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# PARTIAL REGULARITY OF WEAK SOLUTIONS TO MAXWELL'S EQUATIONS IN A QUASI-STATIC ELECTROMAGNETIC FIELD*

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*Delicated to Professor Neil Trudinger on the occasion of his 65th birthday*

**Abstract.** We study Maxwell's equations in a quasi-static electromagnetic field, where the electrical conductivity of the material depends on the temperature. By establishing the reverse Hölder inequality, we prove partial regularity of weak solutions to the non-linear elliptic system and the non-linear parabolic system in a quasi-static electromagnetic field.

**Key words.** Partial regularity, elliptic systems, parabolic systems.

**AMS subject classifications.** 35J45, 35J60, 58E20

**1. Introduction.** In this paper, let $\Omega$ be a domain in $\mathbb{R}^n$ with $n \geq 3$, and let $u(x)$ and $H^i(x)$ for $i = 1, ..., n$ be scalar functions defined on $\Omega$. For any positive integer $k$, let $\Lambda_k(\Omega)$ denote the space of $k$-forms on $\Omega$. We have the usual exterior derivative $d$ of forms with $d : \Lambda_k(\Omega) \to \Lambda_{k+1}(\Omega)$. Consider a 1-form $H = \sum_{i=1}^n H^i(x)dx_i$, which may be regarded as a connection in differential geometry. We define the curvature $F$ of the connection $H$ by

$$F = dH = \sum_{i<j} F^{ij} dx_i \wedge dx_j,$$

where $F^{ij} = \frac{\partial H^j}{\partial x_i} - \frac{\partial H^i}{\partial x_j}$ (e.g. [9]).

Let $*$ be the Hodge star linear operator which assigns to each $k$-form on $\Omega$ an $(n-k)$-form and which satisfies

$$** = (-1)^{k(n-k)}.$$

We have a product $\langle \cdot, \cdot \rangle$ in the $k$-form space $\Lambda_k(\Omega)$

$$\langle a, b\rangle dx_1 \wedge ... \wedge dx_n = a \wedge *b, \quad |a|^2 = \langle a, a\rangle$$

for all $a, b \in \Lambda_k(\Omega)$ (e.g. [15]).

By definition, we have

$$|H|^2 = \langle H, H\rangle = \sum_{i=1}^n (H^i)^2, \quad |dH|^2 = \langle dH, dH\rangle = \frac{1}{2}\sum_{i,j=1}^n (F^{ij})^2.$$

Let $d^*$ be the adjoint operator of $d$ with $d^* = (-1)^{n+nk+1} * d* : \Lambda_k(\Omega) \to \Lambda_{k-1}(\Omega)$ and

$$\int_\Omega \langle da, b\rangle dx = \int_\Omega \langle a, d^*b\rangle dx$$

for $a \in \Lambda_k(\Omega), b \in \Lambda_{k+1}(\Omega)$, where $b$ or $a$ has compact support inside of $\Omega$.

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We consider the following system

$$d^*[\sigma(u)dH] = 0 \quad \text{in } \Omega \tag{1.1}$$

$$-\triangle u = \sigma(u)|dH|^2 \quad \text{in } \Omega \tag{1.2}$$

where $\sigma$ is a positive function defined on $\mathbb{R}$.

We say that a pair $(u, H)$ is a weak solution to the system (1.1)-(1.2) if $u \in W^{1,q}(\Omega)$ for some $q \in (1, \frac{n}{n-1})$ and $H \in W^{1,2}(\Omega; \mathbb{R}^n)$, and the pair $(u, H)$ satisfies the following:

$$\int_\Omega \langle \sigma(u)dH, d\phi \rangle \, dx = 0,$$

$$\int_\Omega \nabla u \cdot \nabla \psi \, dx = \int_\Omega \sigma(u)|dH|^2 \psi \, dx$$

for all $\phi := \sum_{i=1}^n \phi^i(x)dx_i$ for $i = 1, ..., n$, where $\phi^i \in C_0^2(\Omega; \mathbb{R})$ and $\psi \in C_0^2(\Omega; \mathbb{R})$.

Assumption (S). *$\sigma(u)$ is uniformly Hölder continuous in $\mathbb{R}$ and there exist two constants $\sigma_1$ and $\sigma_1$ such that*

$$0 < \sigma_1 \le \sigma(u) \le \sigma_2.$$

Uniform Hölder continuity above can be replaced by the assumption of Hölder continuity of $\sigma(u)$ (see [1]). Without loss of generality, we assume that Assumption (S) holds throughout this paper.

In this paper, we prove the partial regularity of the above weak solution to the system (1.1)-(1.2) in the following:

Theorem A. *Let a pair $(u, H)$ be a weak solution to the system (1.1)-(1.2) with $u \in W^{1,q}(\Omega, \mathbb{R})$ for some $q \in (1, \frac{n}{n-1})$, $H \in W^{1,2}(\Omega; \mathbb{R}^n)$ and $d^*H(x) = 0$ for a.e. $x \in \Omega$. Then there exists an open subset $\Omega_0$ of $\Omega$ such that the solution $(u, H)$ is $C^{1,\alpha}$ locally in $\Omega_0$, and $\mathcal{H}^{n-q_1}(\Omega \backslash \Omega_0) = 0$ for some $q_1 > \frac{n}{n-1}$, where $\mathcal{H}^{n-q_1}$ denotes the $(n - q_1)$-dimensional Hausdorff measure.*

The system (1.1)-(1.2) is not elliptic since it is invariant under the gauge transformation $(u, H) \to (u, H + \nabla\xi)$ for all $\xi \in W^{2,2}(\Omega)$. By a gauge transformation, one can fix a gauge satisfying

$$d^*H = \operatorname{div} H = \sum_i \frac{\partial H^i}{\partial x_i} = 0.$$

The system (1.1)-(1.2) with $d^*H = 0$ on $\Omega$ is a quasi-linear elliptic system which has a natural growth structure. When $n = 3$, Yin in [13], [14] proved the existence of weak solutions $(u, H)$ to (1.1)-(1.2) with $u \in W^{1,q}(\Omega, \mathbb{R})$, $q \in (1, \frac{n}{n-1})$, $H \in W^{1,2}(\Omega; \mathbb{R}^3)$ and $\operatorname{div} H = 0$ in $\Omega$. Moreover, he also proved the regularity of continuous weak solutions to (1.1)-(1.2). However, he also pointed out that the continuity of the weak solution is unknown. For $n > 3$, we have a similar existence result for weak solutions to the system (1.1)-(1.2) using the same proof as in [13] and [14]. Generally, weak solutions of non-linear elliptic systems may have singularities by De Giorgi's example and Giusti-Miranda's example (see [8]). Partial regularity theory for weak solutions of non-linear elliptic systems began around 1968 by Morrey, Giusti-Miranda (e.g. see [1] or [2]). The reader may refer to an excellent book [1] on the further development of the general theory of partial regularity. For many cases of quasi-linear elliptic systems which have natural growth, e.g. harmonic map equations, one usually assumes that weak solutions to (1.1)-(1.2) are in the space $W^{1,2}\cap L^\infty(\Omega)$. From the existence result for weak solutions, we only know $u \in W^{1,q}(\Omega)$ with $q \in (1, \frac{n}{n-1})$, we do not know if $u$ in $W^{1,2}\cap L^\infty(\Omega)$, so the general theory of non-linear elliptic systems in [1] does not apply to our system (1.1)-(1.2). Recently, the partial regularity of non-linear elliptic systems involving forms and maps was studied in [4].

