \tag{3.2}$$

Lemma 3.3 ([29, 35]). *Let C be an open convex cone, and let $C' \Subset C$. Let $V = D_{\xi}^{\gamma}[e^{h_K(\xi)} g(\xi)]$, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$ and $h_K(\xi) = k|\xi|$ for a convex compact set $K = \big[-k, k\big]^n$. Let $V \in H'_{C^*}(\mathbb{R}^n; O)$. Then $f(z) = (2\pi)^{-n} \big\langle V, e^{-i\langle \xi, z \rangle} \big\rangle$ is an element of $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$.*

We now shall define the main space of holomorphic functions with which this paper is concerned. Let C be a proper open convex cone, and let $C' \Subset C$. Let $B(0; r)$ denote an **open** ball of the origin in $\mathbb{R}^n$ of radius r, where r is an arbitrary positive real number. Denote $T(C'; r) = \mathbb{R}^n + i\big(C' \setminus \big(C' \cap B(0; r)\big)\big)$. Throughout this section, we consider functions $f(z)$ which are holomorphic in $T(C') = \mathbb{R}^n + iC'$ and which satisfy the estimate (3.1), with $B[0; r]$ replaced by $B(0; r)$. We denote this space by $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{*o}}$. We note that $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{*o}} \subset \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ for any open convex cone C. Put $\mathscr{U}_c = \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{*o}} / \mathbf{\Pi}$, that is, $\mathscr{U}_c$ is the quotient space of $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{*o}}$ by set of pseudo-polynomials $\mathbf{\Pi}$.

Definition 3.4. *The set $\mathscr{U}_c$ is the subspace of the tempered ultrahyperfunctions generated by $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{*o}}$ corresponding to a proper open convex cone $C \subset \mathbb{R}^n$.*

A useful property of tempered ultrahyperfunctions corresponding to a proper cone is the distributional boundary value theorem concerning analytic functions. The following theorem shows that functions in $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{*o}}$ have distributional boundary values in $\mathfrak{H}'$. Further, it shows that functions in $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{*o}}$ satisfy a strong boundedness property in $\mathfrak{H}'$.

Theorem 3.5 ([36]). *Let C be an open convex cone, and let $C' \Subset C$. Let $V = D_{\xi}^{\gamma}[e^{h_K(\xi)} g(\xi)]$, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$ and $h_K(\xi) = k|\xi|$ for a convex compact set $K = \big[-k, k\big]^n$. Let $V \in H'_{C^*}(\mathbb{R}^n; \mathbb{R}^n)$. Then*

(i) $f(z) = (2\pi)^{-n} \big\langle V, e^{-i\langle \xi, z \rangle} \big\rangle$ *is an element of* $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{*o}}$,

(ii) $\big\{ f(z) \mid y = \operatorname{Im} z \in C' \Subset C, |y| \le Q \big\}$ *is a strongly bounded set in* $\mathfrak{H}'$, *where Q is an arbitrarily but fixed positive real number,*

(iii) $f(z) \to \mathscr{F}^{-1}[V] \in \mathfrak{H}'$ *in the strong (and weak) topology of* $\mathfrak{H}'$ *as* $y = \operatorname{Im} z \to 0$, $y \in C' \Subset C$.

The functions $f(z) \in \mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$ can be recovered as the (inverse) Fourier-Laplace transform[1] of the constructed distribution $V \in H'_{C^*}(\mathbb{R}^n;\mathbb{R}^n)$. This result is a generalization of the Paley-Wiener-Schwartz theorem for the setting of tempered ultrahyperfunctions.

Theorem 3.6 ([36]). *Let $f(z) \in \mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$, where C is an open convex cone. Then the distribution $V \in H'_{C^*}(\mathbb{R}^n;O)$ has a uniquely determined inverse Fourier-Laplace transform $f(z) = (2\pi)^{-n}\langle V, e^{-i\langle \xi,z\rangle}\rangle$ which is holomorphic in $T(C')$ and satisfies the estimate (3.1), with $B[0;r]$ replaced by $B(0;r)$.*

We finish this section with two results proved in Ref. [36], which will be used in the applications of Section 4.

