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A LIOUVILLE TYPE THEOREM FOR MINIMIZING MAPS *

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Abstract. Here we establish a Liouville type theorem for minimizing maps from $\mathbb{R}^2$ (or in general, from $\mathbb{R}^m$) into a compact Riemannian manifold N. As a consequence of this, we prove a local gradient estimate for minimal solutions to a variational problem arise from planar ferromagnetism and anti-ferromagnetism. The latter can be applied to study the asymptotic behavior of entire solutions.

1. Introduction. In [HnL] we studied the following simplified mathematical model for the planar ferromagnetism and anti-ferromagnetism. Let $\Omega \subset \mathbb{R}^2$ be a bounded open connected smooth subset, $S^2 \subset \mathbb{R}^3$ be the standard 2-sphere, S^1 be the horizontal great circle on S^2, and $g : \partial\Omega \to S^1$ be a smooth map. For any $\varepsilon > 0$ and $u \in H^1_g(\Omega, S^2)$, we define

$$I_\varepsilon(u) = \int_\Omega \frac{1}{2}\left[|\nabla u|^2 + \frac{(u^3)^2}{\varepsilon^2}\right] dx. \tag{1.1}$$

We analyzed the asymptotic behavior of the minimizers of I_ε over $H^1_g(\Omega, S^2)$ as $\varepsilon \to 0^+$. One of the crucial step in our proof is gradient estimates for minimizers (see Theorem 1.3 in [HnL]), which was proved by combining a blow-up argument with some Liouville type theorems. The main theme relies on the fact that minimizers of such boundary value problems always lie in a half sphere. In order to study the asymptotic behavior of minimizing solutions or to understand the behavior of general minimizers of (1.1) (without the Dirichlet boundary condition), we lead to the following:

THEOREM 1.1. *Assume $1 < p < \infty$, N is a connected compact Riemannian manifold such that either $1 < p < 2$ or $p \geq 2$ but $\pi_1(N)$ is finite and $\pi_i(N) = 0$ for $2 \leq i \leq [p] - 1$. $m \in \mathbb{N}$, $u \in W^{1,p}_{loc}(\mathbb{R}^m, N)$ is a locally minimizing p-harmonic map.*
- *If $1 < p < m$, then $\int_{B_r} |du|^p \leq c(m, p, N) r^{m-p}$ for any $r > 0$.*
- *If $m \leq p < \infty$, then u must be a constant map.*

From this Liouville type theorem we may deduce the following gradient estimates for minimizing p harmonic maps.

THEOREM 1.2. *Let m, p and N be the same as in Theorem 1.1, $\Omega \subset \mathbb{R}^m$ be an open subset, $u \in W^{1,p}_{loc}(\Omega, N)$ be a locally minimizing p-harmonic map.*
- *If $1 < p < m$, then for any $x \in \Omega$, $0 < r < d(x, \mathbb{R}^m\backslash\Omega)$ we have*

$$\int_{B_r(x)} |du|^p \leq c(m, p, N) \frac{r^{m-p}}{(1 - \frac{r}{d(x,\mathbb{R}^m\backslash\Omega)})^{p-1}}.$$

- *If $m \leq p < \infty$, then $u \in C^1(\Omega, N)$ and*

$$|du(x)| \leq \frac{c(m, p, N)}{d(x, \mathbb{R}^m\backslash\Omega)} \quad \textit{for any } x \in \Omega.$$

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An interesting consequence of Theorem 1.2 is the compactness for p-energy minimizing maps.

Corollary 1.1. *Let m, p, Ω and N be the same as in Theorem 1.2, $u_i \in W^{1,p}_{loc}(\Omega, N)$ be a sequence of locally minimizing p-harmonic maps, then there exists a subsequence $u_{i'}$ and a locally minimizing p-harmonic map $u \in W^{1,p}_{loc}(\Omega, N)$ such that $u_{i'} \to u$ in $W^{1,p}_{loc}(\Omega, N)$.*

The main point in Corollary 1.1 is that, under the topological condition on N, one may drop the condition that the p-energy of the sequence of maps is uniformly bounded as in the Luckhaus compactness theorem for minimizing p-harmonic maps (see [Lu1] and [Lu2]). This fact has already been observed in [HKL] in a special case. We also have the following

Theorem 1.3. *Let m, p and N be the same as in Theorem 1.1, $H_0 = \{x \backslash x \in \mathbb{R}^m, x^m > 0\}$ be the upper half space, and $u \in W^{1,p}_{loc}(\overline{H}_0, N)$ be a locally minimizing p-harmonic map such that $u|_{\partial H_0}$ is a constant map, then u is a constant map.*

When $p = 2$, the topological condition on the target stated in Theorem 1.1 simply says the fundamental group is finite, or equivalently, the universal covering space is compact. Typical examples of Riemannian manifolds with finite fundamental group are compact Riemannian manifolds with strictly positive Ricci curvature. When the fundamental group of the target manifold is infinite, we may have nonconstant minimizing harmonic maps with arbitrary growth rates for the energy. Indeed a lifting argument tells us if N is a complete Riemannian manifold with non-positive sectional curvature, then for $m \geq 2$, any harmonic map from $\mathbb{R}^m$ to N is minimizing. A typical example is the case $N = T^n = S^1 \times \cdots \times S^1$ (n factors). A map $u : \mathbb{R}^m \to T^n$ is a harmonic map if and only if $u = (e^{ih_1}, \cdots, e^{ih_n})$ and $h_1, \cdots, h_n$ are harmonic functions on $\mathbb{R}^m$. This shows Theorem 1.1, Theorem 1.2 and Theorem 1.3 can not be true if we drop the topological condition.

We also would like to point out a few known facts related to our results. It was proved in [SU] that for $n \geq 3$, every stable harmonic map from $\mathbb{R}^2$ to S^n is a constant map (see Theorem 2.9 in [SU]). Note that in Theorem 1.1 one could have $N = S^2$ or $N = S^n$, $n \geq 2$ but with arbitrary smooth Riemannian metric. It is well-known that holomorphic or anti-holomorphic maps from $\mathbb{R}^2$ to S^2 are stable. In fact, a theorem of A. Lichnerowicz says every holomorphic or anti-holomorphic map from a compact Kähler manifold to another Kähler manifold is energy minimizing in its homotopy class (see Theorem 4.2 in [Xi]). If one looks at the proof closely, one can easily show that without the compactness condition on the domain manifold, any holomorphic or anti-holomorphic map is energy minimizing in its homotopy class if only those homotopies supported in compact subsets are considered. In particular, it shows holomorphic or anti-holomorphic maps between Kähler manifolds are always stable harmonic maps. We also note that it was proved in Corollary 6 of [So] that any minimizing harmonic map from $\mathbb{R}^2$ to S^2 which misses a nonempty open subset of S^2 is a constant map. On the other hand, for $m \geq 7$, there exists a nonconstant smooth harmonic map $u : \mathbb{R}^m \to S^m$ with image lying in open upper half sphere (see Example 2.2 in [SU]), and hence it is a minimizing harmonic map by Lemma 2.1 in [SU]. For general p-harmonic maps, we note that if $m - 1 \leq p < m$ or $1 \leq p \leq m - 1$ but $p \in \mathbb{Z}$, then $x/|x| : \mathbb{R}^m \to S^{m-1}$ is a minimizing p-harmonic map. See [AL], [CG], [HLW] and the references therein.

The key concept related to Theorem 1.1, Theorem 1.2 and Theorem 1.3 is the so called *p-extension property* for $1 < p < \infty$ (see Definition 2.1). Based on an important lemma and some techniques from [HrL] (see Section 6 of [HrL]), we may show that a compact Riemannian manifold satisfies p-extension property if and only if it is $([p]-1)$-simply connected (see Theorem 2.1).

Once we show every minimizing harmonic map from $\mathbb{R}^2$ to S^2 is a constant map, we are able to classify blow-up limits of local minimizers of I_ε, $\varepsilon \to 0^+$. We have the following

THEOREM 1.4. *Suppose $u \in C^\infty(\mathbb{R}^2, S^2)$ satisfies*

$$-\triangle u = \left(|\nabla u|^2 + \left(u^3\right)^2\right) u - u^3 e_3 \tag{1.2}$$

on $\mathbb{R}^2$, also assume u locally minimizes I_1, then the image of u lies in upper half sphere or lower half sphere and it satisfies

$$|u^3(x)| \le c(u) e^{-\frac{|x|}{16}}, \quad |\nabla u(x)| \le \frac{c(u)}{|x|}.$$

In addition, either u is a constant in S^1 or the degree of $\frac{(u^1,u^2)}{|(u^1,u^2)|}$ is $+1$ or -1. In the latter case we have $\int_{\mathbb{R}^2} (u^3)^2 = \pi$.

When the base points of blow-ups are somewhat close to the boundary, we get blow-up limits defined on a half plane. Then we have the boundary version of Theorem 1.4, which in some sense corresponds to the fact that the vortices should “stay inside” Ω in Theorem 1.2 of [HnL].