When $n = 3$, the system (1.1)-(1.2) arises from approximating Maxwell's equations in a quasi-stationary electromagnetic field with non-ferromagnetic bodies (e.g. [11]). In the study of the penetration of a magnetic field in a medium, the electrical resistance strongly depends on the temperature. By taking the temperature effect into consideration, the classical Maxwell system in the quasi-static electromagnetic field can be reduced to the following system (see [11], [13] and [14]):

$$\partial_t H + \nabla\times[\sigma(u)\nabla\times H] = 0; \quad (x,t)\in\Omega\times(0,T) \tag{1.3}$$

$$\partial_t u - \triangle u = \sigma(u)|\nabla\times H|^2; \quad (x,t)\in\Omega\times(0,T) \tag{1.4}$$

$$\mathrm{div}H = 0; \qquad (x,t)\in\Omega\times(0,T), \tag{1.5}$$

where $H = (H^1(x,t), H^2(x,t), H^3(x,t))$ and $u(x,t)$ represent the strength of the magnetic field and the temperature respectively, and $\sigma^{-1}(u)$ denotes the electrical conductivity of the material. By changing the notation from vector functions to forms, we can consider the vector function $H$ and its 'curl' $\nabla\times\tilde{H}$ as a 1-form $H(x)$ and its curvature $dH$ respectively.

Now we generalize the Maxwell systems (1.3)-(1.5) to higher dimensional cases; i.e $n > 3$. Let $u = u(x,t)$ and $H = \sum_i H^i(x,t)dx_i$ be a function and a 1-form on $Q_T = \Omega\times[0,T]$ respectively. Then we consider the following system

$$\partial_t H = -d^*[\sigma(u)dH]; \quad \text{in } Q_T \tag{1.6}$$

$$\partial_t u = \triangle u + \sigma(u)|dH|^2; \text{in } Q_T, \tag{1.7}$$

with $d^*H(x,t) = 0$ for a. e. $(x,t)\in Q_T$, where $\sigma$ is a positive function satisfying Assumption (S). The weak solution in $V_q^{1,0}(Q_T)$ to system (1.6)-(1.7) is defined in Section 4.

The second main result of this paper is the following:

THEOREM B. *Let $(u,H)$ be a weak solution to equations (1.6) and (1.7) with $u\in V_q^{1,0}(Q_T)$ for some $q\in(1,\frac{n+2}{n+1})$, $H^i\in V_2^{1,0}(Q_T;\mathbb{R}^n)$ for $i = 1,...,n$ and $d^*H = 0$ for a. e. $(x,t)\in Q_T$. Then when $n\geq 3$, there exists an open subset $\tilde{Q}$ of $Q_T$ such that the solution $(u,H)$ is $C^{1,\alpha}$ in $\tilde{Q}$, and $\mathcal{H}^{n+2-q_3}(Q_T\backslash\tilde{Q}) = 0$ with $q_3 = \frac{(n+2)p}{n+2-2p}$ for some $p > 2$, where $\mathcal{H}^{n+2-q_3}$ denotes the Hausdorff measure.*

The paper is organized as follows. In Section 2, we prove Caccioppoli's inequality for $H$ (Lemma 1) and then obtain $L^p$-estimates (Theorem 3) by applying the reverse Hölder inequality. In Section 3, we prove partial regularity for system (1.1)-(1.2) by applying Theorem 3. Finally, in Section 4, we establish partial regularity of weak solutions for the parabolic problem (1.6)-(1.7)using the analogous techniques as in the elliptic case.

**2. Reverse Hölder inequalities and $L^p$-estimates.** In this section, we establish the Caccioppoli inequality for $H$ and the $L^p$-estimate.

Let $x_0$ be a point in $\Omega$ with $B_R(x_0) \subset \Omega$. For any function $f$, any 1-form $H$ and any measurable set $A$, denote

$$\fint_A f\,dx = \frac{1}{|A|}\int_A f\,dx, \quad f_{x_0,R} = \fint_{B_R(x_0)} f\,dx, \quad (H)_{x_0,R} = H^i_{x_0,R}dx_i.$$

Lemma 1. *(Caccioppoli's inequality for $H$) Assume that $(u,H)$ is a weak solution of (1.1)-(1.2) with $u \in W^{1,q}$, $H \in W^{1,2}$ and $d^*H(x) = 0$ for a.e. $x \in \Omega$. Then there exists a constant $C$ such that for any $x_0 \in \Omega$ and $\rho, R$ with $\rho < R$ with $B_R(x_0) \subset \Omega$,*

$$\int_{B_\rho(x_0)} |\nabla H|^2\,dx \le \frac{C}{(R-\rho)^2}\int_{B_R(x_0)} |H - (H)_{x_0,R}|^2\,dx.$$

*Proof.* Without loss of generality, we assume $x_0 = 0$. Let $\phi$ be a smooth cut-off function with $\phi = 1$ on $B_\rho$, $\phi = 0$ outside $B_R$, $|\phi| \le 1$ on $B_R \backslash B_\rho$, and $|\nabla\phi| \le \frac{C}{R-\rho}$ on $B_R\backslash B_\rho$. Choosing $\phi^2(H - H_{0,R})$ as a test function in (1.1), we have

$$\int_{B_R} \langle d^*[\sigma(u)dH], \phi^2(H - H_{0,R})\rangle\,dx = 0.$$

By Stokes' formula, we obtain

$$\begin{aligned}\int_{B_R} \sigma(u)|dH|^2\phi^2\,dx &= -2\int_{B_R}\langle\sigma(u)dH, \phi d\phi \wedge (H - H_{0,R})\rangle\,dx\\ &\le \varepsilon\int_{B_R}|dH|^2\phi^2\,dx + \frac{C}{(R-\rho)^2}\int_{B_R}|H - H_{0,R}|^2\,dx.\end{aligned}$$

Choosing $\varepsilon$ to be sufficiently small, we have

$$\int_{B_R} |dH|^2\phi^2\,dx \le \frac{C}{(R-\rho)^2}\int_{B_R}|H - H_{0,R}|^2\,dx \tag{2.1}$$

We note

$$\begin{aligned}|dH|^2 &= \frac{1}{2}\sum_{i,j=1}^n \left(\frac{\partial H^i}{\partial x_j} - \frac{\partial H^j}{\partial x_i}\right)^2\\ &= |\nabla H|^2 - \sum_{i,j=1}^n \frac{\partial H^i}{\partial x_j}\frac{\partial H^j}{\partial x_i}.\end{aligned}$$