Theorem 3.7 (Tempered ultrahyperfunction version of edge of the wedge theorem). *et C be an open cone of the form $C = C_1 \cup C_2$, where each C_j, $j = 1,2$, is a proper open convex cone. Denote by $\boldsymbol{ch}(C)$ the convex hull of the cone C. Assume that the distributional boundary values of two holomorphic functions $f_j(z) \in \mathscr{H}_{\boldsymbol{c_j}}^{*\,\boldsymbol{o}}$ $(j = 1,2)$ agree, that is, $U = BV(f_1(z)) = BV(f_2(z))$, where $U \in \mathfrak{H}'(T(O))$ in accordance with the Theorem 3.5. Then there exists $F(z) \in \mathscr{H}_{\boldsymbol{ch(C)}}^{\boldsymbol{o}}$ such that $F(z) = f_j(z)$ on the domain of definition of each $f_j(z)$, $j = 1,2$.*

Theorem 3.8. *Let C be some open convex cone. Let $f(z) \in \mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$. If the distributional boundary value $BV(f(z))$ of $f(z)$ in the sense of tempered ultrahyperfunctions vanishes, then the function $f(z)$ itself vanishes.*

4. Reeh-Schlieder-Type theorem for ultrahyperfunctional quantum fields.

Definition 4.1. *Assume we are given a Hibert space $\mathscr{H}$. According to [33, Proposition 4.1], we define the space of $\mathscr{H}$ valued tempered ultrahyperfunctions to be the set of all continuous linear mapping from $\mathfrak{H}(T(\mathbb{R}^{4m})$ to $\mathscr{H}$.*

Theorem 4.2. *Let Φ be a field operator and Ω_o be the vacuum state. For any non-empty open set $X \subset \mathbb{R}^4$ the set of vectors of the form*

$$\Big\{\Phi(f_1)\cdots\Phi(f_m)\Omega_o \mid \textit{with } f_j(x) \in \mathfrak{H}(T(\mathbb{R}^4)) \textit{ and } x = \operatorname{Re} z \in X,\ m \in \mathbb{N}\Big\}.$$

is dense in $\mathscr{H}$.

For our proof of Theorem 4.2, we shall consider analytic functionals in $\mathfrak{H}'(T(\mathbb{R}^4))$ carried by the real space. In this case, every function $f(z) \in \mathscr{H}_{\boldsymbol{c}}^{*\,\boldsymbol{o}}$, which for each $y \in C'$ as a function of $x = \operatorname{Re} z$ belongs to $\mathfrak{H}'(T(\mathbb{R}^4))$, is a continuous linear functional on the space of **restrictions** to $\mathbb{R}^4$ of functions in $\mathfrak{H}(T(\mathbb{R}^4))$. Then, according to Theorem 3.5(*iii*), $U = BV(f(z))$ the distributional boundary value of $f(z)$ is an element of $\mathfrak{H}'(T(\mathbb{R}^4))$ carried by $\mathbb{R}^4$.

Proof of Theorem 4.2. Denote by D_o the minimal common invariant domain, which is assumed to be dense, of the field operators in the Hilbert space $\mathscr{H}$ of states, *i.e.*, the vector subspace of $\mathscr{H}$ that is spanned by the vacuum state Ω_o and by the set of vectors $\Phi(f_1)\cdots\Phi(f_m)\Omega_o$. Let $\Psi \in \mathscr{H}$ be orthogonal to all vectors of the form $\Phi(f_1)\cdots\Phi(f_m)\Omega_o \in D_o$. Then, it is required to prove that Ψ is identically zero.

[1] The convention of signs in the Fourier transform which is used here one leads us to consider the inverse Fourier-Laplace transform.

