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REMARKS ON REFINED KIRBY CALCULUS FOR THREE-MANIFOLDS OF CYCLIC FIRST HOMOLOGY GROUPS OF ODD PRIME POWER ORDERS

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Abstract

Habiro arranged Kirby moves into a pair so that it preserves linking matrices. The author showed that two framed links of diagonal linking matrices yield homeomorphic 3-manifolds of linking form $(\pm 1/p)$ for an odd prime p if and only if they are related by a sequence of Habiro moves. We generalize this result to 3-manifolds of linking forms $(\pm 1/c)$ for any odd prime power c.

1. Introduction

Every orientable connected closed 3-manifold is obtained by surgery along an integral framed link in S^3 [4, 5]. Two such links yield homeomorphic manifolds if and only if they are related by a sequence of Kirby moves ((de)stabilizations and handle slides) [3]. Here, *stabilization* is introducing a (± 1)-framed trivial component to a framed link and a *handle slide* is deforming a link component as a band connected sum with the curve representing the framing of another component (see [3]). A handle slide changes framing and linking number.

A symmetric integral matrix is called the *linking matrix* of an oriented ordered integral framed link if its diagonal entries denote framings and off-diagonal entries linking numbers. For an integer c, let $\mathscr{H}(c)$ denote the set of unoriented unordered framed links whose linking matrices can be written as

$$\mathrm{diag}(\pm 1, \ldots, \pm 1, c) = (\pm 1) \oplus \cdots \oplus (\pm 1) \oplus (c),$$

where the signs of ± 1 are taken arbitrary.

Every integral homology sphere is obtained from a link in $\mathscr{H}(\pm 1)$. K. Habiro arranged two handle slides into a pair called a *band slide* so that it is

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closed in $\mathscr{H}(\pm 1)$. He proved that two links in $\mathscr{H}(\pm 1)$ yield homeomorphic manifolds if and only if they are related by a sequence of (de)stabilizations and band slides [2].

In the previous paper [1], we extended Habiro's theorem to $\mathscr{H}(\pm p)$ for an odd prime p, that is, two links in $\mathscr{H}(\pm p)$ yield homeomorphic manifolds if and only if they are related by a sequence of (de)stabilizations and band slides. Note that every manifold of linking form $(\pm 1/p)$ is obtained by a link in $\mathscr{H}(\pm p)$. We generalize these results to manifolds of linking forms $(\pm 1/p^s)$ as follows:

Theorem 1.1. *Let p be a positive odd prime and s be a non-negative integer. Two links in $\mathscr{H}(\pm p^s)$ yield homeomorphic 3-manifolds after surgery if and only if they are related by a sequence of (de)stabilizations, band slides and ambient isotopies.*

2. Proofs

Let I_n denote the identity matrix of size n and $E_{\xi,\zeta}$ denote the matrix unit with 1 for (ξ,ζ)-entry and 0 otherwise. We define the following $n \times n$ matrices:

$$P_{\xi,\zeta} := I_n - E_{\xi,\xi} - E_{\zeta,\zeta} + E_{\xi,\zeta} + E_{\zeta,\xi} \quad (1 \le \xi, \zeta \le n, \xi \ne \zeta), \tag{2.1}$$

$$Q_{\zeta} := I_n - 2E_{\zeta,\zeta} \quad (1 \le \zeta \le n), \tag{2.2}$$

$$R_{\xi,\zeta} := I_n + E_{\xi,\zeta} \quad (1 \le \xi, \zeta \le n, \xi \ne \zeta). \tag{2.3}$$

For $1 \le i \le r$ $(\le n/2)$, we regard i' and i'' as functions satisfying $\{i', i''\} = \{2i-1, 2i\}$. Put $\tau_{i',j'} := R_{i',j''}^{-1} R_{j',i''}$ for $1 \le i, j \le r$, $i \ne j$. Let $\langle \tau_{i',j'} \rangle$ denote the group generated by matrices $\tau_{i',j'}$. For vectors $\vec{v}$, $\vec{v}'$, we write $\vec{v} \sim_\tau \vec{v}'$ if $\vec{v}' = S\vec{v}$ for some $S \in \langle \tau_{i',j'} \rangle$. We denote the transposed matrix of M by ${}^{\mathrm{t}}M$. See [1] for detail.

We shall improve the argument of Section A.3 in [1].