Since $H \in W^{1,2}$, we can approximate it by smooth functions $H_k$ in $W^{1,2}$ for

$k = 1, 2, 3, \cdots$. By Stokes' formula, we have

$$\int_{B_R} |dH_k|^2 \phi^2 \, dx = \int_{B_R} |\nabla H_k|^2 \phi^2 \, dx - \sum_{i,j=1}^{n} \int_{B_R} \frac{\partial H_k^i}{\partial x_j} \frac{\partial H_k^j}{\partial x_i} \phi^2 \, dx$$

$$= \int_{B_R} |\nabla H_k|^2 \phi^2 \, dx + 2 \sum_{i,j=1}^{n} \int_{B_R} \frac{\partial H_k^i}{\partial x_j} \phi \frac{\partial \phi}{\partial x_i} [H_k^j - (H_k^j)_{0,R}] \, dx$$

$$+ \int_{B_R} \sum_{i,j=1}^{n} \phi^2 [H_k^j - (H_k^j)_{0,R}] \frac{\partial^2 H_k^i}{\partial x_j \partial x_i} \, dx$$

$$= \int_{B_R} |\nabla H_k|^2 \phi^2 \, dx + 2 \sum_{i,j=1}^{n} \int_{B_R} \frac{\partial H_k^i}{\partial x_j} \phi \frac{\partial \phi}{\partial x_i} [H_k^j - (H_k^j)_{0,R}] \, dx$$

$$- \int_{B_R} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left( \phi^2 [H_k^j - (H_k^j)_{0,R}] \right) \frac{\partial H_k^i}{\partial x_i} \, dx,$$

where we note $\frac{\partial^2 H_k^i}{\partial x_j \partial x_i} = \frac{\partial^2 H_k^i}{\partial x_i \partial x_j}$. As $k \to \infty$, it follows from using $\sum_i \frac{\partial H^i}{\partial x_i} = 0$ that

$$\int_{B_R} |dH|^2 \phi^2 \, dx = \int_{B_R} |\nabla H|^2 \phi^2 \, dx + 2 \sum_{i,j=1}^{n} \int_{B_R} \frac{\partial H^i}{\partial x_j} \phi \frac{\partial \phi}{\partial x_i} [H^j - (H^j)_{0,R}] \, dx$$

Therefore

$$\int_{B_R} |\nabla H|^2 \phi^2 \, dx \le \int_{B_R} |dH|^2 \phi^2 \, dx + \frac{1}{2} \int_{B_R} |\nabla H|^2 \phi^2 \, dx$$
$$+ \frac{C}{(R-\rho)^2} \int_{B_R} |H - (H)_{0,R}|^2 \, dx.$$

Now it follows from (2.1) that

$$\int_{B_R} |\nabla H|^2 \phi^2 \, dx \le \frac{C}{(R-\rho)^2} \int_{B_R} |H - (H)_{0,R}|^2 \, dx.$$

This proves our claim. □

By the Proposition in [1; Chapter V. Proposition 1.1, page 122-123], we have

Proposition 2. *(Reverse Hölder inequalities) Let $\Omega$ be an open domain and let $f$ and $g$ be positive functions. Suppose*

$$⨍_{B_R(x_0)} g^q \, dx \le b \left( ⨍_{B_{2R}(x_0)} g \, dx \right)^q + ⨍_{B_{2R}(x_0)} f^q \, dx + \theta ⨍_{B_{2R}(x_0)} g^q \, dx$$

*for each $x_0 \in \Omega$ and each $R < \frac{1}{2} dist\,(x_0, \partial\Omega) \wedge R_0$, where $R_0$, $b$, $\theta$ are constants with $b > 1$, $R_0 > 0$, $0 \le \theta < 1$. Then $g \in L^p_{loc}(\Omega)$ for $p \in [q, q+\varepsilon)$ and*

$$\left( ⨍_{B_R(x_0)} g^p \, dx \right)^{1/p} \le c \left( ⨍_{B_{2R}(x_0)} g^q \, dx \right)^{1/q} + c \left( ⨍_{B_{2R}(x_0)} f^p \, dx \right)^{1/p}$$

*for $B_{2R} \subset \Omega$, $R < R_0$, where $c$ and $\varepsilon$ are positive constants depending on $b$, $\theta$, $n$.*

THEOREM 3. *($L^p$-estimates) Let $(u, H)$ be a weak solution of (1.1)-(1.2) with $u \in W^{1,q}(\Omega, \mathbb{R})$, $H \in W^{1,2}(\Omega, \mathbb{R}^n)$ and $d^*H(x) = 0$ for a.e. $x \in \Omega$. Then there exists a small positive constant $\varepsilon$ such that $H \in W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$ for some $p \in (2, 2 + \varepsilon)$. More precisely,*

$$\left( \fint_{B_R(x_0)} |\nabla H|^p \, dx \right)^{1/p} \leq c \left( \fint_{B_{2R}(x_0)} |\nabla H|^2 \, dx \right)^{1/2} \tag{2.2}$$

*for all $x_0 \in \Omega$ and all $R$ with $2R < R_0$ with $B_{R_0}(x_0) \subset \Omega$ for some $R_0 > 0$. Moreover $u \in W^{1,q_1}_{loc}$ with $q_1 = \frac{np}{(2n-p)} > \frac{n}{n-1}$ where $p > 2$ is fixed above.*

*Proof.* By the Sobolev-Poincare inequality, we have

$$\int_{B_R} |H - (H)_{x_0,R}|^2 \, dx \leq C R^{2+(1-\frac{2}{q_2})n} \left( \int_{B_R} |\nabla H|^{q_2} \, dx \right)^{2/q_2}$$

for $q_2 = \frac{2n}{n+2} < 2$.

Letting $\rho = R/2$ in Lemma 1, we have

$$\left( \fint_{B_{R/2}(x_0)} |\nabla H|^2 \, dx \right)^{1/2} \leq C \left( \fint_{B_R(x_0)} |\nabla H|^{q_2} dx \right)^{1/q_2}.$$

Applying Proposition 2, there exists a $p > 2$ such that $H \in W^{1,p}(\Omega; R^n)$ and (2.2) holds. Applying the standard $L^p$-theory for equation (1.2), we get $u \in W^{2,p/2}_{loc}(\Omega; \mathbb{R})$. By Sobolev's inequality again, we have $u \in W^{1,\frac{np}{(2n-p)}}$. □

**3. Proof of Theorem A.** In this section, we give a proof of Theorem A.

Let $\Omega(x, \rho) = \Omega \cap B_\rho(x)$ and let $p \geq 1$ and $\lambda \geq 0$. At first, let us define the Morrey space $L^{p,\lambda}(\Omega)$ in the following