According to Ref. [33],

$$\Big[\mathfrak{H}(T(\mathbb{R}^4))\Big]^m \ni (f_1,\ldots,f_m) \to \langle \Psi, \Phi(f_1)\cdots\Phi(f_m)\Omega_o\rangle$$

is a multilinear functional in each $f_j \in \mathfrak{H}(T(\mathbb{R}^4))$ separately with all the others $f_i \in \mathfrak{H}(T(\mathbb{R}^4))$, $i \neq j$, kept fixed. However, then the Theorem 2.6 implies that the functional $\langle \Psi, \Phi(f_1)\cdots\Phi(f_m)\Omega_o\rangle$ has a uniquely determined extension to a tempered ultrahyperfunction $\boldsymbol{F}_\Psi \in \mathscr{U}_c(\mathbb{R}^{4m})$ such that

$$\boldsymbol{F}_\Psi(f^{(m)}) = \int d^4z_1\cdots d^4z_m\ \mathfrak{F}^{(1)}_\Psi(z_1,\ldots,z_m) f^{(m)}(x_1,\ldots,x_m)\ , \tag{4.1}$$

for every $\Psi \in \mathscr{H}$, where $\mathfrak{F}^{(1)}_\Psi(z_1,\ldots,z_m) = \langle \Psi, \Phi(z_1)\cdots\Phi(z_m)\Omega_o\rangle$. According to the arguments of Section IV.C of Ref. [33], the Fourier transform $\widehat{\boldsymbol{F}_\Psi}$ vanishes unless each four-momentum variable lies in the physical spectrum. Hence, we can apply Theorem 3.6 to conclude that $\boldsymbol{F}_\Psi$ is holomorphic in the set $T(V'_+) = \mathbb{R}^{4m} + iV'_+$, with $V'_+ \Subset V_+$. Then, by Theorem 3.5, we have that $\boldsymbol{F}_\Psi|_X$ is the boundary value of $\mathfrak{F}^{(1)}_\Psi$ when $V'_+ \ni y_1 \to 0$, $V'_+ \ni (y_j - y_{j-1}) \to 0$, $j = 2,\ldots,m$. Furthermore the function $\mathfrak{F}^{(2)}_\Psi(z_1,\ldots,z_m) = \overline{\mathfrak{F}^{(1)}_\Psi(\bar{z}_1,\ldots,\bar{z}_m)}$ is holomorphic in the set $T(V'_-) = \mathbb{R}^{4m} + iV'_-$, with $V'_- = -V'_+$ and $\boldsymbol{F}_\Psi|_X$ is the boundary value of $\mathfrak{F}^{(2)}_\Psi$ when $V'_- \ni y_1 \to 0$, $V'_- \ni (y_j - y_{j-1}) \to 0$, $j = 2,\ldots,m$. By hypothesis, $\boldsymbol{F}_\Psi|_X$ vanishes on a non-empty open real set $x_1,\ldots,x_m \in X^m$, since D_o spans the Hibert space $\mathscr{H}$. Therefore we can apply the Edge of the Wedge Theorem 3.7 in order to show that $\mathfrak{F}^{(1)}_\Psi$ and $\mathfrak{F}^{(2)}_\Psi$ have a common analytic continuation $\mathfrak{F}_\Psi$. Since $\mathfrak{F}_\Psi$ vanishes on X^m, it vanishes together with $\mathfrak{F}^{(1)}_\Psi$ identically by Theorem 3.8. This shows that Ψ is even orthogonal to the set $\big\{\Phi(f_1)\cdots\Phi(f_m)\Omega_o \mid f_j(x) \in \mathfrak{H}(T(\mathbb{R}^4)), j = 1,\ldots,m\big\}$. We conclude that $\Psi \in D_o^\perp = \{0\}$. This completes the proof of theorem. □

In what follows, we give Reeh-Schlieder theorem for NCQFT in the setting of tempered ultrahyperfunctions. When referring to NCQFT one should have in mind the deformation of the ordinary product of fields. In terms of complex variables, this deformation is performed through the star product extended for noncoinciding points via the functorial relation

$$\varphi(z_1)\star\cdots\star\varphi(z_n) = \prod_{i<j}\exp\left(\frac{1}{2}\theta^{\mu\nu}\frac{\partial}{\partial z_i^\mu}\frac{\partial}{\partial \bar{z}_j^\nu}\right)\varphi(z_1)\cdots\varphi(z_n)\ , \tag{4.2}$$

where the deformation parameter $\theta_{\mu\nu}$ is an antisymmetric tensor, assumed to be a constant antisymmetric matrix of length dimension 2. This parameter is responsible by breaking of Lorentz invariance, a basic feature of non-commutative theories.