Lemma 2.4. *For a number $s \in \mathbf{N}$, a prime p and any non-zero vector $\vec{v} \in (\mathbf{Z}/p^s\mathbf{Z})^{2r}$ of size $2r \ge 4$, there exist $w, t \in \mathbf{Z}$ with $0 \le t < s$ such that $\vec{v} \sim_\tau {}^{\mathrm{t}}(0, \dots, 0, wp^t, p^t) \pmod{p^s}$.*

Proof. Our proof is similar to that of Lemma A.15 in [1]. Thus, we may assume $r = 2$ and we have $\vec{v} \sim_\tau {}^{\mathrm{t}}(0, a, b, c)$. Take $a', b', c', t \in \mathbf{Z}$ so that

$${}^{\mathrm{t}}(0, a, b, c) = {}^{\mathrm{t}}(0, a'p^t, b'p^t, c'p^t) = {}^{\mathrm{t}}(0, a', b', c')p^t \quad (p \nmid \gcd(a', b', c')). \tag{2.5}$$

We abuse the vector ${}^{\mathrm{t}}(0, a', b', c')$ as one in $(\mathbf{Z}/p^{s-t}\mathbf{Z})^4$. Notice $a' \in (\mathbf{Z}/p^{s-t}\mathbf{Z})^\times$ since we may assume $a' \not\equiv 0 \pmod{p}$ (see [1]). Then, the same deformation as in [1] implies ${}^{\mathrm{t}}(0, a', b', c') \sim_\tau {}^{\mathrm{t}}(0, 0, w, 1)$. We complete the proof. $\square$

Remark 2.6. Lemma A.15 in [1] is obtained by putting $s = 1$. For $s > 1$, we need (2.5) and need $a' \in (\mathbf{Z}/p^{s-t}\mathbf{Z})^\times$ to change c' to 1 in $\mathbf{Z}/p^{s-t}\mathbf{Z}$.

Let $c \neq 0$ be an integer and A be an $n \times n$ integral matrix of the form $A = A' \oplus (c)$ such that A' is an $(n-1) \times (n-1)$ matrix of $\det A' = \pm 1$. We call

$$\mathrm{O}(A; \mathbf{Z}) := \{g \in \mathrm{GL}(n; \mathbf{Z}) \mid {}^{\mathrm{t}}gAg = A\}$$

the *orthogonal group* and $\mathrm{SO}(A; \mathbf{Z})$ denotes the *special orthogonal group*. Let $\vec{e}_n$ denote the unit vector ${}^{\mathrm{t}}(0, 0, \dots, 0, 1)$. The last column vector of $g \in \mathrm{SO}(A; \mathbf{Z})$ is written as $g\vec{e}_n$, which has the following simple form:

LEMMA 2.7. *Let A be a matrix as above. For any matrix $g \in \mathrm{SO}(A; \mathbf{Z})$, its last vector $g\vec{e}_n$ satisfies $g\vec{e}_n = \lambda\vec{e}_n \pmod{c}$ for some integer λ with $\lambda^2 \equiv 1 \pmod{c}$.*
In other words, when we write

$$g = \begin{pmatrix} P & \vec{u} \\ {}^{\mathrm{t}}\vec{v} & \lambda \end{pmatrix}$$

for some column vectors $\vec{u}$ and $\vec{v}$ of size $n-1$ and for some matrix P of size $n-1$, we have $\vec{u} = \vec{0} \pmod{c}$ and $\lambda^2 \equiv 1 \pmod{c}$.

Lemma 2.7 holds also for $c = 0$.

Proof of Lemma 2.7. Since we have ${}^{\mathrm{t}}gAg = A$, we have the following identities:

$$ {}^{\mathrm{t}}PA'P + c\vec{v}\,{}^{\mathrm{t}}\vec{v} = A', \tag{2.8}$$

$$ {}^{\mathrm{t}}PA'\vec{u} + c\lambda\vec{v} = \vec{0}, \tag{2.9}$$

$$ {}^{\mathrm{t}}uA'\vec{u} + c\lambda^2 = c. \tag{2.10}$$

By (2.8), we have ${}^{\mathrm{t}}PA'P = A' \pmod{c}$, and thus P is invertible modulo c. By (2.9), we have ${}^{\mathrm{t}}PA'\vec{u} = \vec{0} \pmod{c}$, and thus $\vec{u} = \vec{0} \pmod{c}$. We apply this result to (2.10), showing $c\lambda^2 \equiv c \pmod{c^2}$. This implies $\lambda^2 \equiv 1 \pmod{c}$ as desired. $\square$

Let p be a positive odd prime and s be a non-negative integer. For $c := p^s$ and $n = 2r + 1$, we consider the $n \times n$ matrix

$$A := \mathrm{diag}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, p^s\Big) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus (p^s). \tag{2.11}$$