DEFINITION A. *(Morrey spaces) We say that $u$ belongs to $L^{p,\lambda}(\Omega)$ if $u \in L^p(\Omega)$ satisfies*

$$\|u\|_{L^{p,\lambda}(\Omega)} = \left\{ \sup_{x_0 \in \Omega, 0 < \rho < \text{diam } \Omega} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u|^p \, dx \right\}^{1/p} < +\infty$$

and the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$

DEFINITION B. *(Campanato space) We say that $u$ belongs to $\mathcal{L}^{p,\lambda}(\Omega)$ if $u \in L^p(\Omega)$ satisfies*

$$[u]_{p,\lambda} = \left\{ \sup_{x_0 \in \Omega, 0 < \rho < \text{diam } \Omega} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^p \, dx \right\}^{1/p} < +\infty,$$

*where $u_{x_0,\rho} = \frac{1}{|\Omega(x,\rho)|} \int_{\Omega(x,\rho)} u(x) dx$.*

Let us recall some results about Morrey and Campanato spaces from [1] and [2]. If there exists a constant $A$ such that $|\Omega(x, \rho)| \geq A\rho^n$ for all $\Omega(x, \rho)$, the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$ is isomorphic to the Morrey space $L^{p,\lambda}(\Omega)$ when $0 \leq \lambda < n$, and moreover, when $n < \lambda \leq n + p$, $\mathcal{L}^{p,\lambda}(\Omega)$ is isomorphic to the Hölder space $C^{0,\alpha}$ with $\alpha = \frac{\lambda - n}{p}$.

LEMMA 4. *Let $(u, H)$ be a weak solution to (1.1)-(1.2). Then $u$ is also a weak solution to the following equation*

$$\triangle u = d^*[\sigma(u)\langle dH, H\rangle], \tag{3.1}$$

*where*

$$\langle dH, H\rangle := \sum_{i,j=1}^{n} F^{ij} H^j dx_i. \tag{3.2}$$

*Proof.* Taking $\phi H$ as a test function in (1.1), we obtain

$$\int_\Omega \langle \sigma(u)dH, d(\phi H)\rangle\, dx = 0,$$

where $\phi$ is a function with $\phi \in C_0^2(\Omega;\mathbb{R})$. Then by the definition in Section 1, we get

$$\begin{aligned}\int_\Omega \phi\sigma(u)|dH|^2\, dx &= -\int_\Omega \langle \sigma(u)dH, d\phi \wedge H\rangle\, dx\\ &= -\int_\Omega \sigma(u)\langle \sum_{i,j} F^{ij} H^j dx_i, \sum_m \frac{\partial \phi}{\partial x_m} dx_m\rangle\, dx\\ &= -\int_\Omega \phi d^*[\sigma(u)\langle dH, H\rangle]\, dx\end{aligned}$$

for all $\phi \in C_0^2(\Omega;\mathbb{R})$, where $\langle dH, H\rangle$ is defined in (3.2). This proves our claim. □

Now we prove partial regularity of the weak solutions $(u, H)$ to the system (1.1)-(1.2).

*Proof of Theorem A.* Under the gauge condition $d^*H = 0$, we know from the Hodge theory that

$$-\triangle H = d^*dH + dd^*H = d^*dH.$$

Let $x_0 \in \Omega$ with $B_{R_0}(x_0) \subset\subset \Omega$ for some $R_0 > 0$. Let a 1-form $H_1 \in W^{1,2}(B_R(x_0))$ be a weak solution of the following Dirichlet problem

$$\sigma(u_{x_0,R}) \triangle H_1 = 0, \forall x \in B_R(x_0), \tag{3.3}$$

$$H_1 - H \in W_0^{1,2}(B_R(x_0), \mathbb{R}^n). \tag{3.4}$$

Then for all $\rho < R \le R_0$, we have

$$\int_{B_\rho(x_0)} |\nabla H_1|^2\, dx \le C\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |\nabla H_1|^2\, dx.$$

and therefore for all $\rho < R \le R_0$ with some $R_0 > 0$

$$\int_{B_\rho(x_0)} |\nabla H|^2\, dx \le C\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |\nabla H|^2\, dx + C\int_{B_R(x_0)} |\nabla (H - H_1)|^2\, dx.$$

Let $W = H - H_1$. Using equations (1.1) and (3.3), $W$ is the weak solution of the following

$$\sigma(u_{x_0,R}) \triangle W = d^*\{[\sigma(u) - \sigma(u_{x_0,R})]dH\}$$

with boundary condition $W = 0$ on $\partial B_R(x_0)$. Using $W$ as a test function in the above equation, we get

$$\sigma(u_{x_0,R}) \int_{B_R} |\nabla W|^2 \, dx = - \int_{B_R} \langle [\sigma(u) - \sigma(u_{x_0,R})] dH, dW \rangle \; dx. \tag{3.5}$$

By the assumption on $\sigma(u)$, there exists a non-negative, bounded function $\omega(t)$ increasing in $t$, concave, continuous with $\omega(0) = 0$, such that for $u, v \in \mathbb{R}$,

$$|\sigma(u) - \sigma(v)| \leq \omega(|u - v|^{q_1}), \tag{3.6}$$

where $q_1 = \frac{np}{2n-p}$ and $p$ is a fixed exponent in $(2, 2+\varepsilon)$ from Theorem 3. Hence we get from (3.5)-(3.6)

$$\int_{B_R(x_0)} |\nabla W|^2 \, dx \leq C \int_{B_R(x_0)} \omega^2(|u - u_{x_0,R}|^{q_1}) |\nabla H|^2 \, dx.$$

By the Sobolev-Poincare inequality, we obtain

$$\int_{B_R} |u - u_{x_0,R}|^{q_1} \, dx \leq C R^{q_1} \int_{B_R} |\nabla u|^{q_1} \, dx.$$

Using the $L^p$-estimate (Theorem 3) and the boundedness and concavity of $\omega$, we have

$$\begin{aligned}
&\int_{B_R(x_0)} \omega^2(|u - u_{x_0,R}|^{q_1}) |\nabla H|^2 \, dx \\
&\leq C \left( \int_{B_R(x_0)} |\nabla H|^p \, dx \right)^{2/p} \left( \int_{B_R(x_0)} \omega^{\frac{2p}{p-2}} (|u - u_{x_0,R}|^{q_1}) \, dx \right)^{\frac{p-2}{p}} \\
&\leq C \left( \int_{B_{2R}(x_0)} |\nabla H|^2 \, dx \right) \left( |B_R(x_0)|^{-1} \int_{B_R(x_0)} \omega(|u - u_{x_0,R}|^{q_1}) \, dx \right)^{\frac{p-2}{p}} \\
&\leq C \omega^{\frac{p-2}{p}} \left( C R^{q_1 - n} \int_{B_R(x_0)} |\nabla u|^{q_1} \, dx \right) \left( \int_{B_{2R}} |\nabla H|^2 \, dx \right),
\end{aligned}$$

where last inequality comes from the concavity of $\omega$ using the Jensen inequality and the Poincare inequality.