THEOREM 1.5. *Let $H_0 = \{x \backslash x \in \mathbb{R}^2, x^2 > 0\}$ be the open upper half plane. Assume $u \in C^\infty(\overline{H_0}, S^2)$ satisfies (1.2) in H_0 and locally minimizes I_1 in $\overline{H_0}$, $u|_{\partial H_0} \equiv e$, $e \in S^1$ is a constant, then $u \equiv e$ in $\overline{H_0}$.*

The ingredients in proving Theorem 1.4 and Theorem 1.5 are the gradient estimate, which follows from a blowing up argument, and energy comparison maps from [Sa2] and Section 6 of [HrL]. We have just learned from Sylvia Serfaty that in [AS] and [Sa1], the authors made a similar investigation as our previous work [HnL]. However, [AS] seems to have missed this key gradient estimate (see page 677 of [AS]). It is also necessary to have this gradient estimates to understand the fine properties of minimizers. An interesting point in Theorem 1.4 and Theorem 1.5 is that we do not have any growth condition on solutions to start with. It remains as an open problem if after translation, rotation, reflection with respect to x^1 axis on $\mathbb{R}^2$ and reflection with respect to the horizontal plane on S^2, a minimizer in Theorem 1.4 is either a constant or the degree 1 radial solution in Proposition 5.2 of [HnL]. For the Ginzburg-Landau model case, the corresponding problem was solved in [Mi].

The paper is written as follows. In Section 2, we study the relation between minimizing p-harmonic maps and the topology of the target manifolds and prove Theorem 1.1, Theorem 1.2 and Theorem 1.3. In Section 3, we classify the blow-up limits of minimizers of I_ε as $\varepsilon \to 0^+$ and prove Theorem 1.4 and Theorem 1.5.

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2. Minimizing p-harmonic maps. In this section we shall study the relations between minimizing p-harmonic maps and the topology of the target manifolds. As mentioned in the introduction, the key concept related to Theorem 1.1 and Theorem 1.2 is the following

DEFINITION 2.1 (p-Extension property). *Assume $1 < p < \infty$, N is a smooth compact manifold. If for any Riemannian metric g on N, any $m \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^m$ open, bounded and piecewisely smooth, there exists a constant $c = c(p, g, \Omega, N)$ such that for any $f \in W^{1-\frac{1}{p},p}(\partial\Omega, (N, g))$, there exists a $u \in W^{1,p}(\Omega, (N, g))$ such that*

$$u|_{\partial\Omega} = f \quad \text{and} \quad \int_\Omega |du|^p \le c(p, g, \Omega, N)\, [f]^p_{W^{1-\frac{1}{p},p}(\partial\Omega,(N,g))}, \tag{2.1}$$

then we say N satisfies the p-extension property.

It is easy to see that once there exists a Riemannian metric g_0 on N such that we may do extensions satisfying (2.1), then N has the p-extension property. We may also define (m, p)-*extension property* by putting the dimension m in, but we don't need this here. The p-extension property is a topological property, in fact one has the following

THEOREM 2.1. *If N is a smooth connected compact manifold, $1 < p < \infty$, then it has the p-extension property if and only if $\pi_i(N) = 0$ for $1 \le i \le [p] - 1$.*

To prove this theorem, we need Lemma 6.1 in [HrL], which is stated below for reader's convenience.

LEMMA 2.1 (Lemma 6.1 in [HrL]). *Let $N^n \subset \mathbb{R}^k$ be a smooth connected compact submanifold, $l \in \mathbb{Z}$, $l \ge 0$. If for any $1 \le i \le l$, $\pi_i(N) = 0$, then there exists a compact $(k-l-2)$-dimensional Lipschitz polyhedron $X \subset \mathbb{R}^k$ and a locally Lipschitz retraction $P : \mathbb{R}^k \backslash X \to N$ such that*

$$\int_{B_R} |dP(x)|^p dx < \infty \quad \text{for any } 1 \le p < l + 2 \text{ and } R > 0. \tag{2.2}$$

Moreover, P is smooth in an open neighborhood of N.

Proof of Theorem 2.1. If N satisfies p-extension property, then, for any $1 \le i \le [p] - 1$, any smooth map $f : S^i = \partial B_1^{i+1} \to N$, there exists a $u \in W^{1,p}(B_1^{i+1}, N)$ such that $u|_{\partial B_1} = f$. If p is not an integer or p is an integer but $i \ne [p] - 1$, then by Sobolev embedding theorem, u is continuous, hence f is homotopic to a constant. If p is an integer and $i = p - 1$, then it follows from [BN] that f is still homotopic to a constant. In any case, $\pi_i(N) = 0$.

Let N be such that $\pi_i(N) = 0$ for $1 \le i \le [p] - 1$. First of all, we may assume there is an embedding $N \subset \mathbb{R}^k$ for some k. From Lemma 2.1 we may find a compact $(k-[p]-1)$-dimensional Lipschitz polyhedron $X \subset \mathbb{R}^k$ and a local Lipschitz retraction $P : \mathbb{R}^k \backslash X \to N$ such that

$$\int_{B_R} |dP(x)|^q dx < \infty \quad \text{for } 1 \le q < [p] + 1 \text{ and } R > 0. \tag{2.3}$$

Moreover, P is smooth in an open neighborhood of N. We may find a $\delta \in (0, 1)$ such that for any $a \in B_\delta^k$, the map $P_a : N \to N$, which is defined by $P_a(y) = P(y - a)$, is a diffeomorphism with $|dP_a^{-1}(y)| \le c(N)$. Now given any open bounded piecewisely smooth subset $\Omega \subset \mathbb{R}^m$ and any $f \in W^{1-\frac{1}{p},p}(\partial\Omega, N)$, let $v : \Omega \to \mathbb{R}^k$ be the harmonic extension of f, then if we denote $R_0 = \sup_{y\in N} |y|$, we have

$$|v(x)| \le R_0 \quad \text{and} \quad \int_\Omega |dv|^p \le c(p, \Omega, N)[f]^p_{W^{1-\frac{1}{p},p}(\partial\Omega)}. \tag{2.4}$$

For any $a \in B_\delta^k$, denote $v_a(x) = P(v(x) - a)$, then

$$\int_{B_\delta^k} da \int_\Omega |dv_a(x)|^p dx \le \int_{B_\delta} da \int_\Omega |dP(v(x)-a)|^p |dv(x)|^p dx \tag{2.5}$$

$$\le \int_\Omega dx \int_{B_{R_0+1}} |dP(y)|^p |dv(x)|^p dy \le c(p,\Omega,N)[f]^p_{W^{1-\frac{1}{p},p}(\partial\Omega)}.$$

Here we used (2.3) with $q = p$ and (2.4). From (2.5) we may find an $a \in B_\delta$ such that

$$\int_\Omega |dv_a(x)|^p dx \le c(p,\Omega,N)[f]^p_{W^{1-\frac{1}{p},p}(\partial\Omega)},$$

then $u = P_a^{-1} \circ v_a$ is the needed extension. □

We note the extension problem without energy estimate was considered in [BD]. In fact, Theorem 5 in [BD] is in the same spirit as the necessary part of Theorem 2.1. To prove Theorem 1.1 and 1.2 we need some technical lemmas.

Definition 2.2. *Let X be a metric space, $k \in \mathbb{Z}$, $k \ge 0$, $E \subset X$. If there exists a sequence of bounded subsets, namely $A_i \subset \mathbb{R}^k$ and a sequence of Lipschitz maps, namely $\phi_i : A_i \to X$ such that $E = \cup_{i=1}^\infty \phi_i(A_i)$, then we say E is* countably k rectifiable.