Lemma 2.7 implies the following proposition:

PROPOSITION 2.12. *Let A be a matrix as in* (2.11). *For any matrix $g \in \mathrm{SO}(A; \mathbf{Z})$, its last vector $g\vec{e}_n$ satisfies $g\vec{e}_n = \lambda\vec{e}_n \pmod{p^s}$ for some integer λ with $\lambda^2 \equiv 1 \pmod{p^s}$.*

Proof. Put $A' := \operatorname{diag}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $c := p^s$. Then, apply Lemma 2.7. □

Remark 2.13. Lemma A.16 in [1] is obtained from Proposition 2.12 by putting $s = 1$.

For an odd integer $n \geq 5$, consider the following set of matrices:

$$P_{2i-1,2j-1}P_{2i,2j} \quad (1 \leq i, j \leq (n-1)/2, i \neq j), \tag{2.14}$$

$$P_{1,2}, \tag{2.15}$$

$$Q_1 Q_2, \tag{2.16}$$

$$Y := I_{n-3} \oplus \begin{pmatrix} c & -2k^2 & -2ck \\ -2 & c & 2c \\ 2 & -c+1 & -2c+1 \end{pmatrix} \quad (c = p^s = 2k+1 > 0), \tag{2.17}$$

$$\tau_{1,3} := R_{1,4}^{-1} R_{3,2}, \tag{2.18}$$

$$\tau_{1,n} := R_{n,2} R_{1,n}^{-2c} R_{n,2}, \tag{2.19}$$

$$Q_n. \tag{2.20}$$

See (2.1)–(2.3) for matrices $P_{\xi,\zeta}$, Q_ζ, $R_{\xi,\zeta}$ and I_n. We obtain the set of matrices (5.6)–(5.12) in [1] from the above one by putting $s = 1$ for Y and $\tau_{1,n}$.

THEOREM 2.21. *For a matrix A as in* (2.11), *suppose* $\operatorname{size}(A) = n \geq 5$. *The orthogonal group* $\mathrm{O}(A;\mathbf{Z})$ *is generated by matrices from* (2.14) *to* (2.20).

Proof. For $g \in \mathrm{SO}(A;\mathbf{Z})$, we have $g\vec{e}_n = \vec{e}_n \pmod 2$ similarly to [1, Lemma A.17]. Proposition 2.12 then implies $g\vec{e}_n = \lambda\vec{e}_n \pmod{2p^s}$. Since $p^s = 8Mp^{2s} + p^s\lambda^2$ for some $M \in \mathbf{Z}$ (see [1] for detail), we have $\lambda^2 \equiv 1 \pmod{2p^s}$. The fact that the multiplicative group $(\mathbf{Z}/p^s\mathbf{Z})^\times$ is cyclic deduces $\lambda \equiv \pm 1 \pmod{2p^s}$. Hence, either $g\vec{e}_n$ or $P_{1,2}Q_n g\vec{e}_n$ equals $\vec{e}_n \pmod{2p^s}$ (similarly to [1, Corollary A.18]). A discussion similar to one after [1, Lemma A.19] delivers a set of generators of $\mathrm{SO}(A;\mathbf{Z})$. Then, the same observation as one after [1, Theorem A.9] completes the proof. □

Remark 2.22. The technique of the above proof is the same as [1]. For $s > 1$, we need Proposition 2.12 and that $(\mathbf{Z}/p^s\mathbf{Z})^\times$ is cyclic (and has an even order).

Proof of Theorem 1.1. We prove it by a method similar to [1, 2]. It suffices to prove for $\mathscr{H}(p^s)$ because the other case follows from the bijection $\Theta : \mathscr{H}(p^s) \to \mathscr{H}(-p^s)$ induced by the orientation reversing involution on S^3. For two links in $\mathscr{H}(p^s)$, after suitable stabilization, we associate them to links with the same linking matrix A. Let Z denote one of those links. It is a key to find a sequence of handle slides relating Z to itself corresponding to each generating matrix of the orthogonal group in Theorem 2.21 (see [1, Proposition 4.4] and Remark 2.23 for detail). It gives a sequence s_0 as in Proof of Theorem 2.3 in [1]. Hence, the same argument after s_0 completes the proof. $\square$

Remark 2.23. In [1], we claim Lemma 5.13 to prove Proposition 4.4 under the condition that p is an odd prime but the lemma holds under that p is an odd integer (and then, so does the proposition). This is because realizations of matrices Y and $\tau_{1,n}$ are done in the same ways.

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