Remark 2. *The functorial relation (4.2) is actually the Moyal-Voros product, that is more precisely what is used in a holomorphic setting.*

For coinciding points $z_1 = z_2 = \cdots = z_n$ the product (4.2) becomes identical to the multiple Moyal $\star$-product. We consider NCQFT in the sense of a field theory on a non-commutative space-time encoded by a Moyal product. In this point, a few comments about the NCQFT are in order. Generalizing the Wightman axioms to NCQFT is not as simple, especially the Poincaré symmetry. It is well known that due to the constant matrix θ, the Poincaré symmetry is not preserved in NCQFT. Furthemore, the existence of hard infrared singularities in the non-planar sector of the theory can destroy the *tempered* nature of the Wightman functions. And more, how can the local commutativity condition be described in a field theory with a fundamental length? The analysis in Ref. [12] has shown that the sequence of vacuum expectation values of a NCQFT in terms of tempered ultrahyperfunctions satisfies a number of specific properties, which actually characterize a NCQFT in terms of tempered ultrahyperfunctions. We summarize these below (for details see [12]):

P₁ $\mathfrak{W}_0^\star = 1$, $\mathfrak{W}_m^\star \in \mathfrak{H}'(T(\mathbb{R}^{4m}))$ for $n \geq 1$, and $\mathfrak{W}_m^\star(f^*) = \overline{\mathfrak{W}_m^\star(f)}$, for all $f \in \mathfrak{H}(T(\mathbb{R}^{4m}))$, where $\mathfrak{W}_m^\star(z_1,\dots,z_m) \stackrel{\text{def}}{=} \langle \Omega_o \mid \Phi(z_1)\star\cdots\star\Phi(z_m) \mid \Omega_o\rangle$ and $f^*(z_1,\dots,z_m) = \overline{f(\bar{z}_1,\dots,\bar{z}_m)}$.

P₂ The Wightman functionals $\mathfrak{W}_m^\star$ are invariant under the *twisted* Poincaré group

P₃ Spectral condition. Since the Fourier transformation of tempered ultrahyperfunctions are distributions, the spectral condition is not so much different from that of Schwartz distributions. Thus, for every $m \in \mathbb{N}$, there is $\widehat{\mathfrak{W}}_m^\star \in H'_{V^*}(\mathbb{R}^{4m},\mathbb{R}^{4m})$ [33], where

$$H'_{V^*}(\mathbb{R}^{4m},\mathbb{R}^{4m}) = \left\{ V \in H'(\mathbb{R}^{4m},\mathbb{R}^{4m}) \mid \operatorname{supp}(\widehat{\mathfrak{W}}_m^\star) \subset V^* \right\}, \tag{4.3}$$

with V^* being the properly convex cone defined by

$$\left\{ (p_1,\dots,p_m) \in \mathbb{R}^{4m} \;\Big|\; \sum_{j=1}^m p_j = 0,\ \sum_{j=1}^k p_j \in \overline{V}_+,\ k = 1,\dots,m-1 \right\},$$

where $\overline{V}_+ = \{(p^0,\boldsymbol{p}) \in \mathbb{R}^4 \mid p^2 \geq 0, p^0 \geq 0\}$ is the closed forward light cone.

P₄ Extended local commutativity condition.