Lemma 2.2. *Let X and Y be metric spaces, $s \ge 0$, $k \in \mathbb{Z}$, $k \ge 0$. If $A \subset X$ satisfies $\mathcal{H}^s(A) = 0$, $B \subset Y$ is countably k rectifiable, then $\mathcal{H}^{k+s}(A \times B) = 0$.*

Proof. We may assume $k > 0$, $s > 0$ and $B = \phi(E)$, where $E \subset [0,1]^k$ and ϕ is a map from E to X with $Lip(\phi) \le L$. Given any $0 < \varepsilon < 1$, we may find $(A_i)_{i=1}^\infty$ such that

$$A \subset \bigcup_{i=1}^\infty A_i, \quad \sum_{i=1}^\infty d(A_i)^s < \varepsilon.$$

Choose $\alpha_i > d(A_i)$ such that $\sum_{i=1}^\infty \alpha_i^s < \varepsilon$, then $0 < \alpha_i < 1$. Set $l_i = [1/\alpha_i] + 1$, then

$$[0,1]^k = \bigcup_{j=1}^{l_i^k} C_{ij}, \quad E_{ij} = C_{ij} \cap E, \quad E = \bigcup_{j=1}^{l_i^k} E_{ij},$$

here C_{ij} is a cube with side length $1/l_i$. We have

$$A \times B \subset \bigcup_{i=1}^\infty A_i \times B = \bigcup_{i=1}^\infty \bigcup_{j=1}^{l_i^k} A_i \times \phi(E_{ij}),$$

$$d(A_i \times \phi(E_{ij})) \le d(A_i) + d(\phi(E_{ij})) \le c(k,L)\alpha_i,$$

which shows

$$\sum_{i=1}^\infty \sum_{j=1}^{l_i^k} d(A_i \times \phi(E_{ij}))^{k+s} \le \sum_{i=1}^\infty c(k,s,L) l_i^k \alpha_i^{k+s} \le c(k,s,L) \sum_{i=1}^\infty \alpha_i^s \le c(k,s,L)\varepsilon.$$

This implies $\mathcal{H}^{k+s}(A \times B) = 0$. $\square$

LEMMA 2.3. *Assume $m \geq 3$, $F \subset \mathbb{R}^m$ is a closed subset such that $\mathcal{H}^{m-2}(F) = 0$, then $\mathbb{R}^m \backslash F$ is simply connected.*

Proof. First we want to show $\mathbb{R}^m \backslash F$ is path connected. In fact, given any two points x_0, x_1 in $\mathbb{R}^m \backslash F$, let $\gamma(t) = (1-t)x_0 + tx_1$ for $0 \leq t \leq 1$. Since F is closed, we may find a $\delta > 0$ such that $B_\delta(x_i) \subset \mathbb{R}^m \backslash F$ for $i = 0, 1$. From Lemma 2.2 we know $\mathcal{H}^{m-1}(F - \gamma([0,1])) = 0$, hence we may find a point $\xi \in B_\delta^m$ such that $\xi \notin F - \gamma([0,1])$. Clearly for any $0 \leq t \leq 1$, $\gamma(t) + \xi \notin F$, this means $x_0 + \xi$ can be connected to $x_1 + \xi$ in $\mathbb{R}^m \backslash F$. On the other hand, it is clear that x_i can be connected to $x_i + \xi$ in $\mathbb{R}^m \backslash F$ by the line segment connecting them for $i = 0, 1$. Hence x_0 can be connected to x_1 in $\mathbb{R}^m \backslash F$.

Since $\mathbb{R}^m \backslash F$ is open and connected, to show it is simply connected, it suffices to show for any Lipschitz map $f : \partial B_1^2 \to \mathbb{R}^m \backslash F$, there exists a Lipschitz map $\bar{f} : \overline{B_1^2} \to \mathbb{R}^m \backslash F$ such that $\bar{f}|_{\partial B_1^2} = f$. In fact for any $f \in Lip(\partial B_1^2, \mathbb{R}^m \backslash F)$, we may find a $\delta > 0$ such that for any $x \in \partial B_1^2$, $B_\delta^m(f(x)) \subset \mathbb{R}^m \backslash F$. On the other hand, we may find a $\tilde{f} \in Lip(\overline{B_1^2}, \mathbb{R}^m)$ such that $\tilde{f}$ is an extension of f. Indeed one may take $\tilde{f}(x) = |x| f(x/|x|)$ for any $x \in \overline{B_1^2}$. Via Lemma 2.2 we know $\mathcal{H}^m(F - \tilde{f}(\overline{B_1^2})) = 0$. Hence we may find a $\xi \in B_\delta^m$ such that $\xi \notin F - \tilde{f}(\overline{B_1^2})$. This implies $\tilde{f}(x) + \xi \notin F$ for any $x \in \overline{B_1^2}$. Define

$$\bar{f}(x) = \begin{cases} \tilde{f}(2x) + \xi, & \text{for } x \in \overline{B_{1/2}^2}; \\ f(x/|x|) + 2(1-|x|)\xi, & \text{for } x \in \overline{B_1^2} \backslash B_{1/2}^2. \end{cases}$$

Clearly $\bar{f} \in Lip(\overline{B_1^2}, \mathbb{R}^m \backslash F)$ is the needed extension of f. $\square$

Proof of Theorem 1.1. Let us first consider the special case when $\pi_i(N) = 0$ for $1 \leq i \leq [p] - 1$. From Theorem 2.1 we know N satisfies the p-extension property, hence for any $f \in W^{1-\frac{1}{p},p}(\partial B_1^m, N)$, there exists a $v \in W^{1,p}(B_1^m, N)$ such that

$$v|_{\partial B_1} = f \quad \text{and} \quad \int_{B_1} |dv|^p \leq c(m,p,N)[f]^p_{W^{1-\frac{1}{p},p}(\partial B_1)}. \tag{2.6}$$

A scaling argument shows for any $r > 0$, any $f \in W^{1-\frac{1}{p},p}(\partial B_r^m, N)$, there exists a $v \in W^{1,p}(B_r^m, N)$ such that

$$v|_{\partial B_r} = f \quad \text{and} \quad \int_{B_r} |dv|^p \leq c(m,p,N)[f]^p_{W^{1-\frac{1}{p},p}(\partial B_r)}. \tag{2.7}$$

The point here is that the constant $c(m,p,N)$ doesn't depend on r. Suppose $u : \mathbb{R}^m \to N$ is a minimizing p-harmonic map, for $r \geq 0$, let $\phi(r) = \int_{B_r} |du|^p$. For any $r > 0$, let $f = u|_{\partial B_r}$ in (2.7), from the minimality of u and (2.7) we have

$$\begin{aligned} \phi(r) = \int_{B_r} |du|^p &\leq c(m,p,N)\,[u|_{\partial B_r}]^p_{W^{1-\frac{1}{p},p}(\partial B_r)} \\ &\leq c(m,p,N) \left(\int_{\partial B_r} |u|^p d\mathcal{H}^{m-1} \right)^{\frac{1}{p}} \left(\int_{\partial B_r} |d\,(u|_{\partial B_r})\,|^p d\mathcal{H}^{m-1} \right)^{1-\frac{1}{p}} \end{aligned} \tag{2.8}$$

$$\leq c(m,p,N)r^{\frac{m-1}{p}}\left(\int_{\partial B_r}|du|^p d\mathcal{H}^{m-1}\right)^{1-\frac{1}{p}} = c(m,p,N)r^{\frac{m-1}{p}}\phi'(r)^{1-\frac{1}{p}}.$$

Assume for some $R > 0$, we have $\phi(R) > 0$, then for any $r \geq R$,

$$\frac{1}{c(m,p,N)r^{\frac{m-1}{p-1}}} \leq \frac{\phi'(r)}{\phi(r)^{\frac{p}{p-1}}}. \tag{2.9}$$

If $p > m$, then integrating (2.9), we obtain

$$\frac{1}{c(m,p,N)}((R')^{\frac{p-m}{p-1}} - R^{\frac{p-m}{p-1}}) \leq \frac{1}{\phi(R)^{\frac{1}{p-1}}} - \frac{1}{\phi(R')^{\frac{1}{p-1}}} \leq \frac{1}{\phi(R)^{\frac{1}{p-1}}}, \tag{2.10}$$

for any $R' \geq R$. Let $R' \to \infty$ in (2.10), we lead to a contradiction. Hence $\phi \equiv 0$, that is u must be a constant map.

If $p = m$, then integrating (2.9) one gets

$$\frac{1}{c(m,N)}\log\frac{R'}{R} \leq \frac{1}{\phi(R)^{\frac{1}{m-1}}} - \frac{1}{\phi(R')^{\frac{1}{m-1}}} \leq \frac{1}{\phi(R)^{\frac{1}{m-1}}}, \tag{2.11}$$

for any $R' \geq R$. Let $R' \to \infty$ in (2.11), we obtain again a contradiction. Hence u is a constant.

If $1 < p < m$, then integrating (2.9),one has

$$\frac{1}{c(m,p,N)}(R^{-\frac{m-p}{p-1}} - (R')^{-\frac{m-p}{p-1}}) \leq \frac{1}{\phi(R)^{\frac{1}{p-1}}} - \frac{1}{\phi(R')^{\frac{1}{p-1}}} \leq \frac{1}{\phi(R)^{\frac{1}{p-1}}}, \tag{2.12}$$

for any $R' \geq R$. Let $R' \to \infty$ in (2.12), we thus conclude

$$\phi(R) \leq c(m,p,N)R^{m-p} \quad \text{whenever } \phi(R) > 0. \tag{2.13}$$

Now let us prove Theorem 1.1 in its full generality. If $1 < p < 2$, this has been proved above because $[p] - 1 = 0$. If $p \geq 2$, then since $\pi_1(N)$ is a finite group, the universal covering space of N, namely $\widetilde{N}$, is compact. Denote π as the natural projection map from $\widetilde{N}$ to N, and let $\widetilde{N}$ be endowed with the induced Riemannian metric $\pi^* g_N$. Note that $\widetilde{N}$ satisfies $\pi_i(\widetilde{N}) = 0$ for $1 \leq i \leq [p] - 1$.