Therefore for all $\rho < R < 2R \leq R_0$ we have

$$\begin{aligned}
\int_{B_\rho(x_0)} |\nabla H|^2 \, dx &\leq C \left( \frac{\rho}{R} \right)^n \int_{B_{2R}(x_0)} |\nabla H|^2 \, dx + \\
&\quad + C \omega^{\frac{p-2}{p}} \left( C R^{q_1 - n} \int_{B_{2R}(x_0)} |\nabla u|^{q_1} \, dx \right) \int_{B_{2R}(x_0)} |\nabla H|^2 \, dx
\end{aligned} \tag{3.7}$$

By Theorem 3, $u$ belongs to $W^{2,p/2}(\Omega)$. Let $v \in W^{2,p/2}(B_R(x_0))$ be a weak solution of the following Dirichlet problem:

$$\begin{aligned}
-\triangle v &= 0, \quad \text{in } B_R(x_0), \\
v|_{\partial B_R} &= u|_{\partial B_R}, \quad x \in \partial B_R(x_0).
\end{aligned}$$

For the harmonic function $v$, it is easy to see that for $\rho \le R < 2R \le R_0$, we obtain

$$\int_{B_\rho(x_0)} |\nabla v|^{q_1}\,dx \le C(\frac{\rho}{R})^n \int_{B_R(x_0)} |\nabla v|^{q_1}\,dx.$$

Let $w = u - v$. Then $w \in W^{2,p/2}(B_R(x_0);\mathbb{R})$ satisfies

$$-\triangle w = \sigma(u)|dH|^2, \quad \text{in } B_R(x_0),$$

$$w = 0 \quad \text{on } \partial B_R(x_0).$$

Then

$$\int_{B_\rho(x_0)} |\nabla u|^{q_1}\,dx \le C(\frac{\rho}{R})^n \int_{B_R(x_0)} |\nabla u|^{q_1}\,dx + C\int_{B_R} |\nabla w|^{q_1}\,dx.$$

We rescale

$$\tilde{u}(x) = u(x_0 + Rx), \tilde{w}(x) = w(x_0 + Rx), \tilde{H}(x) = H(x_0 + Rx) = H^i(x_0 + Rx)dx_i.$$

Then

$$-\triangle \tilde{w} = \sigma(u)|d\tilde{H}|^2, \quad \text{in } B_1, \tag{3.8}$$

$$\tilde{w} = 0; \quad \text{on } \partial B_1, \tag{3.9}$$

where $B_1 = B(0,1)$ is the unit ball in $\mathbb{R}^n$. Applying the standard elliptic $L^p$-theory (see [7]) to (3.8)-(3.9), we obtain

$$\left(\frac{1}{|B_1|}\int_{B_1} |\nabla^2 \tilde{w}|^{p/2}\,dx\right)^{2/p} \le C\left(\frac{1}{|B_1|}\int_{B_1} |\nabla \tilde{H}|^p\,dx\right)^{2/p},$$

where $C$ is a constant independent of $R$.

Rescaling back, we have

$$\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} |\nabla^2 w|^{p/2}\,dx\right)^{2/p} \le C\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} |\nabla H|^p\,dx\right)^{2/p},$$

where $C$ is a constant independent of $R$. By the Sobolev inequality and using $L^p$-estimates, we see

$$\begin{aligned}\left(\fint_{B_R(x_0)} |\nabla w|^{q_1}\,dx\right)^{\frac{1}{q_1}} &\le CR\left(\fint_{B_R(x_0)} |\nabla^2 w|^{p/2}\,dx\right)^{2/p} \\ &\le CR^{1-n}\int_{B_{2R}(x_0)} |\nabla H|^2\,dx.\end{aligned}$$

Therefore for all $\rho < R < 2R \le R_0$, we have

$$\begin{aligned}\int_{B_\rho(x_0)} |\nabla u|^{q_1}\,dx \le & C(\frac{\rho}{R})^n \int_{B_{2R}(x_0)} |\nabla u|^{q_1}\,dx \\ & + CR^{n+q_1(1-n)}\left(\int_{B_{2R}(x_0)} |\nabla H|^2\,dx\right)^{q_1}.\end{aligned} \tag{3.10}$$

For any $x_0 \in \Omega$ and $r$ with $R_0 \geq r > 0$, we denote

$$\Phi(x_0, r) = r^{2-n} \int_{B_r(x_0)} |\nabla H|^2 \, dx, \quad \xi(x_0, r) = r^{q_1 - n} \int_{B_r(x_0)} |\nabla u|^{q_1} \, dx,$$

Note that (3.7) and (3.10) also hold for $R < \rho < 2R \leq R_0$. Then for all $\tau < 1$, we have

$$\Phi(x_0, \tau R) \leq C_1[1 + \omega^{\frac{p-2}{p}}(C_2 \xi(x_0, R))\tau^{-n}]\tau^2 \Phi(x_0, R)$$

and

$$\xi(x_0, \tau R) \leq C_1 \tau^{q_1} \xi(x_0, R) + \tau^{q_1 - n} \Phi^{q_1}(x_0, R)$$

by using $R$ instead of $2R$ in (3.7) and (3.10). For any $\alpha < 1$, choose $\tau < 1$ such that

$$2C_1 \tau^{q_1 \alpha} = 1.$$

There exists a small constant $\varepsilon_0 > 0$ such that if

$$\xi(x_0, R) + \Phi(x_0, R) < \varepsilon_0$$

for some $R < R_0$, then we have

$$\Phi^{q_1 - 1}(x_0, R) < \tau^n, \quad \omega^{\frac{p-2}{p}}(C_2 \xi(x_0, R)) < \tau^n$$

provided that $R$ is less than some $R_0$. Hence

$$\xi(x_0, \tau R) + \Phi(x_0, \tau R) \leq \tau^{2\alpha}[\xi(x_0, R) + \Phi(x_0, R)].$$

Therefore by iteration we obtain

$$\xi(x_0, \tau^k R) + \Phi(x_0, \tau^k R) \leq \tau^{2k\alpha}[\xi(x_0, R) + \Phi(x_0, R)] < \varepsilon_0$$

In conclusion, if $\xi(x_0, R) + \Phi(x_0, 2R) < \varepsilon_0$ for some $R < R_0$, then

$$\xi(x_0, \tau^k R) + \Phi(x_0, \tau^k R) \leq \tau^{2k\alpha} \varepsilon_0.$$

Hence for any $\rho < R_0$, we have

$$\xi(x_0, \rho) + \Phi(x_0, \rho) \leq C(\frac{\rho}{R})^{2\alpha}, \tag{3.11}$$

where $C$ is a constant independent of $\rho$ and $R$.

Note that $\xi(x_0, R)$ and $\Phi(x_0, R)$ are continuous functions of $x_0$. There exits an open $\Omega_0 \subset \Omega$ such that $u$ and $H$ are in $C^{0,\alpha}_{loc}(\Omega_0)$ for every $\alpha < 1$. Moreover, $\Omega \backslash \Omega_0 \subset \Sigma_1 \cup \Sigma_2$, where

$$\Sigma_1 = \{x \in \Omega : \liminf_{R \to 0^+} R^{2-n} \int_{B_R(x)} |\nabla H|^2 \, dx > 0\},$$

$$\Sigma_2 = \{x \in \Omega : \liminf_{R \to 0^+} R^{q_1 - n} \int_{B_R(x)} |\nabla u|^{q_1} \, dx > 0\}.$$

Moreover, since $H \in W^{1,2}(\Omega, \mathbb{R}^n)$ and $u \in W^{1,q_1}_{loc}(\Omega, R^n)$ with $q_1 = \frac{np}{2n-p}$ for some $p > 2$, we have

$$\mathcal{H}^{n-q_1}(\Omega \backslash \Omega_0) = 0$$

where $\mathcal{H}^{n-q_1}$ denote $(n - q_1)$-Hausdorff measure.