P₅ For any finite set $f_o, f_1, \dots, f_N$ of test functions such that $f_o \in \mathbb{C}$, $f_j \in \mathfrak{H}(T(\mathbb{R}^{4j}))$ for $1 \leq j \leq N$, one has

$$\sum_{k,\ell=0}^N \mathfrak{W}_{k+\ell}^\star(f_k^* \otimes f_\ell) \geq 0 .$$

Remark 3. *It should be mentioned that the test function space used is closed under the $\star$-product and that the tempered ultrahyperfunctions $\mathfrak{W}_m^\star \in \mathfrak{H}'(T(\mathbb{R}^{4m}))$ have been commonly called* non-commutative *Wightman functions in [12].*

Theorem 4.3 (Reeh-Schlieder Theorem for NCQFT). *Suppose that the hypotheses of Theorem 4.2 hold except that instead of vectors of the form $\Phi(f_1)\cdots\Phi(f_m)\Omega_o$, we have vectors of the form $\Phi(f_1)\star\cdots\star\Phi(f_m)\Omega_o$. Then the conclusions of Theorem 4.2 again hold.*

Proof. For this purpose, we consider the functional

$$\langle \Psi, \Phi(f_1)\star\cdots\star\Phi(f_m)\Omega_o\rangle = \prod_{i<j} \exp\left(\frac{1}{2}\theta^{\mu\nu}\frac{\partial}{\partial z_i^\mu}\frac{\partial}{\partial \bar{z}_j^\nu}\right)\langle \Psi, \Phi(f_1)\cdots\Phi(f_m)\Omega_o\rangle$$

One first notes that the formula above simplifies considerably the proof of theorem in the case of NCQFT in terms of tempered ultrahyperfunctions, since $\langle\Psi, \Phi(f_1)\cdots\Phi(f_m)\Omega_o\rangle$ is representable by means of holomorphic functions (the holomorphy properties of the functions under consideration are discussed in Ref. [12]). Thus the star product coincides with the regular product of fields

$$\langle\Psi, \Phi(f_1)\star\cdots\star\Phi(f_m)\Omega_o\rangle = \langle\Psi, \Phi(f_1)\cdots\Phi(f_m)\Omega_o\rangle\ . \tag{4.4}$$

This means that a NCQFT in terms of tempered ultrahyperfunctions is unchanged by the deformation of the product. Therefore, the conclusions of Theorem 4.2 again hold. □

Remark 4. *In [10] the Wightman functions were written as follows:*

$$\mathfrak{W}^{\tilde{\star}}_m(z_1,\ldots,z_m) \stackrel{\text{def}}{=} \langle\Omega_o \mid \Phi(z_1)\tilde{\star}\cdots\tilde{\star}\,\Phi(z_m) \mid \Omega_o\rangle\ ,$$

where the meaning of $\tilde{\star}$ depends on the considered case. In particular, if $\tilde{\star} = 1$, we obtain the standard form $\mathfrak{W}_m(z_1,\ldots,z_m) = \langle\Omega_o \mid \Phi(z_1)\cdots\Phi(z_m) \mid \Omega_o\rangle$ adopted in [6]. On the other hand, if $\tilde{\star} = \star$, this choice corresponds to the Wightman functions introduced in [7]. In this case, the non-commutativity is manifested not only at coincident points but also in their neighborhood. The Equation (4.4) reflects the fact that the axiomatic approach to the NCQFT in terms of tempered ultrahyperfunctions is independent of the concrete type of the $\tilde{\star}$-product (similar conclusion was obtained in [10]).

5. Final Considerations. In the present paper, we consider a quantum field theory in terms of the tempered ultrahyperfunctions of Sebastião e Silva corresponding to a convex cone, within the framework formulated by Wightman. Tempered ultrahyperfunctions are representable by means of holomorphic functions. As is well known there are certain advantages to be gained from the representation of distributions in terms of holomorphic functions. In particular, for non-commutative theories the product of fields involving the $\star$-product has the same form as the ordinary product of fields (effects of non-commutativity are nontrivial in the formula with real variables). In light of this result, we show that the Reeh-Schlieder property, proved in the framework of local QFT, also holds for states of quantum fields on non-commutative space-times.

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