Claim 2.1. *If $p \geq 2$, then there exists a minimizing p-harmonic map $\tilde{u} \in W^{1,p}_{loc}(\mathbb{R}^m, \widetilde{N})$ such that $\pi \circ \tilde{u} = u$.*

Proof of Claim 2.1. If $p \geq m$, then from Corollary 2.6 of [HrL] we know $u \in C(\mathbb{R}^m, N)$. Since $\mathbb{R}^m$ is simply connected, we may find a $\tilde{u} \in C(\mathbb{R}^m, \widetilde{N})$ such that $\pi \circ \tilde{u} = u$. It is clear that $\tilde{u} \in W^{1,p}_{loc}(\mathbb{R}^m, \widetilde{N})$.

If $2 \leq p < m$, then from Corollary 2.6 of [HrL] we may find a closed subset $S_u \subset \mathbb{R}^m$ such that $u|_{\mathbb{R}^m \backslash S_u}$ is locally Hölder continuous and $\mathcal{H}^{m-p}(S_u) = 0$. From Lemma 2.3 we know $\mathbb{R}^m \backslash S_u$ is simply connected, hence we may find a $\tilde{u} \in C(\mathbb{R}^m \backslash S_u, \widetilde{N})$ such that $\pi \circ \tilde{u} = u$. It is then clear that $\tilde{u} \in W^{1,p}_{loc}(\mathbb{R}^m, \widetilde{N})$.

For any $r > 0$, any $\tilde{v} \in W^{1,p}(B^m_r, \widetilde{N})$ such that $\tilde{v}|_{\partial B_r} = \tilde{u}|_{\partial B_r}$, then $v = \pi \circ \tilde{v} \in W^{1,p}(B_r, N)$ and $v|_{\partial B_r} = u|_{\partial B_r}$. From the minimality of u and the fact π is a local isometry we know

$$\int_{B_r}|d\tilde{u}|^p = \int_{B_r}|d(\pi \circ \tilde{u})|^p = \int_{B_r}|du|^p \leq \int_{B_r}|dv|^p = \int_{B_r}|d\tilde{v}|^p.$$

Hence $\tilde{u}$ is also a minimizing p-harmonic map. This proves Claim 2.1.

This reduces the general case to the special case we have treated, and hence completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We consider first the case $1 < p < m$.

Claim 2.2. *If* $\Omega = B_1^m$, *then* $\int_{B_r} |du|^p \le \frac{c(m,p,N)r^{m-p}}{(1-r)^{p-1}}$ *for* $0 < r < 1$.

Proof of Claim 2.2. First look at the case $\pi_i(N) = 0$ for $1 \le i \le [p] - 1$. Denote $\phi(r) = \int_{B_r} |du|^p$ for $0 < r < 1$, then the arguments in the proof of Theorem 1.1 gives us (see (2.12)) that, if $\phi(r) > 0$, then for any $r < s < 1$, we have

$$\frac{1}{c(m,p,N)}\left(r^{-\frac{m-p}{p-1}} - s^{-\frac{m-p}{p-1}}\right) \le \frac{1}{\phi(r)^{\frac{1}{p-1}}},$$

Let $s \to 1^-$, we get

$$\phi(r) \le c(m,p,N)\frac{r^{m-p}}{(1 - r^{\frac{m-p}{p-1}})^{p-1}} \le \frac{c(m,p,N)r^{m-p}}{(1-r)^{p-1}}.$$

Hence Claim 2.2 is true under the assumption that the target is $[p] - 1$ simply connected. The general case can be proved by the lifting argument presented above.

When Ω is an arbitrary open subset, the conclusion in Theorem 1.2 follows from Claim 2.2 by a simple scaling.

Next let us look at the case $m \le p < \infty$. In this case it follows from Corollary 2.6 of [HrL] that $u \in C^1(\Omega, N)$. Again by scalings, to prove the gradient estimate, it suffices to show the following

Claim 2.3. *If* $u \in C^1(\overline{B_1^m}, N)$ *is a minimizing* p*-harmonic map, then* $|du(x)| \le \frac{c(m,p,N)}{1-|x|}$ *for any* $x \in B_1^m$.

Proof of Claim 2.3. If the conclusion of Claim 2.3 were false, then we would find a sequence $u_i \in C(\overline{B_1^m}, N)$ such that u_i is a minimizing p-harmonic map and

$$K_i = \max_{x \in \overline{B_1}} (1 - |x|)|du_i(x)| \to \infty \quad \text{as } i \to \infty.$$

Let $x_i \in B_1$ be such that $K_i = (1 - |x_i|)|du_i(x_i)|$. Denote $\sigma_i = 1 - |x_i|$. Define $v_i(x) = u_i(x_i + \frac{\sigma_i}{K_i}x)$ for $x \in B_{K_i}$, then v_i is a minimizing p-harmonic map with $|dv_i(x)| \le \frac{1}{1-|x|/K_i}$ and $|dv_i(0)| = 1$. It follows from Theorem 3.1 of [HrL] that for any $r > 0$, $|v_i|_{C^{1,\alpha}(\overline{B_r})} \le c(m,p,r,N)$ for i large enough, here $\alpha = \alpha(m,p,N) \in (0,1)$. Hence after passing to a subsequence, we may find a $v \in C^{1,\alpha}_{loc}(\mathbb{R}^m, N)$ such that $v_i \to v$ in $C^{1,\alpha/2}_{loc}(\mathbb{R}^m)$. It is clear that v is still a locally minimizing p-harmonic map. By Theorem 1.1, v is a constant map. On the other hand, $|dv(0)| = 1$ because $|dv_i(0)| = 1$ for any i. This gives us a contradiction. We finish the proof of Claim 2.3 and hence also the Theorem 1.2. □

Proof of Theorem 1.3. Again we consider first the case $\pi_i(N) = 0$ for $1 \le i \le [p] - 1$. For any $r \ge 0$, we denote the open upper half ball as $B_r^+ = B_r \cap H_0$, and let $\phi_+(r) = \int_{B_r^+} |du|^p$. Replacing B_r in the proof of Theorem 1.1 by B_r^+ and we observe that

$$\int_{\partial B_r^+} |d(u|_{\partial B_r^+})|^p d\mathcal{H}^{m-1} = \int_{\partial B_r \cap H_0} |d(u|_{\partial B_r^+})|^p d\mathcal{H}^{m-1} \tag{2.14}$$

$$\leq \int_{\partial B_r \cap H_0} |du|^p d\mathcal{H}^{m-1} = \phi_+'(r) \quad \text{for } r > 0.$$

(2.9) remains true if we replace ϕ by ϕ_+. When $m \leq p < \infty$, we prove in the same way as before that $\phi_+ \equiv 0$, that is u is a constant map. If $1 < p < m$, via the proof of Theorem 1.1 we get

$$\int_{B_r^+} |du|^p \leq c(m,p,N) r^{m-p} \quad \text{for } r > 0. \tag{2.15}$$

To show u is a constant map, we need the monotonicity formula.

CLAIM 2.4 (Monotonicity identity). *For almost every $r > 0$, we have*

$$\frac{d}{dr}\left(r^{p-m}\int_{B_r^+} |du|^p d\mathcal{H}^m\right) = pr^{p-m}\int_{\partial B_r \cap H_0} |\partial_r u|^2 |du|^{p-2} d\mathcal{H}^{m-1}. \tag{2.16}$$

Proof of Claim 2.4. See Lemma 4.1 in [HrL].

Define a function ρ_u by $\rho_u(r) = r^{p-m}\int_{B_r^+} |du|^p$ for $r > 0$. From Claim 2.4 and (2.15) we know ρ_u is a bounded increasing function. Hence there is a limit $\rho_u(\infty) \in \mathbb{R}$. For any $\lambda > 0$, we denote $u_\lambda(x) = u(x/\lambda)$ for $x \in H_0$. Then $\rho_{u_\lambda}(r) = \rho_u(r/\lambda)$. From the proof of Corollary 2.8 and Theorem 6.4 in [HrL] or [Lu1], [Lu2] we know there exists a $v \in W^{1,p}_{loc}(\overline{H_0}, N)$ and a sequence of positive numbers $\lambda_i \to 0$ such that $u_{\lambda_i} \to v$ in $W^{1,p}_{loc}(\overline{H_0}, N)$ and v is a minimizing p-harmonic map. By the strong convergence, one has $\rho_v(r) \equiv \rho_u(\infty)$, and hence by (2.16) we get $\partial_r v = 0$. Since v is a constant map on ∂H_0, it follows from Theorem 5.7 of [HrL] that v itself is a constant map. The latter implies $\rho_u(\infty) \equiv \rho_v(r) = 0$, and therefore u is a constant map.