Next we prove $C^{1,\alpha}$-regularity inside $\Omega_0$. We assume that $x_0 \in \Omega$ with $B_{2R}(x_0) \subset \Omega_0$. From the above results, we know that $u$ and $H$ are $C^{0,\alpha}(\Omega_0)$ for every $\alpha < 1$ and

$$R^{q_1-n}\int_{B_R(x_0)} |\nabla u|^{q_1}\,dx \le CR^{2\alpha}, \quad R^{2-n}\int_{B_R(x_0)} |\nabla H|^2\,dx \le CR^{2\alpha},$$

where $C$ is a constant independent of $R$. Note that $H_1$ is the solution to equations (3.3)-(3.4). For any $\rho$ and $R$ with $\rho < R \le R_0$, we have

$$\int_{B_\rho(x_0)} |\nabla H_1 - (\nabla H_1)_{x_0,\rho}|^2\,dx \le C\left(\frac{\rho}{R}\right)^{n+2}\int_{B_R(x_0)} |\nabla H_1 - (\nabla H_1)_{x_0,\rho}|^2\,dx.$$

Repeating the same proof as before (3.7), we get

$$\int_{B_\rho(x_0)} |\nabla W|^2\,dx \le C\omega^{\frac{p-2}{p}}\left(CR^{q_1-n}\int_{B_R(x_0)} |\nabla u|^2\,dx\right)\int_{B_R(x_0)} |\nabla H|^2\,dx$$

for some $p > 2$.

Since $\omega$ is uniformly Hölder continuous, there exist constants $\beta$ and $C$ with $0 < \beta < 1$ such that $\omega(t) \le Ct^\beta$. Therefore

$$\begin{aligned}\int_{B_\rho(x_0)} |\nabla H - (\nabla H)_{x_0,\rho}|^2\,dx \le & C\left(\frac{\rho}{R}\right)^{n+2}\int_{B_{2R}(x_0)} |\nabla H - (\nabla H)_{x_0,R}|^2\,dx \\ & + CR^{n-2+\alpha[2+\beta\frac{p-2}{p}]},\end{aligned}$$

where $\alpha[2+\beta\frac{p-2}{p}] > 2$ by letting $\alpha$ be closing to 1. Then the standard procedure yields that $\nabla H$ is $C^{0,\gamma}$ for some $0 < \gamma < 1$. By applying standard PDE theory to equation (1.2), it is easy to see that $\nabla u$ is also locally in $C^{0,\gamma_1}_{loc}(\Omega_0)$ for some $\gamma_1 > 0$. This proves our claim. $\square$

**4. Partial regularity for the parabolic system.** In this section, we prove the partial regularity of the weak solutions to system (1.5)-(1.6).

Denote $Q_T = \Omega \times (0, T)$ and let $z = (x, t)$ for $x \in \Omega$ and $t \in (0, T)$. We recall some definitions from [9]. $L_{p,r}(Q_T)$ is the Banach space consisting of all measurable functions on $Q_T$ with a finite norm

$$\|u\|_{p,r,Q_T} = \left(\int_0^T\left(\int_\Omega |u(x,t)|^p\,dx\right)^{r/p} dt\right)^{1/r}.$$

We denote $\|u\|_{p,Q_T} = \|u\|_{p,p,Q_T}$. The space $V_p^{1,0}(Q_T)$ is the completion of $C^1(Q_T)$ with respect to the norm

$$|u|_{p;Q_T} = \left\{\int_{Q_T}(|u|^p + |\nabla u|^p)dz\right\}^{1/p}.$$

The space $W_p^{2,1}(Q_T)$ with $p \ge 1$ is the Banach space consisting of the elements of $L_p(Q_T)$ having generalized derivatives of the form $D_t^r D_x^s$ with any $r$ and $s$ satisfying the inequality $2r + s \le 2$. The norm is defined by

$$\|u\|^{(2)}_{q,Q_T} = \sum_{j=0}^{2}\langle\langle u\rangle\rangle^{(j)}_{q,Q_T}$$

with

$$\langle\langle u\rangle\rangle^{(j)}_{p,Q_T} = \sum_{2r+s=j} \|D_t^r D_x^s u\|_{q,Q_T}.$$

We say that a pair $(u, H)$ is a weak solution to equations (1.6)-(1.7) if $u \in V_q^{1,0}(Q_T)$ for some $q \in (1, \frac{n}{n-1})$ and $H^i \in V_2^{1,0}(Q_T; \mathbb{R}^n)$, and the pair $(u, H)$ satisfies the following:

$$\int_{Q_T} [\langle H, \partial_t \phi\rangle + \langle \sigma(u) dH, d\phi\rangle] \, dz = 0, \tag{4.1}$$

$$\int_{Q_T} [-u\psi_t + \nabla u \cdot \nabla \psi] \, dz = \int_{Q_T} \sigma(u)|dH|^2 \psi \, dz \tag{4.2}$$

for all $\phi := \sum_{i=1}^n \phi^i(x,t) dx_i$ for $i = 1, ..., n$ with $d^*H = 0$ in $Q_T$ in the weak sense, where $\phi^i(x,t) \in C_0^2(Q_T; \mathbb{R})$ and $\psi(x,t) \in C_0^2(Q_T; \mathbb{R})$. The existence of weak solutions of (4.1)-(4.2) with $d^*H = 0$ in $Q_T$ was obtained by Yin in [13] and [14].

For any $R > 0$, denote $Q_R(z_0) = B_R(x_0) \times (t_0 - R^2, t_0 + R^2)$ with $z_0 = (x_0, t_0)$. We denote for any function $u(x, t)$

$$u_{z_0,R} = \fint_{Q_R(z_0)} u(z) \, dz.$$

Next, we prove partial regularity of weak solutions to the system (4.1)-(4.2) by modifying the method for elliptic case of Sections 2-3. The first step towards the proof of Theorem B is to establish a Caccioppoli's inequality and $L^p$-estimates for weak solutions to the parabolic system (4.1)-(4.2) by applying the proof of [3] and [6]. More precisely, we have

LEMMA 7. *(Caccioppoli's inequality for parabolic problems) Assume that $(u, H)$ is a weak solution of (4.1)-(4.2) with the assumptions of Theorem B. Then there exists a constant $C$ such that for any $x_0 \in Q_T$ and any $R$ with $2R \le R_0$ with $Q_{R_0}(z_0) \subset Q_T$ for some $R_0 > 0$,*

$$\int_{Q_R(z_0)} |\nabla H|^2 \, dz \le \frac{C}{R^2} \int_{Q_{2R}(z_0)} |H - \tilde{H}_{x_0,2R}(t)|^2 \, dz.$$