Theorem 1.3 in its full generality can be proved by the same lifting argument as that in the proof of Theorem 1.1. □

Proof of Corollary 1.1. This follows from Theorem 1.2 and the Luckhaus Compactness Theorem (see [Lu1] and [Lu2]). □

3. Minimal solutions of a simplified Landau-Lifschitz equation. The aim of this section is to classify all blow-up limits of minimizers of I_ε (see (1.1)). That is we want to study minimal solutions of the simplified Landau-Lifschitz equation

$$-\triangle u = \left(|\nabla u|^2 + \left(u^3\right)^2\right) u - u^3 e_3 \tag{3.1}$$

for a S^2 valued u defined on the entire plane.

To proceed, we need the following gradient estimate.

PROPOSITION 3.1. *Suppose $u \in C^\infty(\overline{B_1}, S^2)$ satisfies (3.1) in B_1. If u minimizes I_1 on $\overline{B_1}$, then $|\nabla u(x)| \leq \frac{c}{1-|x|}$ on B_1, here c is an absolute constant.*

Proof. Otherwise, we would find a sequence $u_j \in C^\infty(\overline{B_1}, S^2)$, minimizing I_1 on $\overline{B_1}$ and

$$K_j = \sup_{x \in \overline{B_1}} (1-|x|)|\nabla u_j(x)| \to \infty.$$

Choose $x_j \in B_1$ such that $(1-|x_j|)|\nabla u_j(x_j)| = K_j$, put $\sigma_j = 1-|x_j|$, and define $v_j(x) = u_j(x_j + \frac{\sigma_j}{K_j}x)$ for $x \in B_{K_j}$. Then

$$-\triangle v_j = \left(|\nabla v_j|^2 + \frac{\sigma_j^2\left(v_j^3\right)^2}{K_j^2}\right) v_j - \frac{\sigma_j^2 v_j^3}{K_j^2} e_3 \text{ on } B_{K_j},\ |\nabla v_j(x)| \leq \frac{1}{1-\frac{|x|}{K_j}},\ |\nabla v_j(0)| = 1,$$

and v_j minimizes $I_{\frac{K_j}{\sigma_j}}$. Hence $|v_j|_{C^{1,\alpha}(\overline{B_r})} \le c(\alpha, r)$. After passing to a subsequence we may assume $v_j \to v$ in $C^\infty(\mathbb{R}^2)$, then $v \in C^\infty(\mathbb{R}^2, S^2)$ and

$$-\triangle v = |\nabla v|^2 v \text{ on } \mathbb{R}^2,\ |\nabla v(0)| = 1,\ |\nabla v(x)| \le 1.$$

Moreover, v is a locally minimizing harmonic map. It follows from Theorem 1.1 that v is a constant, we obtain a contradiction. □

We also need the following edition of Theorem 2.1. The key point here is that the constant doesn't depend on domain Ω.

LEMMA 3.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open subset with Lipschitz boundary, $n \ge 2$, $u \in H^1(\Omega, \mathbb{R}^{n+1})$ such that $u|_{\partial\Omega} \in S^n$, then there exists a $\tilde{u} \in H^1(\Omega, S^n)$ such that*

$$\tilde{u}|_{\partial\Omega} = u|_{\partial\Omega} \quad \textit{and} \quad \int_\Omega |\nabla \tilde{u}|^2 \le c(n) \int_\Omega |\nabla u|^2. \tag{3.2}$$

Proof. For any $a \in B^{n+1}_{\frac{1}{2}}$, let $u_a(x) = \frac{u(x)-a}{|u(x)-a|}$. Since

$$|\nabla u|^2 = |\nabla |u-a||^2 + |u-a|^2 |\nabla u_a|^2,$$

we have $|\nabla u_a|^2 \le \frac{|\nabla u|^2}{|u-a|^2}$. Integrating both a and x, we get

$$\int_{B_{\frac{1}{2}}} da \int_\Omega |\nabla u_a(x)|^2 dx \le \int_{B_{\frac{1}{2}}} da \int_\Omega \frac{|\nabla u|^2}{|u-a|^2} dx$$

$$= \int_\Omega dx \int_{B_{\frac{1}{2}}} \frac{|\nabla u(x)|^2}{|u-a|^2} da \le c(n) \int_\Omega |\nabla u(x)|^2 dx.$$

Hence we may find a $b \in B_{\frac{1}{2}}$ such that $\int_\Omega |\nabla u_b(x)|^2 dx \le c(n) \int_\Omega |\nabla u(x)|^2 dx$. For any $a \in B_{\frac{1}{2}}$, define $P_a : S^n \to S^n$ by $P_a(y) = \frac{y-a}{|y-a|}$, then P_a is a diffeomorphism with $|\nabla_{S^n} P_a^{-1}(y)| + |\nabla_{S^n} P_a(y)| \le c(n)$ for $y \in S^n$. Let $\tilde{u}(x) = P_b^{-1}(u_b(x))$, then $\tilde{u}$ is the needed map. □

We note the method above was introduced in Section 6 of [HrL]. Now we may turn to Theorem 1.4.

Proof of Theorem 1.4. From Proposition 3.1 we deduce

$$|\nabla u(x)| \le c \quad \text{for } x \in \mathbb{R}^2. \tag{3.3}$$

Next we will combine Lemma 3.1 together with the comparison method in [Sa2] to show u has nice decay properties.

CLAIM 3.1. $I_1(u, B_R) \le cR$ *for $R \ge 0$.*

Proof of Claim 3.1. We may assume $R \ge 2$, define

$$u_R(x) = \begin{cases} (R - |x|)e_1 + (|x| - R + 1)u(x), & \text{if } R-1 \le |x| \le R, \\ e_1, & \text{if } |x| \le R-1, \end{cases}$$

here $e_1 = (1, 0, 0)$. From (3.3) we know $|\nabla u_R(x)| \le c$. Hence $\int_{B_R \backslash B_{R-1}} |\nabla u_R|^2 \le cR$. By Lemma 3.1 we may find a $\tilde{u}_R \in H^1(B_R \backslash B_{R_1}, S^2)$ such that $\tilde{u}_R|_{\partial B_R \cup \partial B_{R-1}} =$

$u_R|_{\partial B_R \cup \partial B_{R-1}}$ and $\int_{B_R \setminus B_{R-1}} |\nabla \tilde{u}_R|^2 \leq cR$. Let $\tilde{u}_R = e_1$ in B_{R-1}, then $I_1(u, B_R) \leq I_1(\tilde{u}_R, B_R) \leq cR$. This proves Claim 3.1.

In the next step we want to show indeed the growth of $I_1(u, B_R)$ is sublinear in R.

Claim 3.2. $I_1(u, B_R) \leq cR^{\frac{3}{4}}$ *for* $R \geq 0$.

Proof of Claim 3.2. We may assume $R \geq 4$. Via Claim 3.1 we have $I_1(u, B_{2R} \setminus B_R) \leq cR$. Hence we may find $R_1 \in [R, 2R]$ such that

$$\int_{\partial B_{R_1}} \left(|\nabla u|^2 + (u^3)^2\right) ds \leq c. \tag{3.4}$$

Let $B = \{x \backslash x \in \partial B_{R_1}, |u^3(x)| \geq \frac{1}{2}\}$. By estimates (3.3) and (3.4), and by a covering argument we may find a finite number of unit length arcs on ∂B_{R_1}, namely $I_1, \cdots, I_m$ such that $\cup_{j=1}^m I_j \supset B$ and $m \leq c$. Denote $\tilde{I} = \partial B_{R_1} \setminus \cup_j I_j$. Let $v = \Gamma^{-1} \circ u$ on $\tilde{I}$, Γ is the stereographic projection, that is

$$\Gamma : \mathbb{R}^2 \to S^2 \backslash \{(0,0,-1)\}, \quad \Gamma(y^1, y^2) = \left(\frac{2y^1}{1+|y|^2}, \frac{2y^2}{1+|y|^2}, \frac{1-|y|^2}{1+|y|^2}\right). \tag{3.5}$$

From co-area formula we have

$$\int_{S^1} \# \left(\left(\frac{v}{|v|}\right)^{-1} (\{\xi\}) \right) d\mathcal{H}^1(\xi) = \int_{\tilde{I}} |\partial_\tau \left(\frac{v}{|v|}\right)| d\mathcal{H}^1 \leq c \int_{\tilde{I}} |\partial_\tau v| d\mathcal{H}^1 \tag{3.6}$$

$$\leq c\sqrt{R} \left(\int_{\partial B_{R_1}} |\nabla v|^2 d\mathcal{H}^1 \right)^{\frac{1}{2}} \leq c\sqrt{R} \left(\int_{\partial B_{R_1}} |\nabla u|^2 d\mathcal{H}^1 \right)^{\frac{1}{2}} \leq c\sqrt{R}.$$

Hence we may find a $\xi_0 \in S^1$ such that

$$\# \left(\left(\frac{v}{|v|}\right)^{-1} (\{\xi_0\}) \right) \leq c\sqrt{R}. \tag{3.7}$$

For simplicity we assume $\xi_0 = -1$, then let $J_1, \cdots, J_n$ be those unit length arcs centered at points in $(\frac{v}{|v|})^{-1}(\{-1\})$. Let $G = \partial B_{R_1} \backslash ((\cup_j I_j) \cup (\cup_k J_k))$. On G we write $v(x) = \rho(x) e^{i\alpha(x)}$ with $|\alpha| < \pi$. Fix a $\delta \in (0, \frac{R}{2})$ to be determined later, let $V = \{x \backslash x \in B_{R_1} \backslash B_{R_1 - \delta}, \frac{R_1 x}{|x|} \in G\}$, then for $x \in V$, we set

$$v_R(x) = \left(\frac{R_1 - |x|}{\delta} + \frac{|x| - R_1 + \delta}{\delta} \rho\left(\frac{R_1 x}{|x|}\right) \right) e^{i \frac{|x| - R_1 + \delta}{\delta} \alpha\left(\frac{R_1 x}{|x|}\right)}.$$

Let $\underline{u_R(x)} = \Gamma(v_R(x))$ for $x \in V$, $u_R(x) = u(x)$ for $x \in \partial B_{R_1}$ and $u_R(x) = e_1$ for $x \in \overline{B_{R_1 - \delta}}$. Then we check that

$$\mathrm{Lip}(u_R, V \cup \partial B_{R_1} \cup \partial B_{R_1 - \delta}) \leq c.$$

Hence we may extend u_R to $B_{R_1} \backslash B_{R_1 - \delta}$ such that

$$\mathrm{Lip}(u_R, B_{R_1} \backslash B_{R_1 - \delta}) \leq c. \tag{3.8}$$

A computation using the polar coordinates and the stereographic coordinates yields

$$\int_V |\nabla u_R|^2 + (u_R^3)^2 \le c\frac{R}{\delta}. \tag{3.9}$$

By (3.7) and (3.8) we get

$$\int_{(B_{R_1}\backslash B_{R_1-\delta})\backslash V} |\nabla u_R|^2 \le c\sqrt{R}\delta.$$

From Lemma 3.1 we may find a $\tilde{u}_R \in H^1((B_{R_1}\backslash B_{R_1-\delta})\backslash V, S^2)$ such that

$$\int_{(B_{R_1}\backslash B_{R_1-\delta})\backslash V} |\nabla \tilde{u}_R|^2 \le c\sqrt{R}\delta \quad \text{and} \quad \tilde{u}_R|_{\partial((B_{R_1}\backslash B_{R_1-\delta})\backslash V)} = u_R|_{\partial((B_{R_1}\backslash B_{R_1-\delta})\backslash V)}.$$

Let $\tilde{u}_R$ be equal to u_R on $B_{R_1-\delta}\cup V$, then

$$I_1(u, B_R) \le I_1(u, B_{R_1}) \le I_1(\tilde{u}_R, B_{R_1}) \le c(\frac{R}{\delta} + \sqrt{R}\delta). \tag{3.10}$$

By taking $\delta = R^{\frac{1}{4}}$ in (3.10), we obtain the Claim 3.2.

Now we proceed to show the growth of $I_1(u, B_R)$ is at most of order $\log R$. That is

Claim 3.3. *For R large enough, $I_1(u, B_R) \le c\log R$, here c is an absolute constant.*

Proof of Claim 3.3. Denote $\phi(R) = I_1(u, B_R)$. Given $R > 0$, choose $R_1 \in [R, 2R]$ such that

$$\phi'(R_1) = \min_{R\le r\le 2R} \phi'(r). \tag{3.11}$$

From Claim 3.2 we know

$$\int_R^{2R} \phi'(r)dr \le \phi(2R) \le cR^{\frac{3}{4}}.$$

This and (3.11) imply that

$$\int_{\partial B_{R_1}} \left(|\nabla u|^2 + (u^3)^2\right) ds = 2\phi'(R_1) \le \frac{c}{R^{\frac{1}{4}}}. \tag{3.12}$$

Combining(3.12) and (3.3), one has $|u^3| \le \frac{1}{2}$ on ∂B_{R_1} when R is large enough. Let $v_R(x) = \Gamma^{-1}(u(x))$ for $x \in \partial B_{R_1}$. For each $x \in B_{R_1}\backslash B_{R_1-1}$, set

$$v_R(x) = (R_1 - |x|)\frac{v_R\left(\frac{R_1 x}{|x|}\right)}{|v_R\left(\frac{R_1 x}{|x|}\right)|} + (|x| - R_1 + 1)v_R\left(\frac{R_1 x}{|x|}\right).$$

We have

$$\int_{\partial B_{R_1-1}} |\partial_\tau v_R| ds \le c\sqrt{R}\left(\int_{\partial B_{R_1-1}} |\partial_\tau v_R|^2 ds\right)^{\frac{1}{2}} \le c(R\phi'(R_1))^{\frac{1}{2}} \le c(R\phi'(R))^{\frac{1}{2}}. \tag{3.13}$$

Define a continuous function $\alpha : [0,2\pi] \to \mathbb{R}$ such that $v_R((R_1-1)e^{i\theta}) = e^{i\alpha(\theta)}$ for $0 \le \theta \le 2\pi$. Let $\theta_0 = 0$, choose $\theta_1 \in [0,2\pi]$ be such that $e^{i\alpha(\theta_1)} = e^{i\alpha(0)}$, $|\alpha(\theta_1)-\alpha(0)| = 2\pi$ and $|\alpha(\theta)-\alpha(0)| < 2\pi$ for any $0 \le \theta < \theta_1$. Starting from θ_1, we may inductively define $\theta_2, \theta_3, \cdots, \theta_n$, until we come back to $\theta = 2\pi$. From (3.13) we get a rough bound $n \le c(R\phi'(R))^{\frac{1}{2}}$. $(R_1-1)e^{i\theta_0}, \cdots, (R_1-1)e^{i\theta_n}$ breaks ∂B_{R_1-1} into arcs. First we assume there are some arcs on which the degree of v_R is $+1$ or -1, then after combining neighbored arcs, we may assume on each arc (which could be a union of several original arcs) v_R has degree $+1$ or -1. For every such resulting arc I_j, we let J_j be the circular arc (with the circle's center at the intersection of two tangent lines at ∂I_j) lie inside the ∂B_{R_1-1} and orthogonal to I_j at ∂I_j. I_j and J_j together encloses a domain called Ω_j. Set $v_R|_{B_{R_1-1}\setminus \cup_j \Omega_j} = e^{i\alpha(0)}$. Choose $a_j \in \Omega_j$ such that $B_{2r_0}(a_j) \subset \Omega_j$ for some $r_0 > 0$, an absolute constant. Suppose the degree of v_R on I_j is $+1$, then let

$$v_R|_{\partial\Omega_j} = \frac{x-a_j}{|x-a_j|} e^{i\varphi_j(x)}.$$

Set φ_j in Ω_j as the harmonic extension of the boundary function. Let

$$v_R|_{\Omega_j \setminus B_{r_0}(a_j)} = \frac{x-a_j}{|x-a_j|} e^{i\varphi_j(x)},$$

and $v_R|_{B_{r_0}(a_j)}$ be the harmonic extension of $v_R|_{\partial B_{r_0}(a_j)}$. We may proceed similarly for the degree -1 case. If no arc has nonzero degree, then we have $v_R|_{\partial B_{R_1-1}} = e^{i\varphi}$. Then using the harmonic extension to define φ inside B_{R_1-1}, and let $v_R|_{B_{R_1-1}} = e^{i\varphi}$. Let $u_R = \Gamma \circ v_R$, by a careful computation as in [Sa2], we have

$$\phi(R) \le \phi(R_1) = I_1(u, B_{R_1}) \le I_1(u_R, B_{R_1}) \tag{3.14}$$

$$\le c(R\phi'(R))^{\frac{1}{2}} \log R + c \log R \quad \text{for } R \text{ large enough,}$$

If we put $\tilde\phi(R) = \phi(R) + \log R$, then (3.14) implies

$$\tilde\phi(R) \le c(R\tilde\phi'(R))^{\frac{1}{2}} \log R. \tag{3.15}$$

In other words

$$\frac{1}{cR\log^2 R} \le \frac{\tilde\phi'(R)}{\tilde\phi(R)}^2. \tag{3.16}$$

By integrating on both sides we get for any $\widetilde R > R$,

$$\frac{1}{\tilde\phi(R)} \ge \frac{1}{\tilde\phi(R)} - \frac{1}{\tilde\phi(\widetilde R)} \ge \frac{1}{c}\left(\frac{1}{\log R} - \frac{1}{\log \widetilde R}\right). \tag{3.17}$$

Let $\widetilde R \to \infty$, we get $\tilde\phi(R) \le c \log R$ for R large. This implies $I_1(u, B_R) \le c \log R$ and Claim 3.3 is proved.