*Proof.* Let $z_0 = (x_0, t_0) \in Q_T$. Let $\xi(x)$ be a function in $C_0^\infty(B_2(x_0))$ such that $0 \le \xi \le 1$, $\xi = 1$ in $B_1(x_0)$ and $|\nabla \xi| \le 2$. We also denote by $\xi_R$ the function $\xi_{2R}(x) = \xi(\frac{x}{R})$. As in [6], for a function $H^i(x,t)$, we define the weighted means of $H^i(x,t)$ in $B_{2R}(x_0)$ as

$$\tilde{H}^i_{x_0,2R}(t) = \frac{\int_{B_{2R}(x_0)} H^i(x,t)\xi_{2R}^2 \, dx}{\int_{B_{2R}(x_0)} \xi_{2R}^2(x) \, dx}$$

Then we define

$$\tilde{H}_{x_0,2R}(t) = \sum_i \tilde{H}^i_{x_0,2R}(t) dx_i.$$

Let $\tau \in C^\infty(\mathbb{R},\mathbb{R})$ be a function only in $t$ and satisfy $0 \le \tau \le 1$, $\tau \equiv 1$ on $[t_0 - R^2, t_0]$, $\tau \equiv 0$ on $t < t_0 - (2R)^2$. By the above choice, we note

$$\int_{t_0-4R^2}^{t_0} \left[\int_{B_{2R}(x_0)} (H^i - H^i_{2R}(t))\xi^2\, dx\right] \partial_t \tilde{H}^i_{2R}(t)\tau^2\, dt = 0. \tag{4.3}$$

Let $\mathbb{I}_{(-\infty,t_0)}$ be the characteristic function of the interval $(-\infty, t_0)$. Testing $\phi = (H - \tilde{H}_{2R}(t))\xi^2_{2R}\tau^2\mathbb{I}_{(-\infty,t_0)}$ and noting (4.3), we have

(4.4)
$$\begin{aligned}
&\int_{B(x_0,2R)\times\{t_0\}} |H - \tilde{H}_{x_0,2R}(t)|^2\xi^2\tau^2\, dx + \int_{Q_{2R}(z_0)} \sigma(u)|dH|^2\xi^2\tau^2\, dz\\
&\le 2\int_{Q_{2R}(z_0)} |H - \tilde{H}_{x_0,2R}(t)|^2\xi^2\tau\partial_t\tau\, dz\\
&\quad - 2\int_{Q_{2R}(z_0)} \sigma(u)\langle dH, \xi d\xi \wedge (H - \tilde{H}_{x_0,2R}(t))\rangle\tau^2\, dz.
\end{aligned}$$

It follows from (4.4) that

$$\int_{Q_R} |dH|^2\xi^2\tau^2\, dz \le \frac{C}{R^2}\int_{Q_{2R}(z_0)} |H - \tilde{H}_{x_0,2R}(t)|^2\, dz.$$

A similar argument to Lemma 1 yields

$$\begin{aligned}
&\int_{t_0-R^2}^{t_0}\int_{B_{2R}(x_0)} |dH|^2\xi^2\, dx\tau^2\, dt\\
&= \int_{t_0-R^2}^{t_0}\left(\int_{B_{2R}(x_0)} |\nabla H|^2\xi^2\, dx - \sum_{i,j=1}^n \int_{B_{2R}(x_0)} \frac{\partial H^i}{\partial x_j}\frac{\partial H^j}{\partial x_i}\xi^2\, dx\right)\tau^2\, dt\\
&= \int_{t_0-R^2}^{t_0}\int_{B_{2R}(x_0)}\left(|\nabla H|^2\xi^2 + 2\sum_{i,j=1}^n \frac{\partial H^i}{\partial x_j}\xi\frac{\partial \xi}{\partial x_i}[H^j - \tilde{H}^j_{x_0,2R}(t)]\right) dx\,\tau^2\, dt\\
&+ \int_{t_0-R^2}^{t_0}\int_{B_{2R}(x_0)}\sum_{i,j=1}^n \xi^2[H^j - \tilde{H}^j_{x_0,2R}(t)]\frac{\partial^2 H^i}{\partial x_j\partial x_i}\, dx\,\tau^2\, dt.
\end{aligned}$$

By using $d^*H = 0$, the last term in above identity is zero. This proves our claim. ∎

We have the following $L^p$-estimate:

Lemma 8. *Let $(u, H)$ be a weak solution to the system (4.1)-(4.2) with the assumptions of Theorem B. Then there exists an exponent $p > 2$ such that $\nabla H \in L^p_{loc}(Q_T)$; moreover for all $Q_R(z_0) \subset Q_{4R}(z_0) \subset Q_T$ we have*

$$⨍_{Q_R(z_0)} |\nabla H|^p\, dz \le C\left(⨍_{Q_{4R}(z_0)} |\nabla H|^2\, dz\right)^{\frac{p}{2}}$$

*and $u \in W^{2,1}_{p/2;loc}(Q_T)$.*

For the proof of Lemma 8, the same proof as in [5] gives the desired $L^p$-estimate for $H$ by using the reverse Hölder inequality as in Proposition 3. The fact $u \in$

$W^{2,1}_{p/2;loc}(Q_T)$ follows from Theorem 9.1 of Chapter IV of [10; pages 341-2].

By a slight modification of arguments in [12] (for the details, see [14]), we have

LEMMA 9. *Let $(u, H)$ be a weak solution to the system (4.1)-(4.2) with the assumptions of Theorem B. Then for all $Q_R(z_0) \subset Q_{2R}(z_0) \subset Q_T$, we have*

$$\int_{Q_R(z_0)} |H - H_{R,z_0}|^2 \, dz \le CR^2 \int_{Q_{2R}(z_0)} |\nabla H|^2 \, dz.$$

Now we complete the proof of Theorem B.