Claim 3.3 along with the Pohozaev's identity (which follows from multiplying (3.1) by $x^j\partial_j u$ and integrating by parts, one may see Lemma 4.4 in [HnL])

$$\int_{B_r} (u^3)^2 + \frac{r}{2}\int_{\partial B_r} |\partial_\nu u|^2 ds = \frac{r}{2}\int_{\partial B_r} |\partial_\tau u|^2 ds + \frac{r}{2}\int_{\partial B_r} (u^3)^2 ds, \tag{3.18}$$

yields $\int_{\mathbb{R}^2}(u^3)^2dx < \infty$. Combining the last fact with (3.3), we get $u^3 \to 0$ as $|x| \to \infty$. Further estimates of u^3 and $|\nabla u|$ follow from Proposition 6.1 in [HnL]. To obtain the degree of the map at ∞, we assume for $|x| \geq R_0$, $|u^3(x)| \leq \frac{1}{2}$, then $\Gamma^{-1}\circ u = \rho e^{i(d\theta+\psi)}$, Γ is the stereographic projection defined in (3.5), d is the degree of $\frac{(u^1,u^2)}{|(u^1,u^2)|}$ at ∞. For $R \geq 2R_0$, from the Annulus Lemma (see [BMR] or Lemma 4.1 in [HnL]) we have

$$\frac{1}{2}\int_{B_R\backslash B_{R_0}} |\nabla u|^2 \geq \pi d^2 \log\frac{R}{R_0} - c(u). \tag{3.19}$$

On the other hand, if we set $\tilde{v}(x) = \tilde{\rho}e^{i(d\theta+\tilde{\psi})}$ on $B_R\backslash B_{\frac{R}{2}}$, where

$$\tilde{\rho}(x) = 1 + \frac{|x|-\frac{R}{2}}{\frac{R}{2}}(\rho(x)-1), \quad \tilde{\psi}(x) = \psi_R + \frac{|x|-\frac{R}{2}}{\frac{R}{2}}(\psi-\psi_R), \quad \psi_R = \fint_{B_R\backslash B_{\frac{R}{2}}}\psi,$$

and $\tilde{u} = \Gamma\circ\tilde{v}$, then

$$\frac{1}{2}\int_{B_R\backslash B_{\frac{R}{2}}} |\nabla\tilde{u}|^2 + (\tilde{u}^3)^2 = \int_{B_R\backslash B_{\frac{R}{2}}} \frac{2(|\nabla\tilde{v}|^2 + \frac{(1-|\tilde{v}|^2)^2}{4})}{(1+|\tilde{v}|^2)^2} \leq c(u). \tag{3.20}$$

By the fact that $|\rho-1|$, $|\nabla\rho|$ decay exponentially at ∞, one has via Poincare's inequality that $|\nabla\psi(x)| = O(|x|^{-2})$ (see Proposition 6.1 in [HnL]). We note that $\tilde{u}(x) = e^{i(d\theta+\psi_R)}$ on $\partial B_{\frac{R}{2}}$, from Lemma 4.3 in [HnL] we may choose $\tilde{u}$ on $B_{\frac{R}{2}}$ such that

$$\frac{1}{2}\int_{B_{\frac{R}{2}}} |\nabla\tilde{u}|^2 + (\tilde{u}^3)^2 \leq \pi|d|\log R + c, \tag{3.21}$$

where c is an absolute constant. Combining (3.20) and (3.21), and energy minimizing property of u, we conclude

$$\frac{1}{2}\int_{B_R} |\nabla u|^2 + (u^3)^2 \leq \pi|d|\log R + c(u). \tag{3.22}$$

Applying (3.19) and (3.22), and letting $R \to \infty$, we get $d^2 \leq |d|$. Hence $d = 0$, $+1$ or -1. If $d = 0$, from Proposition 6.1 in [HnL] we know $u^3 \equiv 0$. An estimate for harmonic function and $|\nabla u(x)| \leq \frac{c(u)}{|x|}$ tells us $u \equiv$ *const.*.

Claim 3.4. *$\tilde{u} = (u^1, u^2, |u^3|)$ is locally minimizing I_1.*

Proof of Claim 3.4. For $R \geq R_0$, define

$$w_R(x) = \Pi(u(x) + (R+1-|x|)(\tilde{u}(x)-u(x))) \quad \text{for } x \in B_{R+1}\backslash B_R.$$

Here $\Pi(\xi) = \frac{\xi}{|\xi|}$ for $\xi \in \mathbb{R}^3\backslash\{0\}$. From the estimates for u, one easily verifies

$$I_1(w_R, B_{R+1}\backslash B_R) = o(1), \quad I_1(u, B_{R+1}\backslash B_R) = o(1) \quad \text{as } R \to \infty.$$

For any $v \in H^1(B_{R_1}, S^2)$, $v|_{\partial B_{R_1}} = \tilde{u}|_{\partial B_{R_1}}$, $R_1 \geq R_0$, pick up a $R > R_1$, extend v to B_{R+1} by setting $v|_{B_R\backslash B_{R_1}} = \tilde{u}$, $v|_{B_{R+1}\backslash B_R} = w_R$. Via minimizing property of u we know $I_1(v, B_{R+1}) \geq I_1(u, B_{R+1})$. But

$$I_1(v,B_{R+1})-I_1(u,B_{R+1}) = I_1(v,B_{R_1})-I_1(u,B_{R_1})+I_1(w_R,B_{R+1}\backslash B_R)-I_1(u,B_{R+1}\backslash B_R)$$

$$= I_1(v, B_{R_1}) - I_1(\tilde{u}, B_{R_1}) + o(1).$$

Let $R \to \infty$, we get $I_1(v, B_{R_1}) \geq I_1(\tilde{u}, B_{R_1})$, hence Claim 3.4 is proved.

From Claim 3.4 we know $\tilde{u}$ is smooth and satisfies (3.1). Since $\tilde{u}^3 \geq 0$, from the equation of third component we know either $\tilde{u}^3 > 0$ or $\tilde{u}^3 \equiv 0$. The first case implies $u^3 > 0$ or $u^3 < 0$. The second case implies $u \equiv const$. $\square$

REMARK 3.1. For any $c \in \mathbb{R}$, $|c| \leq 1$, $u(x) = (\sqrt{1-c^2}\cos x^1, \sqrt{1-c^2}\sin x^1, c)$ is a solution to (3.1). Clearly these are not local minimizers.

REMARK 3.2. It is of interest to prove that under translation, rotation, and reflection with respect to the x^1 axis and the horizontal plane, the degree 1 radial solution in Proposition 5.2 in [HnL] is the unique nonconstant local minimizer. In the Ginzburg-Landau model case, the corresponding problem was solved in [Mi].

To prove Theorem 1.5, we need the following boundary version of Proposition 3.1.

PROPOSITION 3.2. *Denote $B_1^+ = B_1 \cap H_0$, $L_1 = B_1 \cap \partial H_0$, where H_0 is the open upper half plane. Suppose $u \in C^\infty(\overline{B_1^+}, S^2)$ satisfies (3.1) and it locally minimizes I_1 in B_1^+, $u|_{L_1} \equiv const$, then $|\nabla u(x)| \leq \frac{c}{1-|x|}$, here c is an absolute constant.*

Proof. The proof goes almost the same as the one for Proposition 3.1, except in case we get half plane in the blow-up limit, we use Theorem 1.3 to find a contradiction. One may refer to the proofs of Theorem 3.1 and Proposition 6.3 in [HnL]. $\square$

Proof of Theorem 1.5. Without losing of generality we may assume $e = e_1 = (1, 0, 0)$. From Proposition 3.1 and Proposition 3.2 we get

$$|\nabla u(x)| \leq c \quad \text{for any } x \in \overline{H_0}. \tag{3.23}$$

Here c is an absolute constant. Denote $B_R^+ = B_R \cap H_0$, we may show as for Theorem 1.4 that, for R large enough,

$$I_1(u, B_R^+) \leq c \log R \tag{3.24}$$

for some absolute constant c.

By Pohozaev's identity (see Lemma 4.4 in [HnL]) we have

$$\int_{B_R^+} (u^3)^2 + \frac{R}{2}\int_{\partial B_R \cap H_0} |\partial_\nu u|^2 ds = \frac{R}{2}\int_{\partial B_R \cap H_0} (|\partial_\tau u|^2 + (u^3)^2) ds. \tag{3.25}$$

From (3.24) one may find a sequence $R_j \to \infty$ such that

$$R_j \int_{\partial B_{R_j} \cap H_0} \left(|\nabla u|^2 + (u^3)^2\right) ds \leq c. \tag{3.26}$$

(3.25) and (3.26) together imply

$$\int_{H_0} (u^3)^2 \leq c, \tag{3.27}$$

here c is an absolute constant. Next, using (3.23), one has $u^3 \to 0$ as $|x| \to \infty$. Choose $R_0 > 0$ such that $|u^3(x)| \leq \frac{1}{2}$ for $x \in H_0 \backslash B_{R_0}^+$, then $u' = u^1 + iu^2 = \rho e^{i\varphi}$ with

$\rho = |u'| \geq \frac{\sqrt{3}}{2}$, $\varphi = 2d\theta + \psi$. d is the degree of $\frac{u'}{|u'|}$ on ∂B_R^+, $\psi(x^1, 0) \equiv 0$. A simple computation shows $\operatorname{div}(\rho^2 \nabla\varphi) = 0$.