*Proof of Theorem B.* For any $z_0 \in Q_T$, choose $R_0$ with $Q_{R_0}(z_0) \subset Q_T$. Let $S_R(z_0)$ be the parabolic boundary of $Q_R(z_0)$ defined by

$$S_R = B_R(x_0, t_0 - R^2) \cup [\partial B_R(x_0) \times (t_0 - R^2, t_0 + R^2)].$$

Let a 1-form $H_1 \in V_2^{1,0}(Q_R(z_0))$ be the weak solution of the following parabolic problem:

$$\partial_t H_1 = \sigma(u_{z_0,R}) \triangle H_1, \quad \text{in } Q_R(z_0), \tag{4.5}$$

$$H_1|_{S_R(z_0)} = H|_{S_R(z_0)}, \quad \text{on } S_R(z_0). \tag{4.6}$$

For all $\rho < R \le R_0$, we have

$$\int_{Q_\rho(z_0)} |\nabla H|^2 \, dz \le C \left(\frac{\rho}{R}\right)^{n+2} \int_{Q_R(z_0)} |\nabla H|^2 \, dz + C \int_{Q_R(x_0)} |\nabla W|^2 \, dz$$

with $W = H - H_1$. By a similar proof as in Section 3, we have

$$\int_{Q_R(z_0)} |\nabla W|^2 \, dz \le C\omega^{\frac{p-2}{p}} \left( CR^{-n} \int_{Q_{4R}(z_0)} |u - u_{z_0,R}|^{p/2} \, dz \right) \int_{Q_{2R}(z_0)} |\nabla H|^2 \, dz.$$

Let $v \in W^{2,1}_{p/2}(Q_R(z_0))$ be a weak solution of

$$\partial_t v = \triangle v, \quad \text{in } Q_R(z_0),$$

$$v|_{S_R(z_0)} = u|_{S_R(z_0)}, \quad \text{on } S_R(z_0).$$

Then for all $\rho < R \le R_0$, we have

$$\int_{Q_\rho(z_0)} |\partial_t u|^{p/2} \, dz \le C(\frac{\rho}{R})^{n+2} \int_{Q_R(z_0)} |\partial_t u|^{p/2} \, dz + C \int_{Q_R(z_0)} |\partial_t w|^{p/2} \, dz$$

and

$$\int_{Q_\rho(z_0)} |\nabla u|^{p/2} \, dz \le C(\frac{\rho}{R})^{n+2} \int_{Q_R(z_0)} |\nabla u|^{p/2} \, dz + C \int_{Q_R(z_0)} |\nabla w|^{p/2} \, dz,$$

where $w = u - v$ satisfies

$$\partial_t w = \triangle w + \sigma(u)|dH|^2, \quad \text{in } Q_R(z_0),$$

$$w = 0 \quad \text{on } S_R(z_0).$$

Since $|\nabla H|^2$ is locally in $L^{p/2,p/2}(Q_T)$, we have from Theorem 9.1 of [10; Chapter IV] and Lemma 8 that

(4.7)
$$\int_{Q_R(z_0)} \left(|\nabla^2 w|^{p/2} + |\partial_t u|^{p/2}\right) dz \le C \int_{Q_R(z_0)} |\nabla H|^p \, dz$$
$$\le CR^{n+2} \left( \fint_{Q_{4R}(z_0)} |\nabla H|^2 \, dx \right)^{p/2}.$$

By the Sobolev inequality and using $L^p$-estimates in (4.7), we know

$$\int_{Q_R(z_0)} |\nabla w|^{p/2} \, dz \le CR^{p/2} \int_{Q_R(z_0)} |\nabla^2 w|^{p/2} \, dz$$
$$\le CR^{n+2-\frac{p}{2}(n+1)} \left( \int_{Q_{4R}(z_0)} |\nabla H|^2 \, dx \right)^{p/2}.$$

By a version of the Sobolev-Poincare inequality, we have

(4.8)
$$\int_{Q_R(z_0)} |u - u_{z_0,R}|^{p/2} \, dz \le C \left[ R^{p/2} \int_{Q_R(z_0)} |\nabla u|^{p/2} \, dz + R^p \int_{Q_R(z_0)} |\partial_t u|^{p/2} \, dz \right].$$

For any $z_0 \in Q_T$ and $r$ with $Q_{z_0,r} \subset Q_T$, we denote

$$\Phi(z_0, r) = r^{-n} \int_{Q_r(z_0)} |\nabla H|^2 \, dz, \quad \xi(z_0, r) = r^{-n-2+p/2} \int_{Q_r(z_0)} |\nabla u|^{p/2} \, dz,$$

$$\eta(z_0, r) = r^{-n-2+p} \int_{Q_r(z_0)} |\partial_t u|^{p/2}. \, dz.$$

Then for all $\tau < 1$, we have

$$\Phi(z_0, \tau R) \le C_1[1 + \omega^{\frac{p-2}{p}} (C_2[\xi(z_0, R) + \eta(z_0, R)])\tau^{-(n+2)}\tau^2 \Phi(x_0, R),$$

$$\xi(z_0, \tau R) \le C_1 \tau^2 \xi(x_0, R) + \tau^{\frac{p}{2}-(n+2)} \Phi^{\frac{p}{2}}(x_0, R)$$

and

$$\eta(z_0, \tau R) \le C_1 \tau^p \eta(z_0, R) + \tau^{p-(n+2)} \Phi^{\frac{p}{2}}(z_0, R).$$

If there exists a constant $\varepsilon_0$ such that $\Phi(z_0, r) + \xi(z_0, r) + \eta(z_0, r) < \varepsilon_0$ for some $r \le R_0$, then a similar iteration step as in Section 3 yields

$$\phi(z_0, \rho) + \xi(z_0, \rho) + \eta(z_0, \rho) \le C\rho^{2\alpha}$$

for all $\alpha < 1$ and $\rho \le r \le R_0$. Using the Sobolev inequality (4.8) and Lemma 9, we obtain through the Campanato space that $u(x,t)$ and $H(x,t)$ are Hölder continuous in $\alpha$ locally in $\tilde{Q}$ where $\tilde{Q}$ is an open subset of $Q_T$. A similar argument as in Section 3 yields that $u(x,t)$ and $H(x,t)$ are also in $C^{1,\gamma}_{loc}(\tilde{Q})$ for some $\gamma < 1$.

Since $u$ is in $W^{2,1}_{p/2;loc}(Q_T)$, we have $\nabla u \in L_{q_3,q_3;loc}(Q_T)$, $q_3 = \frac{(n+2)p}{n+2-2p}$ by the parabolic type Sobolev inequality (see [10; Lemma 3.3, page 80 ]). Moreover, Hölder's inequality gives

$$\xi(z_0, R) \le \left( R^{q_3-n-2} \int_{Q_{z_0,R}} |\nabla u|^{q_3}\, dz \right)^{\frac{p}{2q_3}}.$$

We have $Q_T \backslash \tilde{Q} \subset \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ where

$$\Sigma_1 = \{z_0 \in Q_T : \liminf_{R \to 0^+} R^{-n} \int_{Q_{z_0,R}} |\nabla H|^2\, dz > 0\},$$

$$\Sigma_2 = \{z_0 \in Q_T : \liminf_{R \to 0^+} R^{q_3-n-2} \int_{Q_{z_0,R}} |\nabla u|^{q_3}\, dz > 0\},$$

and

$$\Sigma_3 = \{z_0 \in Q_T : \liminf_{R \to 0^+} R^{p-n-2} \int_{Q_{z_0,R}} |\partial_t u|^{p/2}\, dz > 0\}.$$

Sine $\nabla H \in L_{2;loc}(Q_T, \mathbb{R}^n)$ and $\partial_t u \in L_{p/2}(Q_T, \mathbb{R}^n)$, we have

$$\mathcal{H}^{n+2-q_3}(Q_T \backslash \tilde{Q}) = 0,$$

where $\mathcal{H}^{n+2-q_3}$ denotes $(n+2-q_3)$-Hausdorff measure. This proves our claim. $\square$

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