CLAIM 3.5. $\int_{H_0 \backslash B_{R_0}^+} |\nabla\psi|^2 < \infty$.

Proof of Claim 3.5. Denote $A_R^+ = B_R^+ \backslash \overline{B_{R_0}^+}$, then

$$\int_{A_R^+} \rho^2 (2d\nabla\theta + \nabla\psi) \cdot \nabla\psi = \int_{A_R^+} \operatorname{div}(\psi\rho^2\nabla\varphi)$$

$$= (\int_{\partial B_R \cap H_0} - \int_{\partial B_{R_0} \cap H_0}) \rho^2 \frac{\partial\varphi}{\partial\nu} \psi ds = \int_{\partial B_R \cap H_0} \rho^2 \frac{\partial\psi}{\partial\nu} \psi ds - c(u).$$

Since $\int_{\partial B_r \cap H_0} \nabla\theta \cdot \nabla\psi ds = 0$, we have

$$\int_{A_R^+} \rho^2 |\nabla\psi|^2 \leq \int_{\partial B_R \cap H_0} |\frac{\partial\psi}{\partial\nu}||\psi| ds + \int_{A_R^+} (1-\rho^2) \frac{2|d|}{r} |\nabla\psi| + c(u).$$

By the Poincare and Holder inequalities, we have

$$\int_{\partial B_R \cap H_0} |\frac{\partial\psi}{\partial\nu}||\psi| ds \leq \frac{R}{2} \int_{\partial B_R \cap H_0} |\nabla\psi|^2 ds,$$

and

$$\int_{A_R^+} (1-\rho^2) \frac{2|d|}{r} |\nabla\psi| \leq c(u) \int_{A_R^+} (u^3)^2 dx \leq c(u) < \infty.$$

Here we use the fact $|\nabla\psi| \leq c$, which follows from (3.23), also we use (3.27). We, therefore, obtain

$$\int_{A_R^+} |\nabla\psi|^2 \leq cR \int_{H_0 \cap \partial B_R} |\nabla\psi|^2 ds + c(u).$$

Since $\int_{A_R^+} |\nabla\psi|^2 \leq c(u) \log R$, by choosing a sequence of generic radius $R_j \to \infty$, the right hand side with $R = R_j$ remains bounded, we get $\int_{H_0 \backslash B_{R_0}^+} |\nabla\psi|^2 \leq c(u) < \infty$. This proves Claim 3.5.

Multiplying the third component's equation by u^3 and integrating by parts we get

$$\int_{A_R^+} |\nabla u^3|^2 + (u^3)^2 = \int_{A_R^+} (u^3)^2 (|\nabla u|^2 + (u^3)^2) + \int_{\partial B_R \cap H_0} u^3 \frac{\partial u^3}{\partial\nu} ds - \int_{\partial B_{R_0} \cap H_0} u^3 \frac{\partial u^3}{\partial\nu} ds.$$

Combining (3.23) with (3.27) we have

$$\int_{A_R^+} |\nabla u^3|^2 \leq c(u) + c(R \int_{\partial B_R \cap H_0} |\nabla u^3|^2 ds)^{\frac{1}{2}}.$$

By choosing $R_j \to \infty$ such that $R_j \int_{\partial B_{R_j} \cap H_0} |\nabla u^3|^2 ds \leq c$, we obtain

$$\int_{H_0} |\nabla u^3|^2 \leq c(u) < \infty. \tag{3.28}$$

For ρ we have

$$\int_{H_0\backslash B^+_{R_0}} |\nabla\rho|^2 = \int_{H_0\backslash B^+_{R_0}} \frac{(u^3)^2|\nabla u^3|^2}{1-(u^3)^2} \leq c\int_{H_0} (u^3)^2 < \infty. \tag{3.29}$$

Using (3.25) one has

$$\int_{B^+_R} (u^3)^2 = 2\pi d^2 + \mathcal{R},$$

where

$$|\mathcal{R}| \leq c(u)\left(R\int_{\partial B_R\cap H_0} (|\nabla\rho|^2 + |\nabla\psi|^2 + |\nabla u^3|^2 + (u^3)^2)ds + (R\int_{\partial B_R\cap H_0} |\nabla\psi|^2 ds)^{\frac{1}{2}}\right).$$

Since $\int_{H_0\backslash B^+_{R_0}} |\nabla\rho|^2 + |\nabla\psi|^2 + |\nabla u^3|^2 + (u^3)^2 < \infty$, we may find $R_j \to \infty$ such that

$$R_j\int_{\partial B_{R_j}\cap H_0} (|\nabla\rho|^2 + |\nabla\psi|^2 + |\nabla u^3|^2 + (u^3)^2)ds \to 0.$$

Hence

$$\int_{H_0} (u^3)^2 = 2\pi d^2. \tag{3.30}$$

Next we want to derive a lower bound for the energy. We have

$$\frac{1}{2}\int_{B^+_R\backslash B^+_{R_0}} |\nabla u|^2 \geq \frac{1}{2}\int_{B^+_R\backslash B^+_{R_0}} |\nabla u'|^2 \geq \frac{1}{2}\int_{B^+_R\backslash B^+_{R_0}} \rho^2|\nabla\varphi|^2 \tag{3.31}$$

$$\geq \frac{1}{2}\int_{B^+_R\backslash B^+_{R_0}} |\nabla\varphi|^2 - c(u) \geq 2\pi d^2\log\frac{R}{R_0} + 2d\int_{B^+_R\backslash B^+_{R_0}} \partial_\theta\psi - c(u) = 2\pi d^2\log\frac{R}{R_0} - c(u).$$

Let Γ be the stereographic projection defined in (3.5), we may write $\Gamma^{-1}\circ u = \rho_1 e^{i(2d\theta+\psi)}$, $\rho_1 = \frac{\rho}{1+u^3}$. Set $\tilde{v} = \tilde{\rho}e^{i(2d\theta+\tilde{\psi})}$ on $B^+_R\backslash B^+_{\frac{R}{2}}$, where

$$\tilde{\rho}(x) = 1 + \frac{|x|-\frac{R}{2}}{\frac{R}{2}}(\rho_1(x)-1), \quad \tilde{\psi}(x) = \frac{|x|-\frac{R}{2}}{\frac{R}{2}}\psi(x),$$

and $\tilde{u} = \Gamma\circ\tilde{v}$, then

$$\frac{1}{2}\int_{B^+_R\backslash B^+_{\frac{R}{2}}} |\nabla\tilde{u}|^2 + (\tilde{u}^3)^2 = \int_{B^+_R\backslash B^+_{\frac{R}{2}}} \frac{2\left(|\nabla\tilde{v}|^2 + \frac{(1-|\tilde{v}|^2)^2}{4}\right)}{(1+|\tilde{v}|^2)^2} \leq c(u). \tag{3.32}$$

Here one uses the fact that $\int_{H_0\backslash B^+_R} |\nabla\rho_1|^2 < \infty$, $\int_{H_0\backslash B^+_{R_0}} (\rho_1-1)^2 < \infty$ and $\int_{H_0} |\nabla\psi|^2 < \infty$. Note that we have $\tilde{u}(x) = e^{2id\theta}$ on $\partial B^+_{\frac{R}{2}}\cap H_0$, from Lemma 4.3 in [HnL] we may choose $\tilde{u}$ on $B^+_{\frac{R}{2}}$ such that

$$\frac{1}{2}\int_{B^+_{\frac{R}{2}}} |\nabla\tilde{u}|^2 + (\tilde{u}^3)^2 \leq \pi|d|\log R + c(d). \tag{3.33}$$

Via (3.32) and (3.33), we get

$$\frac{1}{2}\int_{B_R^+}|\nabla u|^2+(u^3)^2\le\frac{1}{2}\int_{B_R^+}|\nabla\tilde{u}|^2+(\tilde{u}^3)^2\le\pi|d|\log R+c(u). \tag{3.34}$$

Combining (3.31) and (3.34) and letting $R\to\infty$, we see $2d^2\le|d|$. Hence $d=0$ and by (3.30), $u^3\equiv 0$. Thus $u(x)=(e^{i\varphi(x)},0)$, φ is a harmonic function on $\overline{H_0}$ with $\varphi|_{\partial H_0}\equiv 0$. Since $|\nabla\varphi|\le c$, we have $\varphi(x)=c_2x^2$ for some $c_2\in\mathbb{R}$. Now it follows from (3.24) that $c_2=0$, hence $u\equiv(1,0,0)$. $\square$

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